SOLUTIONS

Chapter 4

Section 1

4.1.1 This question should have been moved to the section on permutations, which used to be at the beginning... Well, here is the answer anyway, though you'll have to look ahead to understand the notation.

(a) This is true, and clearly so if you think about the definition.

(b) This too is true. It is clear from part **(a)** and the formula(5.2) for the determinant given in section 4.5

4.1.3 Computing determinants: If we could just cancel out that 2 in the third row, second column, well then, we'd have that block matrix structure that would let us apply Theorem 2. So, let's cancel it out.

Muliplying the first row though by 2, and the second row though by a we get

$$\begin{bmatrix} 2a & 4 & 0 & 0 \\ 2a & a^2 & 2a & 0 \\ 0 & 2 & a & 2 \\ 0 & 0 & 2 & a \end{bmatrix}$$

Subtracting the first row from the second, we get

$$\begin{bmatrix} 2a & 4 & 0 & 0 \\ 0 & a^2-4 & 2a & 0 \\ 0 & 2 & a & 2 \\ 0 & 0 & 2 & a \end{bmatrix}$$

Now multiply the second row through by 2 and the third row through by $a^2 - 4$ to get

$$\begin{bmatrix} 2a & 4 & 0 & 0 \\ 0 & 2a^2-8 & 4a & 0 \\ 0 & 2a^2-8 & a^3-4a & 2a^2-8 \\ 0 & 0 & 2 & a \end{bmatrix}$$

Subtracting the second row from the third, we get

$$\begin{bmatrix} 2a & 4 & 0 & 0 \\ 0 & 2a^2-8 & 4a & 0 \\ 0 & 0 & a^3-8a & 2a^2-8 \\ 0 & 0 & 2 & a \end{bmatrix}$$

By Theorem 2, the determinant of this matrix is

$$2a(2a^2-8)(a^4-12a^2+16)\ .$$

Since we multiplied rows through by a total of $2a(2a^2 - 8)$, the determinant of the original matrix is

$$a^4 - 12a^2 + 16\ .$$

Section 2

4.2.1 Area of a parallelogram: By Theorem 6, The area is $\left|\det\left(\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}\right)\right| = 7.$

4.2.3 Volume of a parallelepiped: By Theorem 6, the volume is $\left|\det\left(\begin{bmatrix} 1 & 0 & 3 \\ -2 & -1 & -1 \\ 1 & 2 & 0 \end{bmatrix}\right)\right| = 7.$

4.2.5 Swaping of columns and determinant: This is clear from Theorem 3 since the transpose does not change the value of the determinant, and since it interchanges row and columns, and since the determinant changes sign when rows are interchanged. So: Take the transpose. No change. Swap two rows, and there is a change of sign. But this is like swapping columns in the original matrix. Taking the transpose again, we have the determinant of the column swapped matrix.

Section 3

4.3.1

(a) Solution for cross products $\mathbf{u} \times \mathbf{v}$, $\mathbf{u} \times \mathbf{w}$, $\mathbf{v} \times \mathbf{w}$:

Let us start with the cross product $\mathbf{u} \times \mathbf{v}$, for example. Recall that $\mathbf{u} \times \mathbf{v} = \det(A)$ where A is the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \end{bmatrix}.$$

Calculating this determinant you conclude that $\mathbf{u} \times \mathbf{v} = -\mathbf{e_1} - \mathbf{3e_2} + \mathbf{2e_3}$ that is its coordinate vector with respect to the basis $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ is $\begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$.

How do you make sure that your result makes sense?

You know that the cross product of $\mathbf{u} \times \mathbf{v}$ is a vector orthogonal to $\mathbf{u}$ and $\mathbf{v}$. Well, that is easy to check:

$$\begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 0 \qquad \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = 0$$

At this point, you can certainly conclude that your result makes sense! But we can make an even more thorough check! We know the length of $\mathbf{u} \times \mathbf{v}$, and we can figure out its exact direction by the right hand rule.

