SOLUTIONS

Chapter 5

Section 1

5.1.1 (a) From the definition of characteristic polynomial you know that

$$p_A(\mu) = \det(\mathrm{A} - \mu \mathrm{I}) = \det\left(\begin{bmatrix} 1-\mu & 1 & 0 \\ 1 & -\mu & 1 \\ 0 & 1 & 1-\mu \end{bmatrix}\right)$$

Computing the determinant you obtain

$$(1-\mu)(\mu^2 - \mu - 2)$$

To find the roots of the characteristic polynomial you just need to factor it, and since you have already written it as the product of a factor of degree 1 and a polynomial of degree 2, you just have to determine the roots of the last one. You find

$$p(\mu) = (1-\mu)(\mu-2)(\mu+1)$$

Therefore the only solutions to

$$\det(A - \mu I) = 0$$

hence the roots of the characteristic polynomial, are $\mu = 1,\ \mu = -1$ and $\mu = 2$.

(b) Eigenvalues: In part (a) you have found that the roots of the characteristic polynomial of A are $\mu = 1,\ \mu = -1$ and $\mu = 2$. Using the definition of eigenvalues of a matrix you conclude that the eigenvalues of A are $1, -1$ and 2.

Eigenspaces: In order to find bases for the eigenspaces you have to determine the kernel of the matrix $A - \mu I$ where μ takes the values $1,\ -1$ and 2.

Eigenspace associated to $\mu = 1$: You have to find the kernel of the matrix $A - I$ which means to solve

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

After row-reducing the coefficient matrix indicated above, you see that the kernel is the one-dimensional subspace of R^3 consisting of all multiples of the vector $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Let E_μ denote the subspace of R^3 consisting of vectors $\mathbf{v}$ with $A\mathbf{v} = \mu\mathbf{v}$. Then E_1 is the eigenspace corresponding to $\mu = 1$ and

$$E_1 = \left\{ t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} : t \in R \right\} .$$

Note: E_μ is not exactly the "set of eigenvectors with eigenvalues μ". As a subspace, sit contains 0 and by definition 0 is not an eigenvector. The "set of eigenvectors with eigenvalues μ" is E_μ with 0 removed.

Eigenspace associated to $\mu = -1$: You have to find the kernel of the matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

After solving the system $(A + I)\mathbf{x} = 0$ you find that the kernel of the matrix $(A + I)$ consists of all vectors multiple of the vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. The set $\{\mathbf{v}_2\}$ where $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is a basis for this space.

Eigenspace associated to $\mu = 2$: Following the same steps as previously you find that the kernel of the matrix $A - 2I$ consists of all multiples of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, hence a basis for this eigenspace is the set $\{\mathbf{v}_3\}$ where $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(c) Each of the eigenspaces has multiplicity one, both geometric and algebraic.

5.1.3 (a) The characteristic polynomial of A is

$$p(\mu) = \det(\mathrm{A} - \mu\mathrm{I}) = (\mu^2 - 5\mu)$$

Its roots are 0 and 5.

(b) Eigenvalues: From part (a) you know that

$$p(\mu) = (\mu^2 - 5\mu) \Rightarrow \mu = 0 \quad \text{or} \quad \mu = 5$$

Hence the eigenvalues of A are 0 and 5.

Eigenspace associated to $\mu = 0$: You have to find the kernel of the matrix A. The kernel is the one-dimensional subspace of R^2 consisting of all multiples of the vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and

$$E_0 = \left\{ t \begin{bmatrix} -2 \\ 1 \end{bmatrix} : t \in R \right\}$$

where E_0 denotes the eigenspace associated to the eigenvalue 0. The set $\{\mathbf{u}_1\}$ where $\mathbf{u}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a basis for this space.

Eigenspace associated to $\mu = 5$: After solving the system $(A + 5I)\mathbf{x} = 0$ you find that the set $\{\mathbf{u}_2\}$ where $\mathbf{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a basis for this space.

(c) Again, each of the eigenspaces has multiplicity one, both geometric and algebraic.

5.1.5 (a) The trace of A is 2, and the traceo f B is 1.

(**b**) The determinant of A is -4 and the determinant of B is -9.

(**c**) They cannot be similar since similar matrices have the same traces and the same determinants.

5.1.7 Eigenvalues of an Idempotent matrix: Since $A^2 = A$ we have that $\det(A^2 - A) = 0$. Using the properties of the determinant, this implies that $\det(A)\det(A - I) = 0$ and this is true if and only if $\det(A) = 0$ or $\det(A - I) = 0$. Hence there are exactly two eigenvalues of A, namely 0 and 1.

5.1.9 - Let $\mathbf{v}$ be an eigenvector of the matrix B corresponding to the eigenvalue λ and $A\mathbf{v} \neq 0$. We have

$$B(A\mathbf{v}) = (BA)\mathbf{v} = (AB)\mathbf{v} = A(B\mathbf{v}) = A(\lambda\mathbf{v})$$

where we have used the associativity property of the product of matrices, the fact that by the hypothesis $AB = BA$ (A and B commute) and the definition of eigenvector, respectively. But this imples that $B(Av) = \lambda Av$, that is, Av is an eigenvector of the matrix B associated to the eigenvalue λ.

Remark: Why do we need the condition $A\mathbf{v} \neq 0$? This is because if $A\mathbf{v} = 0$ then $A\mathbf{v}$ could not be an eigenvector, since eigenvectors are non–zero vectors by definition.

Section 2

5.2.1 We have already found the eigenvalues and eigenvectors in problem **5.1.1**. Moreover, the three eigenvalues are distinct, and so you know that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are the vectors you have determined is linearly independent, and hence is a basis of R^3. Now, to diagonalize A with an orthogonal matrix U, you need an orthonormal basis for R^3 consisting of eigenvectors. We have one if and only if the eigenspaces are orthogonal. Are they? Yes, so we can do it! In section 5 we will see that this was not an accident! It is a consequence of the fact that $A = A^t$.

