THE POLAR REPRESENTATION THEOREM FOR LINEAR HAMILTONIAN SYSTEMS

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Let $n = 1, 2, \ldots$. If M is a real matrix, we shall denote M^* its transpose. I_n is the identity $n \times n$ matrix.

Consider the time-dependent linear Hamiltonian system

$$\dot{Q} = BQ + CP, \quad \dot{P} = -AQ - B^* P, \tag{1}$$

where A, B and C are time-dependent $n \times n$ matrices. A and C are symmetric. The dot means time derivative, the derivative with respect to τ. The time variable τ belongs to an interval. Without loss of generality we shall assume that this interval is $[0, T[$, $T > 0$. T can be ∞. In the following t, $0 < t < T$, is also a time variable and $\tau \in [0, t]$.

If (Q_1, P_1) and (Q_2, P_2) are solutions of (1) one denotes $W(Q_1, P_1; Q_2, P_2)$ the Wronskian (which is constant)

$$W(Q_1, P_1; Q_2, P_2) \equiv W = P_1^* Q_2 - Q_1^* P_2.$$

A solution (Q, P) of (1) is called isotropic if $W(Q, P; Q, P) = 0$. From now on (Q_1, P_1) and (Q_2, P_2) will denote two isotropic solutions of (1) such that $W(Q_1, P_1; Q_2, P_2) = I_n$. This means that

$$P_1^* Q_2 - Q_1^* P_2 = I_n, \quad P_1^* Q_1 = Q_1^* P_1, \quad P_2^* Q_2 = Q_2^* P_2.$$

These relations express precisely that, for each $\tau \in [0, T[$ the $2n \times 2n$ matrix

$$\Phi = \begin{bmatrix} Q_2 & Q_1 \\ P_2 & P_1 \end{bmatrix} \tag{2}$$

is symplectic. Its left inverse and, therefore, its inverse, is given by

$$\Phi^{-1} = \begin{bmatrix} P_1^* & -Q_1^* \\ -P_2^* & Q_2^* \end{bmatrix}.$$

As it is well-known the $2n \times 2n$ symplectic matrices form a group, the symplectic group.

Then, one has

$$P_1 Q_2^* - P_2 Q_1^* = I_n, \quad Q_1 Q_2^* = Q_2 Q_1^*, \quad P_1 P_2^* = P_2 P_1^*,$$

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and, therefore,

$$Q_2^* P_1 - P_2^* Q_1 = I_n, \quad Q_2 P_1^* - Q_1 P_2^* = I_n,$$

and the following matrices, whenever they make sense, are symmetric:

$$P_2 Q_2^{-1}, \quad Q_1 P_1^{-1}, \quad Q_2 P_2^{-1}, \quad P_1 Q_1^{-1},$$
$$Q_2^{-1} Q_1, \quad P_2^{-1} P_1, \quad Q_1^{-1} Q_2, \quad P_1^{-1} P_2.$$

Denote by J, S and M, the following $2n \times 2n$ matrices

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \quad S = \begin{bmatrix} A & B^* \\ B & C \end{bmatrix},$$

and $M = -JS$. J is symplectic and S is symmetric.

Equation (1) can then be written

$$\dot{\Phi} = M\Phi.$$

Remark that, if Φ is symplectic, Φ^* is symplectic, and

$$\Phi^{-1} = -J\Phi^* J, \quad \Phi^* J\Phi = J, \quad \Phi J\Phi^* = J.$$

When we have a C^1 function $\tau \longmapsto \Phi(\tau)$, $\dot{\Phi} J\Phi^* + \Phi J\dot{\Phi}^* = 0$. Hence, $\dot{\Phi} J\Phi^*$ is symmetric and one can recover M:

$$M = \dot{\Phi}\Phi^{-1} = -\dot{\Phi} J\Phi^* J.$$

This means that from Φ one can obtain A, B, and C:

$$A = \dot{P}_1 P_2^* - \dot{P}_2 P_1^*, \quad C = \dot{Q}_1 Q_2^* - \dot{Q}_2 Q_1^*,$$
$$B = -\dot{Q}_1 P_2^* + \dot{Q}_2 P_1^* = Q_1 \dot{P}_2^* - Q_2 \dot{P}_1^*.$$

The proof of the following theorem on a polar representation can be found in [1]. See also [2], [3], [4].

Theorem 1. *Assume that $C(\tau)$ is always > 0 (or always < 0) and of class C^1. Consider two isotropic solutions of (1), (Q_1, P_1) and (Q_2, P_2), such that $W = I_n$. Then, there are C^1 matrix-valued functions $r(\tau)$, $\varphi(\tau)$, for $\tau \in [0, T[$, such that: a) $\det r(\tau) \neq 0$ and $\varphi(\tau)$ is symmetric for every τ; b) the eigenvalues of φ are C^1 functions of τ, with strictly positive (negative) derivatives; c) one has*

$$Q_2(\tau) = r(\tau)\cos\varphi(\tau) \quad \text{and} \quad Q_1(\tau) = r(\tau)\sin\varphi(\tau).$$

Remark that φ is not unique and that

$$\frac{d}{d\tau} Q_2^{-1} Q_1 = Q_2^{-1} C Q_2^{*-1}, \tag{3}$$

whenever $\det Q_2(\tau) \neq 0$ (see [1]).