This will give a more conclusive check – after all, there are other vectors that pass the first check, but are not equal to $\mathbf{u} \times \mathbf{v}$. For example, the vector $\begin{bmatrix} 2 \\ 6 \\ -4 \end{bmatrix}$, for example, is also orthogonal to $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and to $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

First let's check the length:

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \angle(\mathbf{u}, \mathbf{v}) \qquad \cos \angle(\mathbf{u}, \mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

and calculating first the angle between $\mathbf{u}$ and $\mathbf{v}$, the length of $\mathbf{u}$ and $\mathbf{v}$, you obtain

$$\cos \angle(\mathbf{u}, \mathbf{v}) = \frac{\mathbf{-1}}{\sqrt{\mathbf{5}}\sqrt{\mathbf{3}}} \qquad \sin \angle(\mathbf{u}, \mathbf{v}) = \sqrt{\mathbf{1} - \frac{\mathbf{1}}{(\mathbf{5})(\mathbf{3})\mathbf{3}}} = \sqrt{\frac{\mathbf{14}}{\mathbf{15}}}$$

Finally, substituting on

$$|\mathbf{u} \times \mathbf{v}| = \sqrt{\mathbf{5}}\sqrt{\mathbf{3}}\sqrt{\frac{\mathbf{14}}{\mathbf{15}}} = \sqrt{\mathbf{14}}$$

Now you know that $\mathbf{u} \times \mathbf{v}$ is a vector with the "direction" of the vector $\begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$ and should have length $\sqrt{14}$.

We easily see that $|\begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}| = \sqrt{14}$. You know know that the vector you computed has the correct the length. Now you can be pretty sure you are right!

But wait – there is still room for doubt. Monsieur Poirot would not find you argument airtight. How to convince him? Use the right hand rule! The only other vector that has the same *direction* and *length* as $\mathbf{u} \times \mathbf{v}$ is the vector $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$! Could it be it? You just need to recall what we said about the *orientation* of the vector $\mathbf{u} \times \mathbf{v}$ that is the right hand rule.

$$\det\left(\begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -3 \\ 1 & 1 & 2 \end{bmatrix}\right) = 14 > 0$$

Now you are completely sure that $\mathbf{u} \times \mathbf{v} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$!! Monsieur Poirot is convinced!

As before, you have

$$\mathbf{u} \times \mathbf{w} = \det\left(\begin{bmatrix} 2 & 0 & 1 \\ 2 & 0 & -2 \\ \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \end{bmatrix}\right) = \mathbf{6e_2} \qquad \mathbf{v} \times \mathbf{w} = \det\left(\begin{bmatrix} -1 & 1 & 1 \\ 2 & 0 & -2 \\ \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \end{bmatrix}\right) = -\mathbf{2e_1} - \mathbf{2e_3}$$

Check your result as we did for $\mathbf{u} \times \mathbf{v}$. In (1) we check the *direction*, in (2) we checked *length* and in (3) the *orientation*. And that is it! Let us do it one more time, for $\mathbf{u} \times \mathbf{w}$, for example. Fowing the same steps as before you find that

1 - The vector $\mathbf{u} \times \mathbf{v}$ has the *direction* of the vector $\begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$.

2 - The *length* of the vector $\mathbf{u} \times \mathbf{v}$ is $\sqrt{36}$. Now there are only two vectors that satisfy (1) and (2). They are $\begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -6 \\ 0 \end{bmatrix}$.

3 - The *orientation* should be such that the right hand rule is satisfied. In fact

$$\det\left(\begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 6 \\ 1 & 2 & 0 \end{bmatrix}\right) = 36 > 0$$

(b) Solution for surface area of a parallelipeped :

The paralleliped has six faces: two faces with area $|\mathbf{u} \times \mathbf{v}|$, two more faces with area $|\mathbf{u} \times \mathbf{w}|$ and two more with area $|\mathbf{v} \times \mathbf{w}|$. We have for the surface area:

$$2|\mathbf{u} \times \mathbf{v}| + \mathbf{2}|\mathbf{u} \times \mathbf{w}| + \mathbf{2}|\mathbf{v} \times \mathbf{w}| = \mathbf{2}(\sqrt{\mathbf{14}} + \mathbf{6} + \mathbf{2}\sqrt{\mathbf{2}}) \ .$$

We compute the three cross products. The absolute value of each of these is the area of one face of the parallelepiped. The faces come in pairs (think of a cube), and so the area in twice their total absolute value.

4.3.3 (a) The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, and hence to any vector in their span. Hence the equation of the plane is

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{x} = 0 \ .$$

(b) The matrix $\begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & -1 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & -3 \end{bmatrix}$. The variable z is free. chosing $z = 1$, we get from the second eqaution $y = -1/3$. Then from the first equation, $x = -3y - 2z = 1 - 2 = -1$. Hence the span of $\begin{bmatrix} -1 \\ -1/3 \\ 1 \end{bmatrix}$ is $\text{Ker}\left(\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}\right)$. This tells us that the equation of the plane is

$$-3x - y + 3z = 0 \ ,$$

which is equivalent to what we found before.