Anyway, since $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are orthogonal to each other, we can normalize these vectors to obtain the basis $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ with

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \qquad \mathbf{u}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad \mathbf{u}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence A is diagonalizable and

$$A = UDU^{-1} \qquad \text{where} \qquad U = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Discussion: Are the matrices U and D unique? Of course you could have also written, for example

$$U = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

5.2.3 The characteristic polynomial of this matrix A is

$$\det(A - tI) = (1 - t)^3 \ ,$$

and hence the only eigenvalue is $\mu = 1$, repeated three times. Since

$$A - I = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \ ,$$

The null space of $A - I$, which is the eigenspace E_1, is spanned by $\mathbf{e}_1$. This is just one dimensional, and so there is no basis of R^3 consisting of eigenvectors of A. By Theorem 2, A is not diagonalizable.

5.2.5 We considered this matrix earlier in problem **5.1.3**. There we found that the eigenvalues we 0 and 5. Since these are distinct, the matrix A is diagonalizable: By Theorem 5.2.1, the corresponding pair of eigenvectors will be linearly independent and hence a basis for R^2. These eigenvectors were determined in problem **5.1.3**, and they happen to be orthogonal. Again, as we shall see, this is because $A = A^t$.

In any case, normalizing them, we obtain the orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$. Then you conclude that A is diagonalizable and

$$A = UDU^{-1} \qquad \text{where} \qquad U = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

Discussion: In this case there is only one other choice for U and D, that is

$$U = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$$

For an $ntimesn$ diagonalizable matrix with distinct eigenvalues how many different choices for U and D do you have? You will have $n!$. How do you check that your matrix is an orthogonal matrix? You have to recall the definition of an orthogonal matrix given in section 3.4 and see if $U^{-1} = U^t$.

5.2.7 (a) One of these matrices obviously has just one single eigenvalue. Which one? You've got it! Since A is a triangular matrix, its eigenvalues are the entries in main diagonal, which in this case is 0. Looking at the other matrices you cannot conclude much without computing the roots of the characteristic polynomial. Let's do it.

$$p_B(t) = t(t-2) \quad p_C(t) = t^2 - 5t + 3 \quad p_D(t) = t^2 - 2t - 3$$

None of these expressions are perfect squares so B, C and D have two eigenvalues.

(b) As in **5.1.1** and **5.1.3** using theorems 2 and 3 you can conclude immediately that B, C and D are diagonalizable.

One way to know whether A is diagonalizable or not, is to determine a basis for the eigenspace corresponding to A. But you immediately see that rank(A) = 1, therefore dim(ker(A)) = 1 which means the eigenspace has dimension one. Therefore you cannot find a basis of R^2 consisting of eigenvectors of A. The theorems you know give you the right answer without much work! Cool, or no?!

(c) You have

$$D^2 - 2D - 3I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

In part (a) you saw that the characteristic polynomial of D is $p_D(t) = t^2 - 2t - 3$. The expression you were asked to compute, $D^2 - 2D - 3I$, is exactly the characteristic polynomial but where t is substituted by D and that is always 0!

(d) Since B is diagonalizable you can write

$$B = UDU^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \frac{1}{4} \begin{bmatrix} -2 & 1 \\ 2 & 1 \end{bmatrix}$$

Therefore

$$B^{15} = UD^{15}U^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2^{15} \end{bmatrix} \frac{1}{4} \begin{bmatrix} -2 & 1 \\ 2 & 1 \end{bmatrix} = 2^{13} \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

5.2.9 We compute the charateristic polynomial of A, and then the from this the eigenvalues, finding the complex conjugate pair

$$\mu_\pm = 5 \pm 4i \ .$$

We the have

$$A - \mu_+ I = \begin{bmatrix} -4i & -8 \\ 2 & -4i \end{bmatrix} \ .$$

The corresponding eigenvector must be orthogonal to both rows of this matrix – by **V.I.F. 2**. Since the bottom row is $2\begin{bmatrix} 1 \\ -2i \end{bmatrix}$, the vector

$$\mathbf{v}_+ = \begin{bmatrix} 1 \\ -2i \end{bmatrix}^\perp = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

is an eigenvector with the eigenvalue μ_+.

To get the other eigenvalue, just take the complex conjugate. It is $\mathbf{v}_- = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$.

Now let

$$V = [\mathbf{v}_+, \mathbf{v}_-] = \begin{bmatrix} 2i & -2i \\ 1 & 1 \end{bmatrix} \ .$$

The inverse is

$$V^{-1} = \frac{1}{4}\begin{bmatrix} -i & 2 \\ i & 2 \end{bmatrix} \ .$$

Finally, let $D = \begin{bmatrix} 5+4i & 0 \\ 0 & 5-4i \end{bmatrix}$. Then $A = VDV^{-1}$, and so $A^k = VD^kV^{-1}$. Since

$$D^k = \begin{bmatrix} (5+4i)^k & 0 \\ 0 & (5-4i)^k \end{bmatrix} \ ,$$

we can compute the product explicitly to obtain

$$\frac{1}{4}\begin{bmatrix} 2(5+4i)^k + 2(5-4i)^k & 4i(5+4i)^k - 4i(5-4i)^k \\ -i(5+4i)^k + i(5-4i)^k & 2(5+4i)^k + 2(5-4i)^k \end{bmatrix} \ .$$

Leaving the answer in this form, it is quite compact, though it may look like the entries of A^k are somehow complex. This cannot be the case. Is our answer right?

It is. Actually, the numbers

$$(5+4i)^k + 2(5-4i)^k \qquad \text{and} \qquad 4i(5+4i)^k - 4i(5-4i)^k$$

are real. to see this, consider the first one, and apply the binomial formula. For the first of these we get

$$\begin{aligned} (5+4i)^k + (5-4i)^k &= \sum_{j=0}^{k} \binom{k}{j} 5^{k-j}(4i)^j + \sum_{j=0}^{k} \binom{k}{j} 5^{k-j}(-4i)^j \\ &= \sum_{j=0}^{k} \binom{k}{j} 5^{k-j} \left[(4i)^j + (-4i)^j\right] \ . \end{aligned}$$

When j is odd, $\left[(4i)^j + (-4i)^j\right] = 0$, and when j is even, say $j = 2\ell$,

$$\left[(4i)^j + (-4i)^j\right] = -2 \times 4^{2\ell} = -2^{2j+1} \ .$$

Hence

$$(5+4i)^k + (5-4i)^k = - \sum_{\substack{j=0 \\ j \text{ even}}}^{k} \binom{k}{j} 5^{k-j} 2^{2j+1} \ .$$

The same can be done for the second of the apparently complex numbers in our matrix. Try it!

