Solutions for Chapter 2.5 and 2.6

Section 5

2.5.1 To find the inverse of the matrix A means to find a matrix C of the same size such that $CA = I$, in which case $AC = I$ automatically follows. We find C by keeping a record of the row operation needed to row reduce A all the way to the identity: Write the matrices A and I side by side

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ 4 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Now you see that multiplying the first row of $[A|I]$ by -2 and adding to the second row you will transform the first entry of the second row into a 0. Multiplying the first row by -4 and adding to the third row you will obtain a zero for the first entry of the original matrix. That is

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ 4 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 0 & -7 & -2 & 1 & 0 \\ 0 & -7 & -14 & -4 & 0 & 1 \end{array}\right]$$

Now we interchange the rows 2 and 3 to obtain the row reduced form of $[A|I]$, and hence A. (We also divide the second and third rows by 7 to make the leading entries 1.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{4}{7} & 0 & -\frac{1}{7} \\ 0 & 0 & 1 & \frac{2}{7} & -\frac{1}{7} & 0 \end{array}\right]$$

You see that there are pivots in each column of A, so the rank is 3, and A is invertible.

To find the inverse, just keep going, and clean out the the entries that are above the diagonal: Cleaning up the third column, we get:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -\frac{1}{7} & \frac{4}{7} & 0 \\ 0 & 1 & 0 & 0 & \frac{2}{7} & -\frac{1}{7} \\ 0 & 0 & 1 & \frac{2}{7} & -\frac{1}{7} & 0 \end{array}\right]$$

One more operation cleans up the second column. Subtracting twice the second row from the first, we finally get A^{-1}:

$$A^{-1} = \begin{bmatrix} -\frac{1}{7} & 0 & \frac{2}{7} \\ 0 & \frac{2}{7} & -\frac{1}{7} \\ \frac{2}{7} & -\frac{1}{7} & 0 \end{bmatrix}$$

Now for B. We form $[B|I]$, and row reduce it, finding in three operations,

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -4 & -8 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right] .$$

There is no pivot in the third column. Thus the rank of B is 2, and it is not invertible.

(b) It is never poossible for $A\mathbf{x} = \mathbf{b}$ to have infinitely many solutions; there is always just one solution, namely $A^{-1}\mathbf{b}$.

(c) It is possible to find a vector $\mathbf{b}$ so that $A\mathbf{x} = \mathbf{b}$ has infinintely many solutions since there is no pivot in the third column when we row reduce B. This means that z will be a free variable. We just need to choose $\mathbf{b}$ so that when we row reduce, we do not get a pivot in the final column. This would mean that there is no solution, even if z is free.

Here is a cheap way to do this: Take $\mathbf{b} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$. This is the first column of B and so $\mathbf{e}_1$ is a solution of $B\mathbf{x} = \mathbf{b}$. (For *any* matrix C, $C\mathbf{e}_j$ is the jth column of C). Hence, with this choice of $\mathbf{b}$, $B\mathbf{x} = \mathbf{b}$ has at least one solution. Since the rank of B is 2, this solution cannot be unique, so there are infinitely many solutions.

2.5.3 Since B is already row reduced and all of the diagonal entries are 1, this is easy. It just takes 6 simple row operations to clean out the 6 upper right entries of B. Doing the row operations, we transform $[B|I]$ to

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] .$$

Thus,

$$B^{-1} = \begin{bmatrix} -2 & 1 & 0 & \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

2.5.5 This has already been done in part (b) of problem 2.5.1

2.5.7 Let $\mathbf{v}$ be the third column of A^{-1}. Then $A\mathbf{v} = \mathbf{e}_3$, the third column of the identity matrix. So the vector we are looking for is the solution of $A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

To solve this, row reduce $\begin{bmatrix} -1 & 0 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 2 & 1 & 5 & | & 1 \end{bmatrix}$. The result is

$$\begin{bmatrix} -1 & 0 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 6 & | & 1 \end{bmatrix}$$

This is equivalnet to the system of equations

$$-x + z = 0$$

$$y + z = 0$$

$$6z = 1$$

Hence $z = 1/6$, $y = -1/6$ and $x = 1/6$. That is, the third column is $\mathbf{v} = \frac{1}{6}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

2.5.8 You can write $C^2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ as

$$C\left(C\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

But since you know that

$$C\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

the given information tells us what C does to two input vectors:

$$C\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad C\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \tag{$*$}$$

To find C, combing the input vectors into the 2×2 matrix $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, and the two outputs into the 2×2 matrix $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ Then (*) is the same as

$$C\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

and so

$$C = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1}$$

Therefore

$$C = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} & \frac{5}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

2.5.9 You just need to do the same type of reasoning that you did in the previous problem. The matrix C is such that

$$C \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

That is

$$C = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}^{-1}$$

Which property did you use in order to conclude that? You used the fact that if you multiply, on the right, both sides of the equation $CA = B$ by the matrix A^{-1} you obtain $C = BA^{-1}$. Do not forget that you would obtain a completely different result if you have multiplied $CA = B$, on the left, by the matrix A^{-1}. In that case you would have obtained $A^{-1}CA = A^{-1}B$.