Theorem 1 can be extended in the following way:

Theorem 2. *Assume that $C(\tau)$ is of class C^1. Consider two isotropic solutions of (1), (Q_1, P_1) and (Q_2, P_2), such that $W = I_n$. Then, there are C^1 matrix-valued functions $r(\tau)$, $\varphi(\tau)$, for $\tau \in [0, t]$, such that: a) $\det r(\tau) \neq 0$ and $\varphi(\tau)$ is symmetric for every τ; b) the eigenvalues of φ are C^1 functions of τ; c) one has*

$$Q_2(\tau) = r(\tau)\cos\varphi(\tau) \quad \textit{and} \quad Q_1(\tau) = r(\tau)\sin\varphi(\tau).$$

Proof. Let us first remark that $Q_2Q_2^* + Q_1Q_1^* > 0$. This is proved noticing that, as $P_1Q_2^* - P_2Q_1^* = I_n$, one has $\left(P_1^*x, Q_2^*x\right) - \left(P_2^*x, Q_1^*x\right) = |x|^2$, which implies that $\ker Q_1^* \cap \ker Q_2^* = \{0\}$. Hence, $\left(Q_2^*x, Q_2^*x\right) + \left(Q_1^*x, Q_1^*x\right) > 0$, for every $x \neq 0$.

Define now

$$\Phi = \begin{bmatrix} Q_2 & Q_1 \\ P_2 & P_1 \end{bmatrix}, \qquad \Psi = \begin{bmatrix} \cos(k\tau) I_n & \sin(k\tau) I_n \\ -\sin(k\tau) I_n & \cos(k\tau) I_n \end{bmatrix},$$

M as before, $\Phi_1 = \Phi\Psi$ and $M_1 = \dot{\Phi}_1\Phi_1^{-1}$. The constant k is > 0. Then, one has

$$M_1 = M + \Phi\dot{\Psi}\Psi^{-1}\Phi^{-1}.$$

Let the $n \times n$ matrices, that are associated with M_1, be A_1, B_1 and C_1. Then

$$C_1 = C + k\left(Q_2Q_2^* + Q_1Q_1^*\right).$$

Hence, as $Q_2Q_2^* + Q_1Q_1^* > 0$, for k large enough, we have that $C_1(\tau) > 0$, for every $\tau \in [0, t]$. We can then apply Theorem 1. There are C^1 matrix-valued functions $r_1(\tau)$, $\varphi_1(\tau)$, for $\tau \in [0, t]$, such that

$$\begin{aligned} \cos(k\tau) Q_2(\tau) - \sin(k\tau) Q_1(\tau) &= r_1(\tau)\cos\varphi_1(\tau) \\ \sin(k\tau) Q_2(\tau) + \cos(k\tau) Q_1(\tau) &= r_1(\tau)\sin\varphi_1(\tau). \end{aligned}$$

From this, we have

$$\begin{aligned} Q_2(\tau) &= r_1(\tau)\, \cos\left(\varphi_1(\tau) - k\tau I_n\right) \\ Q_1(\tau) &= r_1(\tau)\, \sin\left(\varphi_1(\tau) - k\tau I_n\right). \end{aligned}$$

□

The generic differential equations for r and φ are easily derived from equations (15), (17) and (18) in [1].

Consider (r_0, s), with s symmetric, such that

$$\dot{r}_0 = Br_0 + Cr_0^{*-1}s, \qquad \dot{s} = sr_0^{-1}Cr_0^{*-1}s + r_0^{-1}Cr_0^{*-1} - r_0^*Ar_0.$$

Then r is of the form $r = r_0\Omega$, where Ω is any orthogonal, $\Omega^{-1} = \Omega^*$, and time-dependent C^1 matrix. From this one can derive a differential equation for rr^*.

The function φ verifies the equations

$$\frac{\cos C_\varphi - I}{C_\varphi}\dot{\varphi} = -\Omega^*\dot{\Omega}, \quad \frac{\sin C_\varphi}{C_\varphi}\dot{\varphi} = r^{-1}Cr^{*-1}, \tag{4}$$

where $C_\varphi\dot{\varphi} = [\varphi, \dot{\varphi}] = \varphi\dot{\varphi} - \dot{\varphi}\varphi$, $\left(C_\varphi\right)^2\dot{\varphi} \equiv C_\varphi^2\dot{\varphi} = [\varphi, [\varphi, \dot{\varphi}]]$, and so on.

As in Theorem 1, φ is not unique. Remark that $r(\tau) = r_1(\tau)$ and $\varphi(\tau) = \varphi_1(\tau) - k\tau I_n$, with k large enough and φ_1 such that its eigenvalues are C^1 functions of τ, with strictly positive derivatives.

Remark 1. *If one considers Φ^* instead of Φ, then Q_2 is replaced by Q_2^* and Q_1 is replaced by P_1^*. Then, Theorem 2 gives*

$$Q_2^*(\tau) = r(\tau)\cos\varphi(\tau) \quad \textit{and} \quad P_2^*(\tau) = r(\tau)\sin\varphi(\tau),$$

or

$$Q_2(\tau) = \cos\varphi(\tau)\, r^*(\tau) \quad \textit{and} \quad P_2(\tau) = \sin\varphi(\tau)\, r^*(\tau).$$

In this case the matrix $\varphi(\tau)$ is a generalization of the so-called Prüfer angle.

References

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