5.2.11 (a) If $A\mathbf{v} = \mu\mathbf{v}$, then

$$A^k\mathbf{v} = A^{k-1}(A\mathbf{v}) = A^{k-1}(\mu\mathbf{v}) = \mu A^{k-1}\mathbf{v} \ .$$

Using the relation $A^k\mathbf{v} = \mu A^{k-1}\mathbf{v}$ over and over we get

$$A^k\mathbf{v} = \mu A^{k-1}\mathbf{v} = \mu^2 A^{k-2}\mathbf{v} = \cdots = \mu^k\mathbf{v} \ .$$

Hence μ^k is an eigenvalue of A, and $\mathbf{v}$ is an eigenvector for this eigenvalue.

(b) As we see for the first part, every eigenvector $\mathbf{v}$ os A is an eigenvector of A^k; it is just that the iegenvalue is now μ^k instead of μ. The converse is not true though. That is, for $k > 0$, it is not always the case that an eigenvector of A^k is also an eigenvector of A. For example, consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence *every* non zero vector in R^2 is an eigenvector of A^2 (with the eigenvalue 0), but the only eigenvalues of A are the non zero multiples of $\mathbf{e}_1$.

5.2.13 (a) If A has real entries, its characterisitic polynomial is real. Complex roots of real polynomials come in complex conjugate pairs, and so it cannot be that all three roots are complex. One must be real, and so A has at least one real eigenvalue.

(b) If A has at least one complex eigenvalue, then it has exactly two, by what was explained in part **(a)**. These are distinct, since they are a complex conjugate pair. There is, as explained in part **(a)** a third eigenvalue that is real, and hence distinct from the first two. Since A has 3 distinct eigenvalues, by Theorem 5.2.1, any set of corresponding eigenvectors is linearly independent, and therefore a basis of R^3. By Theroem 5.2.2, A is therefore diagonalizable.

Section 3

5.3.1 (a) Using the definition of derivative of a vector valued function you obtain $x'(t) = \begin{bmatrix} 2t \\ 1 \\ 3t^2 \end{bmatrix}$

(b) Analogously to part (a) you obtain $y'(t) = \begin{bmatrix} -t^{-2} \\ 1 \\ -2t^{-3} \end{bmatrix}$

(c) You have

$$\frac{d}{dt}|x(t)| = \frac{d}{dt}(t^4 + t^2 + t^6)^{1/2} == \frac{1}{2}(t^4 + t^2 + t^6)^{1/2}(4t^3 + 2t + 6t^5)$$

(d) You have

$$\frac{d}{dt}|y(t)| = \frac{d}{dt}(t^{-2} + t^2 + t^{-4})^{1/2} == \frac{1}{2}(t^{-2} + t^2 + t^{-4})^{-1/2}(-2t^{-1} + 2t - 4t^{-3})$$

(e) Using Theorems 1 and 2 we have

$$(x(t)\cdot Ay(t))' = \begin{bmatrix} 2t \\ 1 \\ 3t^2 \end{bmatrix} \cdot \begin{bmatrix} t^{-1} \\ 2t^{-1}+3t \\ t^{-1}+3t+3t^{-2} \end{bmatrix} + \begin{bmatrix} t^2 \\ t \\ t^3 \end{bmatrix} \cdot \begin{bmatrix} t^{-1} \\ 8t^{-1}+9 \\ t^{-2}+3t+3t^{-2} \end{bmatrix}$$

5.3.3 Superposition Principle in action: The differential equation can be written as

$$\mathbf{x\prime(t)} = A\mathbf{x(t)} \qquad \text{where} \qquad A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}.$$

The characteristic polynomial of A is $p(\lambda) = \lambda^2 - 2\lambda - 3$ and the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$ with corresponding eigenvectors $\mathbf{v_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Introducing the matrix

$$V = [\,\mathbf{v_1} \quad \mathbf{v_2}\,] = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

and the vector $\mathbf{a} = \begin{bmatrix} a \\ b \end{bmatrix}$ we should solve

$$V\mathbf{a} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

We have

$$\begin{bmatrix} a \\ b \end{bmatrix} = V^{-1} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

and we obtain $a = -2$ and $b = -1$, therefore we can write $\mathbf{x_0} = -2\mathbf{v_1} - \mathbf{v_2}$. Hence the solution $\mathbf{x(t)}$ is given by

$$\mathbf{x(t)} = -2e^{3t}v_1 - e^{-t}v_2 = -2e^{3t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} - e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{3t} - e^{-t} \\ -2e^{3t} - e^{-t} \end{bmatrix}.$$

and $x(t) = 2e^{3t} - e^{-t}$ and $y(t) = -2e^{3t} - e^{-t}$.

5.3.5 Superposition Principle in action: The differential equation can be written as

$$\mathbf{x\prime(t)} = A\mathbf{x(t)} \qquad \text{where} \qquad A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The characteristic polynomial of A is $p(\lambda) = (1-\lambda)(\lambda^2 - \lambda - 2)$ and the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 1$ and $\lambda_3 = 2$ with $\mathbf{v_1} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as corresponding eigenvectors *, respectively. We introduce the matrix

$$V = [\,\mathbf{v_1} \quad \mathbf{v_2} \quad \mathbf{v_3}\,] = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

and the vector $\mathbf{a} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Now we have to solve

$$V\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

* Recall that any non–zero multiples of these vectors are also eigenvectors of A.