Getting back to our initial problem you find that

$$C = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{7}{2} & \frac{3}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 & -8 & 3 \\ -1 & -3 & 2 \\ 0 & -5 & 1 \end{bmatrix}$$

Section 6

2.6.1 Write down the augmented matrix

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 2 & 5 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Cleaning out the first column leads to

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & -7 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right].$$

Cleaning out the second column leads to

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & -7 & -2 & 1 & 0 \\ 0 & 0 & -10 & -3 & 1 & 1 \end{array}\right] = [U|R] .$$

Hence

$$R = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix} \qquad \text{and} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -7 \\ 0 & 0 & -10 \end{bmatrix} .$$

Next,

$$R\mathbf{b} = R \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} .$$

Since R is invertible, and $A = RU$, $A\mathbf{x} = \mathbf{b}$ if and only if

$$U\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} .$$

This is equivalent to the system

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 1 \\ x_2 - 7x_3 &= -1 \\ -10x_3 &= -1 \end{aligned}$$

This is solved by $x_3 = 1/10$, $x_2 = -3/10$ and $x_1 = 6/5$, as one sees by back substitution. that is the unique solution of $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = \frac{1}{10} \begin{bmatrix} 12 \\ -3 \\ 1 \end{bmatrix} .$$

2.6.3 The matrix here is the matrix we worked with in problem (2.6.1). There found $RA = U$, and so with $L = R^{-1}$, we have $A = LU$. Therefore, we compute R^{-1} where R is the matric found in problem (2.6.1). Form the augmented matrix

$$[R|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ -3 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

In three easy steps, the usual reduction procedure leads to

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

Hence

$$L = R^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} .$$

Then, with $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as before, $L\mathbf{y} = \mathbf{b}$ is solved by

$$y = L^{-1}\mathbf{b} = R\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

as before. Then $U\mathbf{x} = \mathbf{y}$ is solved uniquely by

$$\mathbf{x} = \frac{1}{10} \begin{bmatrix} 12 \\ -3 \\ 1 \end{bmatrix} ,$$

where the entries are found by solving the exact same disentangled system we solved in problem (2.6.1).

2.6.5 Write down the augmented matrix

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ 4 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Cleaning out the first column leads to

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 0 & -7 & -2 & 1 & 0 \\ 0 & -7 & -14 & -4 & 0 & 1 \end{array}\right] .$$

Evidently, we have to swap rows 2 and 3. Therefore, let P be the permutation matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} ,$$

so that

$$PA = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 1 & 2 \\ 2 & 4 & 1 \end{bmatrix} .$$

Now start over again with this matrix, and everything will work out fine: Form the augmented matrix

$$[PA|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

Cleaning out the first columns automatically cleans out the second column this time, and in just two steps this reduces to

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & -7 & -14 & -4 & 1 & 0 \\ 0 & 0 & -7 & -2 & 0 & 1 \end{array}\right] = [U|R] \ .$$

Now since $R = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, we have $L = R^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$. All together,

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad , \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -7 & -14 \\ 0 & 0 & -7 \end{bmatrix} \ .$$

Now with $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$,

$$R\mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix} \ .$$

Then $U\mathbf{x} = R\mathbf{b}$ is equaivalent to the system

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 1 \\ -7x_2 - 14x_3 &= -3 \\ -7x_3 &= -1 \end{aligned}$$

The unique solution of this system is $x_3 = 1/7$, $x_2 = 1/7$ and $x_3 = 1/7$. Hence the unique solution to $U\mathbf{x} = R\mathbf{b}$, which is also the unique solution to $A\mathbf{x} = \mathbf{b}$, is

$$\mathbf{x} = \frac{1}{7} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{7}\mathbf{b} \ .$$

This could have been seed directly: Notice that the sum of the entries in each row is the same; that is, 7. Hence $A\mathbf{x} = 7\mathbf{x}$. Therefore, $A\left((1/7)\mathbf{x}\right) = \mathbf{b}$.