(c) This would be $|\mathbf{a} \times \mathbf{b}|$.

4.3.5 Let $C = [\mathbf{u}, \mathbf{v}, \mathbf{w}]$. Then

$$\det(C) = (\mathbf{u} \times \mathbf{v} \cdot \mathbf{w}) \ .$$

But then

$$\det(BC) = (B\mathbf{u} \times B\mathbf{v}) \cdot B\mathbf{w} \ ,$$

and

$$\det(BC) = \det(B)\det(C) \ .$$

It follows that, as long as $\det(C) \neq 0$,

$$(\mathbf{u} \times \mathbf{v} \cdot \mathbf{w}) = (B\mathbf{u} \times B\mathbf{v}) \cdot B\mathbf{w}$$

if and only if $\det(B) = 1$.

4.3.7 Let $\mathbf{a} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$ and let $\mathbf{b} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$. Then, using the given properties,

$$\mathbf{a} \times\!\!\times \mathbf{b} = (ay - xb)\mathbf{v}_3 + (bz - cy)\mathbf{v}_1 + (cx - za)\mathbf{v}_2 \ .$$

Here is the formula, obtained using only the given information.

(a) Just plug the coefficients of $\mathbf{a}$ and $\mathbf{b}$ in to the fromula found above.

(c) With this definition, $B\mathbf{e}_j = \mathbf{e}_j$ for $j = 1, 2, 3$. Therefore, since

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3 \ ,$$

we have

$$B(\mathbf{e}_1 \times \mathbf{e}_2) = B\mathbf{e}_3 = \mathbf{v}_3 = \mathbf{e}_1 \times\!\!\times \mathbf{e}_2 \ .$$

The same thing happens for the other pairs, so by the linearity we have

$$B(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times\times \mathbf{b}$$

in general.

4.3.9 The matrix in this problem should be antisymmetric; there is a wrong sign in the second entry of the bottom row, so it should be

$$A_{\mathbf{x}} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} .$$

We will answer for this matrix.

(a) $A_{\mathbf{x}}\mathbf{y} = \mathbf{x} \times \mathbf{y}$.

(b) Just compute both of them – you find the same result.

Section 4

4.4.1 (a) Solution for $D(\sigma_j)$ for $j = 1, 2, 3$:

To compute $D(\sigma_j)$ for $j = 1, 2, 3$ we have to find the number of pairs that are put out of order by σ. i such that i *precedes* j and $i > j$ where $i = 1, ..., 6$. Let us compute $D(\sigma_1)$:

We see that in the bottom row of the expression for σ that:

3 *precedes* 1 and $3 > 1$

3 *precedes* 2 and $3 > 2$.

4 *precedes* 2 and $4 > 2$.

5 *precedes* 2 and $5 > 2$.

6 *precedes* 2 and $6 > 2$.

Therefore we have the pairs $(3, 1), (3, 2), (4, 2), (5, 2)$ and $(6, 2)$. That is $D(\sigma_1) = 5$.

Following the same procedure you will find that for σ_2 we have to consider the pairs

$$(2, 1), (3, 2), (3, 1), (4, 3), (4, 2), (4, 1), (5, 2), (5, 1), (5, 5), (6, 2) \quad \text{and} \quad (6, 1) .$$

Therefore $D(\sigma_2) = 11$. In the case of σ_3 we have to consider the pairs of permutations $(4, 1)$, $(4, 2)$, $(4, 3)$, $(5, 1)$, $(5, 2)$, $(5, 3)$, $(6, 1)$, $(6, 2)$ and $(6, 3)$. Then $D(\sigma_3) = 9$.

(b) Solution for σ_j as a product of pair permutations for $j = 1, 2, 3$:

Let us look at σ_1. You want to transform $(1, 2, 3, 4, 5, 6)$ to $(3, 1, 4, 5, 6, 2)$ through pair permutations. Of course there are many different ways to do this. For example:

$$(1, 2, 3, 4, 5, 6) \to (1, 6, 3, 4, 5, 2) \to (1, 5, 3, 4, 6, 2) \to (1, 4, 3, 5, 6, 2) \to (1, 3, 4, 5, 6, 2) \to (3, 1, 4, 5, 6, 2)$$

The strategy was to get the correct number in the last place, then the correct number in the next to last place, and so on. We used 5 pair permutations, so $\chi(\sigma_1) = (-1)^5 = -1$.