We have

$$\begin{bmatrix} 1 & -1 & 1 & | & 1 \\ -2 & 0 & 1 & | & 2 \\ 1 & 1 & 1 & | & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 & | & 1 \\ 0 & -2 & 3 & | & 4 \\ 0 & 0 & 3 & | & 3 \end{bmatrix}$$

and we obtain $a = 0$, $b = 1$ and $c = 2$, that is $\mathbf{x_0} = \mathbf{v_2} + 2\mathbf{v_3}$. The solution $\mathbf{x(t)}$ is given by

$$\mathbf{x(t)} = e^t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 2e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -e^t + 2e^{2t} \\ 2e^{2t} \\ e^t + 2e^{2t} \end{bmatrix}.$$

and we conclude that $x(t) = -e^t + 2e^{2t}$, $y(t) = 2e^{2t}$ and $z(t) = e^t + 2e^{2t}$.

5.3.7 Superposition Principle in action: The equation can be written as

$$\mathbf{x\prime(t)} = A\mathbf{x(t)} \qquad \text{where} \qquad A = \begin{bmatrix} 5 & -8 \\ 2 & 5 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = 5 + 4i$ and $\lambda_2 = 5 - 4i$ and corresponding eigenvectors are $\mathbf{v_1} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} -2i \\ 1 \end{bmatrix}$. Let $V = [\,\mathbf{v_1} \quad \mathbf{v_2}\,]$ and $\mathbf{a} = \begin{bmatrix} a \\ b \end{bmatrix}$. Solving $V\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we obtain $a = \frac{1}{2} - \frac{i}{4}$ and $b = \frac{1}{2} + \frac{i}{4}$. We can write

$$\mathbf{x_0} = (\frac{1}{2} + \frac{i}{4})\mathbf{v_1} + (\frac{1}{2} - \frac{i}{4})\mathbf{v_2}$$

and the solution $\mathbf{x(t)}$ is given by

$$\mathbf{x(t)} = (\frac{1}{2} + \frac{i}{4})e^{(5+4i)t} \begin{bmatrix} 2i \\ 1 \end{bmatrix} + (\frac{1}{2} - \frac{i}{4})e^{(5-4i)t} \begin{bmatrix} -2i \\ 1 \end{bmatrix}.$$

Using the Euler's formula for $e^{(5+4i)t}$ and $e^{(5-4i)t}$ we have

$$\mathbf{x(t)} = \begin{bmatrix} e^{5t}(-2\sin(4t) - \cos(4t)) \\ e^{5t}(-\frac{1}{2}\sin(4t) + \cos(4t)) \end{bmatrix}.$$

We conclude that $x(t) = e^{5t}(-2\sin(4t) - \cos(4t))$ and $y(t) = e^{5t}(-\frac{1}{2}\sin(4t) + \cos(4t))$.

5.3.9 Solving a difference equation: We can write the equation in the form

$$a_{n+1} = \frac{3}{2}a_n + \frac{1}{2}a_{n-1}$$

Take $\mathbf{x_n} = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}$. Then the previous equation can be written as $\mathbf{x_{n+1}} = A\mathbf{x_n}$, that is

$$\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}.$$

As before, it is easy to see that $\mathbf{x_n} = A^{n-2}\mathbf{x_2}$. Therefore we should diagonalize A in order to compute A^{n-2}, as we did in Problem **5.2.7 (d)**. The eigenvalues of A are $\lambda_+ = \frac{1}{4}(3 + \sqrt{17})$ and $\lambda_- = \frac{1}{4}(3 - \sqrt{17})$ where corresponding eigenvectors are $\mathbf{v_1} = \begin{bmatrix} \lambda_+ \\ 1 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} \lambda_- \\ 1 \end{bmatrix}$, are corresponding eigenvectors , respectively. Write

$$V = \begin{bmatrix} \lambda_+ & \lambda_- \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$$

and compute V^{-1}.

$$V^{-1} = \frac{2}{\sqrt{17}} \begin{bmatrix} 1 & -\lambda_- \\ -1 & \lambda_+ \end{bmatrix}.$$

Again by Theorem 2 $A = VDV^{-1}$ and the solution $\mathbf{x_n}$ is given by

$$\mathbf{x_n} = VD^{n-2}V^{-1} = \begin{bmatrix} \lambda_+ & \lambda_- \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (\lambda_+)^{n-2} & 0 \\ 0 & (\lambda_-)^{n-2} \end{bmatrix} \frac{2}{\sqrt{17}} \begin{bmatrix} 1 & -\lambda_- \\ -1 & \lambda_+ \end{bmatrix}$$

Hence

$$x_n = \frac{2}{\sqrt{17}}[(\lambda_+)^{n-1}(1-\lambda_) + (\lambda_-)^{n-1}(1-\lambda_+)]$$

and finally

$$x_n = \frac{2^{1-2n}}{\sqrt{17}}[(3+\sqrt{17})^{n-1}(1+\sqrt{17}) + (3-\sqrt{17})^{n-1}(1-\sqrt{17})]$$

5.3.11 The eigenvalues of A are $\mu_1 = -1$ and $\mu_2 = 3$. Let corresponding eigenvectors be denoted $\mathbf{v}_1$ and $\mathbf{v}_2$. Then by the superposition principle, the general solution of the equation is

$$ae^{-t}\mathbf{v}_1 + be3t\mathbf{v}_2$$

for some numbers a and b. Now, as t goes to infinity, e^{-t} goes to zero, but e^{3t} blows up. Hence for the solution to go to zero, it is necessary and sufficient that $b = 0$ which means that $\mathbf{x}_0 = a\mathbf{v}_1$. In other words, the condition is that $\mathbf{x}_0$ must belong to the eigenspace E_{-1}. Hence the set in question is a subsapce, and $\{\mathbf{v}_1\}$ is a basis. you can check that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a valid choice for $\mathbf{v}_1$.