2.6.7 (a) The entries of $\mathbf{y} = P\mathbf{x}$ are the same numbers as the entries of $\mathbf{x}$, just in a different order. Therefore,

$$|P\mathbf{x}|^2 = |\mathbf{y}|^2 = \sum_{j=1}^{n} y_j^2 = \sum_{j=1}^{n} x_j^2 = |\mathbf{x}|^2 \ .$$

(b) By the first part, P is an isometry, and so by Theorem 4 of Section 5 of Chapter 1, $P^tP = I$, so P has a left inverse. But P is square, so by Theorem 4 of Section 5 of this chapter, P is invertible.

Alternatively, it is pretty clear that any permutaion, as a transformation, has an inverse: Just put things back in the original order. But the correspondence between matrices and transformations is such that the matrix has a matrix inverse if and only if the transformation has an inverse.

2.6.9 (a) If A is multiplied on the left by R, the rows of the new matrix will be linear combinations of the rows of A. To see what linear cobinations these are, consider the case $A = I$. Then $RA = R$, and since the third row of R is the sum of twice the second row of I and the third row of I, and so forth we read off the result:

The third row of the new matrix is the sum of twice the second row and the third row of A, and the fifth row of the new matrix is the sum of twice the fourth row and the fifth row of A. The other rows are the same as the corresponding rows of A.

(b) The inverse of R should undo what R did, so it should subtract twice the first row back out of the third, and twice the fourth back out of the fifth. Hence:

(c)

$$R^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{bmatrix} .$$

(d) If A is multiplied on the right by R, the columns of the new matrix are linear combinations of the columns of A. This result corresponds to the remark at the begining of part (a) throught the transpose. Thus, taking $A = I$ as in part (a), we see that in genral:

The second column of the new matrix is the sum of twice the third column and the second column of A, and the fourth column of the new matrix is the sum of twice the fifth column and the fourth column of A. The other columns are the same as the corresponding columns of A.

(e) With the revised definition of R, if we multiply A on the left by R, *The second row of the new matrix is the sum of twice the first row and the second row of A, the third row of the new matrix is the sum of twice the second row and the third row of A, the fourth row of the new matrix is the sum of twice the third row and the fourth row of A, and the fifth row of the new matrix is the sum of twice the fourth row and the fifth row of A. The first row is the same as the first row of A.*

To recover the second row of A from RA, we would subtract twice the first row from the second. To recover the third row, we would subtract twice the second row of A from the third row of RA, and by what we have just said above, this amounts to subtracting twice the second row of RA, but then adding back in 4 times the first row of RA, and so on. Proceeding with this gives us

$$R^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 4 & -2 & 1 & 0 & 0 \\ -8 & 4 & -2 & 1 & 0 \\ 16 & -8 & 4 & -2 & 1 \end{bmatrix} .$$

The difference is that row operations on the second and third rows only "don't interfere" with row operations on the fourth and fifth rows only. Therefore, we could invert these operarions separately. In the second case, we have different row operations acting on the same row, and this makes things more complicated.

2.6.11 (a) To find C, we row reduce $[A|I]$ to the from $[U|RU]$. Doing the row reduction, we find that

$$[U|R] = \left[\begin{array}{cccc|cccc} 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right] .$$

Since the bottom two rows of U are zero, C consists of the bottom two rows of R. That is,

$$C = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix} .$$

(b) With $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$, $C\mathbf{b} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. Since $C\mathbf{b} \neq 0$, there is no solution of $A\mathbf{x} = \mathbf{b}$.

(c) With $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 5 \end{bmatrix}$, $A\mathbf{x} = \mathbf{b}$ is certainly solvable: Note that $\mathbf{b}$ is just the third column of A, so that $A\mathbf{e}_3 = \mathbf{b}$. Also, you can check that $C\mathbf{b} = 0$.

Since we already know a particular solution, namely $\mathbf{e}_3$, we can use Theorem 5 of Section 2.4 to find the general solution easily. (You caould also start from scratch, but let's do it this way as an example of finding a kernel).

To find the general vector with in $\text{Ker}(A)$, look at the row reduced form U of A. We see that we have to solve

$$x + y + 2z + w = 0$$

$$y + z = 0 \ .$$

The variables x and y are pivotal, while z and w are free. The second equation says

$$y = -z$$

and then the first says

$$x = -y - 2z - w = -z - w \ .$$

Hence the general vector in $\text{Ker}(A)$ has the form

$$\begin{bmatrix} -z - w \\ -z \\ z \\ w \end{bmatrix} = z \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \ .$$

Combining this with out particular solution $\mathbf{e}_3$, the general solution is

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \ .$$