In the case of σ_2 one way to transform $(1, 2, 3, 4, 5, 6)$ to $(3, 1, 4, 5, 6, 2)$ through pair permutations is:

$$(1, 2, 3, 4, 5, 6) \to (6, 2, 3, 4, 5, 1) \to (6, 5, 3, 4, 2, 1) \to (6, 4, 3, 5, 2, 1) \to (3, 4, 6, 5, 2, 1) \to (4, 3, 6, 5, 2, 1)$$

For σ_3, let's use a different strategy: Let's get the 1 in the right place, and then the 2 in the right place, and so on. Actually, it turns out that once you swap the 1, 2 and 3 into the right places, everything is in the right place.

$$(1,2,3,4,5,6) \to (4,2,3,1,5,6) \to (4,5,3,1,2,6) \to (4,5,6,1,2,3)$$

Try to think of some other strategies!

(c) Solution for $\chi(\sigma_j)$ **for** $j = 1,2,3$:

Just using the definition and the results you obtained above, namely in part (a), you have $\chi(\sigma_1) = -1$ for all $j = 1,2,3$.

(d) Solution for $\chi(\sigma_1 \circ (\sigma_2 \circ \sigma_3)^{-1})$: Using the product property,

$$\chi(\sigma_1 \circ (\sigma_2 \circ \sigma_3)^{-1}) = \frac{\chi(\sigma_1)}{\chi(\sigma_2)\chi(\sigma_3)} = -1 \ .$$

We can check this by hand, and learn to appreciate the use of the product property: First , compute $\sigma_2 \circ \sigma_3$ and then $(\sigma_2 \circ \sigma_3)^{-1}$: We have

$$\sigma_2 \circ \sigma_3 = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 5 & 2 & 1 & 4 & 3 & 6 \end{matrix} \qquad (\sigma_2 \circ \sigma_3)^{-1} = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 2 & 5 & 4 & 1 & 6 \end{matrix}$$

At this point you can ask: *how do I check my result?* Well, you just need to recall the definition of the inverse of a permutation! You see that

$$(\sigma_2 \circ \sigma_3) \circ (\sigma_2 \circ \sigma_3)^{-1} = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} .$$

Finally you obtain

$$(\sigma_1 \circ (\sigma_2 \circ \sigma_3)^{-1}) = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 6 & 3 & 2 & 5 & 4 & 1 \end{matrix}$$

To determine $\chi(\sigma_1 \circ (\sigma_2 \circ \sigma_3)^{-1})$ you do the same reasoning you did in part (a) and you see that you have to consider the following pair permutations: $(6,3),(6,2),(6,5),(6,4),(6,1),(3,2)(3,1),(2,1),(5,4),(5,1)$, that is 10 permutations. Therefore $\chi(\sigma_1 \circ (\sigma_2 \circ \sigma_3)^{-1}) = 1$.

4.4.3 Solution for (a) Yes, this is true.

This type of question can seem tricky at first, but the only point is to check whether you really understand some definition. In this case, it is the definition of the product $\sigma(A)$. So the first step is to read the definition again.

By definition, $\sigma(P_\tau)$ is the product of the diagonal entries of

$$P_\sigma^t P_\tau = P_{\sigma^{-1}} P_\tau = P_{\sigma^{-1} \circ \tau} \ .$$

Now, $\sigma^{-1} \circ \tau$ is the identity if and only if $\sigma = \tau$. If it is the identity, then $P_{\sigma^{-1} \circ \tau} = I$, and the product of the diagonal entries is 1. Otherwise, at least some pair of rows of the identiy have been swapped out of place, and there are at least two zeros on the diagonal, so in this case the product of the diagonal entries is zero.

Solution for (b) Yes, this is true, and you can use part (a): By the formula,

$$\begin{aligned}\det(P_\tau) &= \sum_\sigma \chi(\sigma)\sigma(P_\tau) \\ &= \chi(\tau^{-1})1 \\ &= \chi(\tau)\ ,\end{aligned}$$

since by part (a), $\sigma(P_\tau) = 0$ unless $\sigma = \tau^{-1}$, and since $\chi(\tau^{-1}) = \chi(\tau)$. Hence we have that for every τ, $\det(P_\tau) = \chi(\tau)$.