Section 4

5.4.1 (a) Computing the characteristic polynomial $p(t)$ of A we find

$$p(t) = t^2 - 6t - 1$$

and so the eigenvalues of A are $\mu_\pm = 3 \pm \sqrt{10}$. To find the eigenvectors of A, we compute that

$$A - \mu_+ I = \begin{bmatrix} -1-\sqrt{10} & 3 \\ 3 & 1-\sqrt{10} \end{bmatrix}.$$

Now by **V.I.F. 2**, a vector $\mathbf{v}$ is an eigenvector of A with eigenvalue μ_+ if and only if $\mathbf{v}$ is orthogonal to both rows of $A - \mu_+ I$. This can only be the case if the rows are parallel, so we need only check one row. Since the second row is

$$\mathbf{r} = \begin{bmatrix} 3 \\ 1-\sqrt{10} \end{bmatrix},$$

an eigenvector is

$$\mathbf{v} = \mathbf{r}^\perp = \begin{bmatrix} \sqrt{10}-1 \\ 3 \end{bmatrix}.$$

Since we are looking for an orthonormal basis, we normalize $\mathbf{v}$. We compute that

$$|\mathbf{v}| = \sqrt{(\sqrt{10}-1)^2 + 9} = \sqrt{20 - 2\sqrt{10}}\ .$$

Hence we have our first normalized eigenvector; we define

$$\mathbf{u}_+ = \frac{1}{\sqrt{20 - 2\sqrt{10}}} \begin{bmatrix} \sqrt{10} - 1 \\ 3 \end{bmatrix} .$$

Now for the other one. *We do not need to start over.* Theorem 5.4.1 ensures that any eigenvector for the eigenvalue μ_- must point in a direction orthogonal to $\mathbf{u}_+$. Since we are in R^2, there is only one direction orthogonal to $\mathbf{u}_+$. So we can take

$$\mathbf{u}_- = (\mathbf{u}_+)^\perp = \frac{1}{\sqrt{20 - 2\sqrt{10}}} \begin{bmatrix} -3 \\ \sqrt{10} - 1 \end{bmatrix} .$$

(b) Now that we have found the eigenvectors and eigenvalues, we can apply Theorem 5.2.2 to obtain the diagonalization. Since the matrix built out of the eigenvectors is orthogonal by virtue of Theorem 5.4.1, it is orthogonal, and so its inverse will be its transpose. The diagonal matrix D will be

$$D = \begin{bmatrix} \mu_+ & 0 \\ 0 & \mu_- \end{bmatrix} = \begin{bmatrix} 3 + \sqrt{10} & 0 \\ 0 & 3 - \sqrt{10} \end{bmatrix} .$$

The orthogonal matrix U is

$$U = [\mathbf{u}_+, \mathbf{u}_-] = \frac{1}{\sqrt{20 - 2\sqrt{10}}} \begin{bmatrix} \sqrt{10} - 1 & -3 \\ 3 & \sqrt{10} - 1 \end{bmatrix} .$$

5.4.3 (a) Computing the characteristic polynomial $p(t)$ of A we find

$$p(t) = t^3 - 4t - t + 4 .$$

This is a cubic polynomial, so we cannot use the quadratic formula to find its roots.

One of the majors successes of Italian renaissance mathematics was the development of a method for solving the general cubic equation. This was accomplished by Girolamo Cardano (1501-1576). You are not expected to know Cardano's formula; these days few people do*

So what do we do with this cubic? If we can find one root μ, we can then divide by $t - \mu$ and get a quadratic. Then apply the quadratic formula to find the other two roots. How do we find one root?

In this course, that is easy: Try *simple things* like 1, 0 and -1. In fact,

$$p(1) = 1 - 4 - 1 + 4 = 0 .$$

Doing the long division and dividing $p(t)$ by $(t - 1)$ we find

$$\frac{t^3 - 4t - t + 4}{t - 1} = t^2 - 3t + 4 = (t - 4)(t + 1) .$$

Hence the eigenvalues are

$$\mu_1 = 1 \qquad \mu_2 = 4 \qquad \text{and} \qquad \mu_3 = -1 .$$

---

* If you want to be one of the few, just do a web search on "Cardono", "cubic" and "formula".

Now, looking for vectors in $\mathrm{Ker}(A - \mu_j I)$, $j = 1, 2, 3$, we find the corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} .$$

Normalizing, we find the corresponding unit eigenvectors

$$\mathbf{u}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad \mathbf{u}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{u}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} .$$

(b) Now that we have found the eigenvectors and eigenvalues, we can apply Theorem 5.2.2 to obtain the diagonalization. Since the matrix built out of the eigenvectors is orthogonal by virtue of Theorem 5.4.1, it is orthogonal, and so its inverse will be its transpose. The diagonal matrix D will be

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix} .$$

The orthogonal matrix U is

$$U = [\mathbf{u}_2, \mathbf{u}_2, \mathbf{u}_3] .$$

5.4.5 It is easy to compute the norms of these matrices using Theorem 5.4.5!

First, you have

$$A^t A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 12 & -2 \\ 0 & -2 & 1 \end{bmatrix} .$$

The block structure of this matrix makes it easy to compute the characteristic polynomial of $A^t A$ which is

$$(t - 2)(t^2 - 13t + 8) .$$

Reducing the quadratic factor, we find that the largest (in absolute value) eigenvalue of $A^t A$ is $(13+\sqrt{137})/2$. Hence

$$\|A\| = \sqrt{(13 + \sqrt{137})/2} .$$

Next, you have

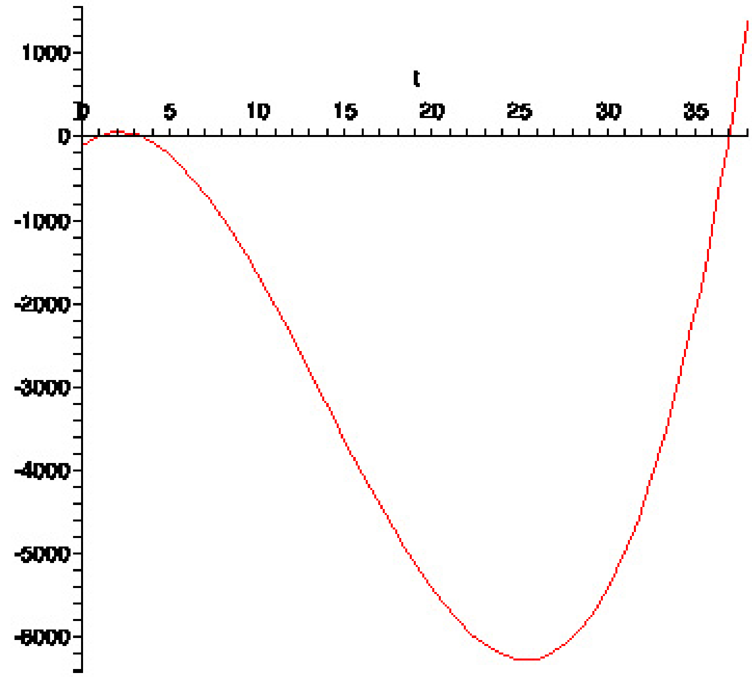
$$B^t B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 13 & 15 \\ 3 & 15 & 27 \end{bmatrix} .$$

Computing the characteristic polynomial $p(t)$ we find

$$p(t) = t^3 - 41t^2 + 153t - 81 .$$

This is a nasty of a cubic! Just try to find its roots algebraically. Go ahead, just try it!

Actually, we do not suggest wasting your time this way. Instead, graph the cubic.



You see from the graph that there are three roots near 1, 3 and 37. Now that we know that the largest roots is near 37, we use this as a starting approximation for Newton's method to find an arbitrarily accurate value for the root: To 10 decimal places, we find that the largest roots is

$$36.91475691\ldots\ ,$$

and hence

$$\|B\| = \sqrt{36.91475691\ldots} = 6.075751551\ldots\ .$$

The exact value of the largest root of this polynomial is in fact

$$\frac{1}{3}\left(41786 + 541\sqrt{26997}\right)^{1/3} + \frac{1222}{3\left(41786 + 541\sqrt{26997}\right)^{1/3}} + \frac{41}{3}\ ,$$

hence the exact value of the norm is

$$\|B\| = \sqrt{\frac{1}{3}\left(41786 + 541\sqrt{26997}\right)^{1/3} + \frac{1222}{3\left(41786 + 541\sqrt{26997}\right)^{1/3}} + \frac{41}{3}}\ .$$

What you see from this problem is that already with 3×3 matrices, exact calculation with characteristic polynomials gets *very complicated*. Eventually, being practical, one would have to resort to approximation methods, like Newton's method, to find the roots of the characteristic polynomial. We will not discuss them here, but actually, there a more direct ways of extracting decimal values of eigenvalues. In the mean time, at least we have *a way* to get our answers if we want them.

5.4.7 (This was list as 5.4.6 in the text, following a 5.3.6. It should have been numbered 5.4.6).

(a) The answer to this question is the unit vector corresponding to the largest eigenvalue of A. Since A is a diagonal matrix, its eigenvalues are the diagonal entries, that is $\lambda_1 = 2$ and $\lambda_2 = 1$. An eigenvector corresponding to 2 is $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. It is already a unit vector!! Easy no?!

Remark: Of course you could choose any non–zero multiple of the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and normalize it!

(b) An eigenvector corresponding to the smallest eigenvalue is $\mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ The vector $\mathbf{u}$ that satisfies the inequality is $\mathbf{u} = \begin{bmatrix} -\frac{1}{sqrt2} \\ \frac{1}{sqrt2} \end{bmatrix}$.

5.4.11 Nilpotent matrices:

(a) Let $\mathbf{v}$ be an eigenvector of A corresponding to the eigenvalues λ and this implies by Problem 5.2.11 that $A^n\mathbf{v} = \lambda^n\mathbf{v}$. Since A is nilpotent, there is an n for which $A^n\mathbf{v} = 0$. But, by the definition of eigenvalue, this implies that for that n, $\lambda^n = 0$ and therefore $\lambda = 0$.

(b) Let A be a nilpotent matrix. We know from **(a)** that the only eigenvalue of A is the zero eigenvalue and using Theorem 2 that implies that A is the zero matrix. So the only symmetric and idempotent matrix is the zero matrix.

Section 5

Problem 5.5.1 (a) We first find the eigenvalues and eigenvectors of A in the usual way: There are three eigenvalues, namely 2, 1 and -1. The corresponding eigenspaces are one dimensional and are:

$$E_2 = \left\{ t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} , t \in R \right\}$$

$$E_1 = \left\{ t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} , t \in R \right\}$$

$$E_{-1} = \left\{ t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} , t \in R \right\}$$

Notice that A is symmetric, and consequently the eigenvecors found above are orthogonal. Using this, you can digaonalize A and find

$$e^{tA} = \frac{1}{6} \begin{bmatrix} 3e^t + 2e^{2t} + e^{-t} & 2e^{2t} - 2e^{-t} & 2e^{2t} + e^{-t} - 3e^t \\ 2e^{2t} - 2e^{-t} & e^{2t} + 4e^{-t} & 2e^{2t} - 2e^{-t} \\ 2e^{2t} + e^{-t} - 3e^t & 2e^{2t} - 2e^{-t} & 3e^t + 2e^{2t} + e^{-t} \end{bmatrix} .$$

(b) Method 1: Multiply the vector of initial data $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ on the left by e^{tA} to get

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = e^{tA} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 4e^{2t} - e^{-t} \\ 4e^{2t} + 2e^{-t} \\ 4e^{2t} - e^{-t} \end{bmatrix} .$$

(b) Method 2: (This is faster if you were only asked to solve the equation for one set of initial data). You can easily expand the initial data as a linear combintation of eigenvectors. Let $\mathbf{x}_0$ be the initial data vector, and let

$$\mathbf{u}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{u}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis of eigenvectors with eigenvalues 2, 1 and -1 respectively in the given order.

It is easy to expand in terms of an orthonormal basis:

$$\begin{aligned}\mathbf{x}_0 &= (\mathbf{x}_0 \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{x}_0 \cdot \mathbf{u}_2)\mathbf{u}_2 + (\mathbf{x}_0 \cdot \mathbf{u}_3)\mathbf{u}_3 \\ &= \frac{4}{\sqrt{3}}\mathbf{u}_1 - \frac{2}{\sqrt{6}}\mathbf{u}_3 \ .\end{aligned}$$

Since $e^{2t}\mathbf{u}_1$ and $e^{-t}\mathbf{u}_3$ are solutions of our system, the superposition principle says that the solution starting from $\mathbf{x}_0$ is

$$\frac{4}{\sqrt{3}}e^{2t}\mathbf{u}_1 - \frac{2}{\sqrt{6}}e^{-t}\mathbf{u}_3$$

which is just what we found above.

Remark: Notice that in the superposition principle approach we did all of the steps explicitly, and this full solution wasn't longer than the sketched solution with the exponenetial. The superposition principle is usually much faster if you just want to solve your equation for one specific set of initial data. But if you will be considering the same equation for many different cases of initial data, there is considerable value in working out the exponential matrix. Indeed, in problems involving probability theory for example, the entries of matrices e^{tB} arising there are called "transition probabilities" and are themselves important in applications. The same sort of thisng is true in other applications as well.

5.5.3 (a) Computing the exponential of a matrix: As in Example 1, the first step is to compute the eigenvalues of A. They are $\lambda_1 = 0$ and $\lambda_2 = 5$ where $\mathbf{v_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are corresponding eigenvalues, respectively. Since A is a symmetric matrix, we know from Theorem 3 that there is an orthonormal matrix U such that $A = UDU^t$. The columns of U are the vectors of a basis for R^2 consisting of normalized eigenvectors, so you just need to normalize the eigenvectors of A. Let us consider

$$U = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

Then

$$U^{-1} = U^t = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

and finally

$$e^{tA} = \frac{1}{5}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{5t} \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 + e^{5t} & -2 + 2e^{5t} \\ -2 + 2e^{5t} & 1 + 4e^{5t} \end{bmatrix}.$$

(b) Solving a system of differential equations using matrix exponentials: As in Example 3 we just have to compute $e^{tA}\mathbf{x_0}$.

$$\mathbf{x(t)} = \frac{1}{5}\begin{bmatrix} 4 + e^{5t} & -2 + 2e^{5t} \\ -2 + 2e^{5t} & 1 + 4e^{5t} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} e^{5t} \\ 2e^{5t} \end{bmatrix}$$

5.5.5 Computing the exponential of a matrix: The eigenvalues of A are $\lambda_1 = 1 + i\sqrt{2}$ and $\lambda_2 = 1 - i\sqrt{2}$. The eigenspace $E_{1+i\sqrt{2}}$ is the line spanned by the vector $\mathbf{v_1} = \begin{bmatrix} i\sqrt{2} \\ 1 \end{bmatrix}$ and the eigenspace $E_{1-i\sqrt{2}}$ is the line spanned by the vector $\mathbf{v_2} = \begin{bmatrix} -i\sqrt{2} \\ 1 \end{bmatrix}$. Form the matrix $V = [\,\mathbf{v_1} \quad \mathbf{v_2}\,]$. From Theorem 2 of Section 2 we have

$$A = VDV^{-1} = \begin{bmatrix} i\sqrt{2} & -i\sqrt{2} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 + i\sqrt{2} & 0 \\ 0 & 1 - i\sqrt{2} \end{bmatrix}\begin{bmatrix} -\frac{i}{2\sqrt{2}} & \frac{1}{2} \\ \frac{i}{2\sqrt{2}} & \frac{1}{2} \end{bmatrix}$$

and then

$$e^{tA} = \begin{bmatrix} i\sqrt{2} & -i\sqrt{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{(1+i\sqrt{2})t} & 0 \\ 0 & e^{(1-i\sqrt{2})t} \end{bmatrix} \begin{bmatrix} -\frac{i}{2\sqrt{2}} & \frac{1}{2} \\ \frac{i}{2\sqrt{2}} & \frac{1}{2} \end{bmatrix}$$

Using the Euler's formula we obtain

$$e^{tA} = \begin{bmatrix} \frac{1}{2}e^t \cos\sqrt{2}t & -\sqrt{2}e^t \sin\sqrt{2}t \\ \frac{1}{\sqrt{2}}e^t \sin\sqrt{2}t & \frac{1}{2}e^t \cos\sqrt{2}t \end{bmatrix}$$

5.5.7 (a) Algebraic and geometric multiplicity of eigenvalues: The characteristic polynomial of A is $p(\lambda) = (\lambda + 1)(1 - \lambda^2)$.

The eigenvalues of the matrix A are $\lambda_1 = -1$ with algebraic multiplicity 2 and $\lambda_2 = 1$ with algebraic muliplicity equal to 1. The only possible eigenvalue with geometric multiplicity less than its algebraic multiplicity is -1. To check it, we just need to know the rank of the matrix $A + I$. We have

$$A + I = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -2 & 3 \end{bmatrix}$$

Clearly $\mathrm{rank(A + I)} = 2$, therefore $\mathrm{ker(A + I)} = 1$ what is exactly the dimension of the eigenspace corresponding to -1, that is, its geometric multiplicity.

(b) Computing generalized eigenvectors: Since both eigenspaces of A have dimension 1, we need to find a generalized eigenvector of A.

We find

$$A - I = \begin{bmatrix} -1 & -1 & 2 \\ 0 & -2 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$

that has rank 2. Solving $(A - I)\mathbf{v_1} = 0$ we see that the eigenspace E_1 is the line spanned by the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Let us compute the eigenspace E_{-1}. We have

$$A + I = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -2 & 3 \end{bmatrix}$$

Solving $(A + I)\mathbf{v_2} = 0$ we see that this eigenspace is the line spanned by the vector $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

Right now we have two eigenvectors for the basis we are looking for and we just need to find a third vector. Using Theorem 3 we know that since the algebraic multiplicity of 1 is 1 we do not get anything new from this eigenvalue and we look for a generalized eigenvector at the kernel of $(A + I)^2$.

$$(A + I)^2 = \begin{bmatrix} 2 & -3 & 5 \\ 2 & -3 & 5 \\ 2 & -3 & 5 \end{bmatrix}$$

Since $(A + I)^2$ has rank 1, its kernel has dimension 2 and solving $(A + I)^2\mathbf{v} = 0$ we see that its eigenspace is the plane spanned by the vectors $\begin{bmatrix} -5 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$. Lets choose, for example $\mathbf{v_3} = \begin{bmatrix} -5 \\ 0 \\ 2 \end{bmatrix}$. We now have our basis $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$, consisting of generalized eigenvectors.

(c) Computing the exponential of a matrix using generalized eigenvectors: Form $V = [\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}]$ with the vectors you found in part (b). Since

$$e^{tA}V = [\, e^{tA}\mathbf{v_1} \quad e^{tA}\mathbf{v_2} \quad e^{tA}\mathbf{v_3} \,]$$

the first step is to compute $e^{tA}\mathbf{v_1}$,$e^{tA}\mathbf{v_2}$ and $e^{tA}\mathbf{v_3}$. From $A^k\mathbf{v_1} = \mathbf{v_1}$ we conclude that

$$e^{tA}\mathbf{v_1} = e^t\mathbf{v_1} = \begin{bmatrix} e^t \\ e^t \\ e^t \end{bmatrix}$$

In the same vein, from $A^k\mathbf{v_2} = -\mathbf{v_2}$, we conclude

$$e^{tA}\mathbf{v_2} = e^{-t}\mathbf{v_2} = \begin{bmatrix} -e^{-t} \\ e^{-t} \\ e^{-t} \end{bmatrix}$$

Now we have to compute $e^{tA}\mathbf{v_3}$. Since $A^k\mathbf{v_3}$ is not a multiple of $\mathbf{v_3}$, we have to proceed in a slightly different way. We have that

$$e^{tA}\mathbf{v_3} = e^{-tI}e^{t(A+I)}\mathbf{v_3} = e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{k!}(A+I)^k\mathbf{v_3} = e^{-t}[I + t(A+I) + \sum_{k=2}^{\infty}\frac{t^k}{k!}(A+I)^k\mathbf{v_3}]$$

But because $(A+I)^k\mathbf{v_3} = 0$ for all $k \geq 2$, we have just to compute $(A+I)\mathbf{v_3}$, and we obtain

$$(A+I)\mathbf{v_3} = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} -5 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{v_2}.$$

Clearly

$$e^{tA}\mathbf{v_3} = e^{-t}(\mathbf{v_3} + t\mathbf{v_2}) = \begin{bmatrix} e^{-t}(-5-t) \\ te^{-t} \\ e^{-t}(2+t) \end{bmatrix}$$

and

$$e^{tA}V = \begin{bmatrix} e^t & -e^{-t} & -5e^{-t} - te^{-t} \\ e^t & e^t & te^{-t} \\ e^t & e^t & 2e^{-t} + te^t \end{bmatrix}$$

Therefore

$$e^{tA} = (e^{tA}V)V^{-1} = \begin{bmatrix} e^t & -e^{-t} & -5e^{-t} - te^{-t} \\ e^t & e^t & te^{-t} \\ e^t & e^t & 2e^{-t} + te^t \end{bmatrix}\frac{1}{4}\begin{bmatrix} 2 & -3 & 5 \\ -2 & 7 & -5 \\ 0 & -2 & 2 \end{bmatrix}$$

and

$$e^{tA} = \frac{1}{4}\begin{bmatrix} 2e^t + 2e^{-t} & -3e^t + 3e^{-t} + 2te^{-t} & 5e^t - 5e^{-t} - 2te^{-t} \\ 2e^t - 2e^{-t} & 4e^t - 2te^{-t} & 2te^{-t} \\ 2e^t - 2e^{-t} & 4e^t - 4e^{-t} - 2t4e^{-t} & 4e^{-t} + 2te^{-t} \end{bmatrix}.$$

(d) Solving a system of differential equations: Following previous examples we just need to calculate $e^{tA}\mathbf{x(0)}$ and we obtain

$$e^{tA}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} == \frac{1}{4}\begin{bmatrix} 2e^t + 2e^{-t} & -3e^t + 3e^{-t} + 2te^{-t} & 5e^t - 5e^{-t} - 2te^{-t} \\ 0 & 4e^t - 2te^{-t} & 2te^{-t} \\ 0 & 4e^t - 4e^{-t} - 2t4e^{-t} & 4e^{-t} + 2te^{-t} \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

and

$$\mathbf{x(t)} = \frac{1}{4}\begin{bmatrix} 7e^{-t}+5e^{t}+2te^{-t} \\ 8e^{t}-2te^{-t} \\ 8e^{t}-4e^{-t}-2te^{-t} \end{bmatrix}$$

The solution of the given equation with $\mathbf{x(0)}$ as its initial condition is $x(t) = \frac{7}{4}e^{-t} + \frac{5}{4}e^{t} + \frac{1}{2}te^{-t}$, $y(t) = 2e^{t} - \frac{1}{2}te^{-t}$ and $z(t) = 2e^{t} - e^{-t} - \frac{1}{2}te^{-t}$.

Section 6

5.6.1 - Classification of quadratic forms:

(a) We can solve this problem using different methods.

Method 1: Completing the square we have

$$x^2 + 2xy + 3y^2 = (x+y)^2 + 2y^2$$

Since the sum of two squares is always a positive number we imediately see that $x^2 + 2xy + 3y^2 \geq 0$. it is easy to see that $x^2 + 2xy + 3y^2 = 0$ if and only if $(x+y) = 0$ and $y = 0$, that is, when $x = 0$ and $y = 0$. Therefore the quadratic form is definite positive.

Method 2: The quadratic form can be written as

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

The eigenvalues of the matrix are $\lambda_1 = 2 + \sqrt{2}$ and $\lambda_2 = 2 - \sqrt{2}$, two positive real numbers and using Theorem 1 we conclude that the quadratic form is definite positive.

(b) Computing the eigenvalues of the matrix associated to the quadratic form we obtain $\lambda_1 = 2 + \sqrt{5}$ and $\lambda_1 = 2\sqrt{5}$. Using Theorem 1 we conclude it is an hyperbolic quadratic form.

(c) We have

$$2xy - x^2 - 2y^2 = -[(x-y)^2 + y^2 0$$

It is equal to zero if and only if $x = y = 0$. It is a definite negative quadratic form.