ANSWERS

Chapter 1

Section 1

1.1.1 (a) $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} x+y \\ x-y \end{bmatrix}$.

1.1.3 (a) To make the table for $f \circ g$, note that $g(1) = 2$ and $f(2) = 4$, therefore $f \circ g(1) = f(g(1)) = f(2) = 4$. Likewise, $f \circ g(2) = f(g(2)) = f(2) = 4$. Continuing in this way, we find

$$g \circ f \ : \ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 4 & 5 & 5 & 4 \end{array}$$

for $f \circ g$. In the same way we find

$$g \circ f \ : \ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 4 & 4 & 2 & 2 \end{array} \qquad h \circ g \ : \ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 2 & 2 & 4 & 4 \end{array} \qquad h \circ h \ : \ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 & 5 \end{array}$$

Note that $h \circ h$ is the identity map.

(b) We find

$$h \circ (g \circ f) \ : \ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 2 & 2 & 4 & 4 \end{array} \qquad \text{and} \qquad (h \circ g) \circ f \ : \ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 2 & 2 & 4 & 4 \end{array}$$

These are the same, so the associativity, $h \circ (g \circ f) = (h \circ g) \circ f$.

(c) Only f and h are one to one, onto and invertible, while g is none of these things. We have seen above that $h \circ h$ is the identity map, so $h^{-1} = h$. To find f^{-1}, just run it backwards: since $f(5) = 1$, $f^{-1}(1) = 5$. since $f(1) = 2$, $f^{-1}(2) = 1$, and so on, with the reuslt that

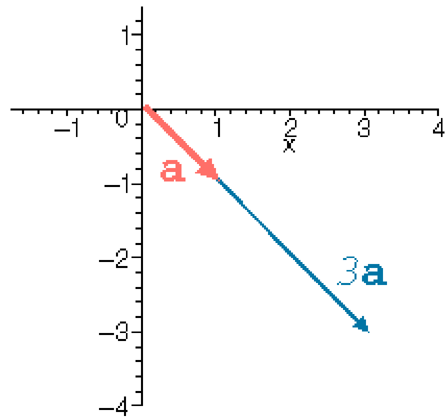
$$f^{-1} \ : \ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 5 & 1 & 3 & 2 & 4 \end{array}$$

Section 2

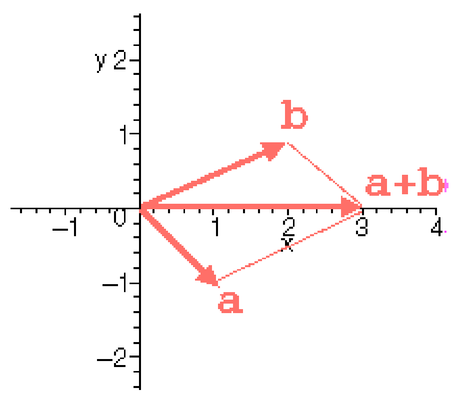
1.2.1 (a)

$$3\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}3\\-3\end{bmatrix}\ ,\quad \begin{bmatrix}1\\-1\end{bmatrix}+\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}3\\0\end{bmatrix}\ ,\quad \begin{bmatrix}1\\-1\end{bmatrix}-\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}-1\\-2\end{bmatrix}$$

(b)



(c)



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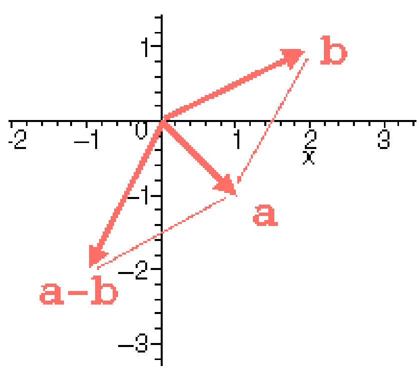


1.2.3 The only linear transformation is f. To see this, we check homogeneity and additivity. In fact, it turns out that only f is homogeneous:

First, for any number a, and any vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$,

$$f\left(a\begin{bmatrix} x \\ y \end{bmatrix}\right) = f\left(\begin{bmatrix} ax \\ ay \end{bmatrix}\right) = \begin{bmatrix} ax + ay \\ ay - ax \end{bmatrix} = a\begin{bmatrix} x + y \\ y - x \end{bmatrix} = af\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) .$$

Hence f is homogenous.

However,

$$g\left(a\begin{bmatrix} x \\ y \end{bmatrix}\right) = g\left(\begin{bmatrix} ax \\ ay \end{bmatrix}\right) = \begin{bmatrix} |a||x| \\ ay - ax \end{bmatrix}$$

If $a > 0$, so $|a| = a$, then this equals

$$a\begin{bmatrix} |x| \\ y - a \end{bmatrix} = ag\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

but otherwise not. Sine the homenity requires equality for *every* a, g is not homogeneous, and therefore not linear.

Also,

$$h\left(a\begin{bmatrix} x \\ y \end{bmatrix}\right) = h\left(\begin{bmatrix} ax \\ ay \end{bmatrix}\right) = \begin{bmatrix} a^2x^2 - a^2y^2 \\ a^2x^2 + a^2y^2 \end{bmatrix} = a^2\begin{bmatrix} x^2 - y^2 \\ x^2 + y^2 \end{bmatrix} = a^2h\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) .$$

Taking any value of a other than $a = 1$, we see that h is not homogeneous. (It looks *almost* homogeneous, and functions that behave like h are sometimes called "homogeneous of degree two"). In any case, we don't have $h(a\mathbf{x}) = ah(\mathbf{x})$ for all $\mathbf{x}$ and a, so h is not homogeneous and therefore not linear.

We now check to see that f is additive. Let $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}$. Then

$$f(\mathbf{x} + \mathbf{u}) = f\left(\begin{bmatrix} x + u \\ y + v \end{bmatrix}\right) = \begin{bmatrix} x + u + y + v \\ y + v - x - u \end{bmatrix} = \begin{bmatrix} x + y \\ y - x \end{bmatrix} + \begin{bmatrix} u + v \\ v - u \end{bmatrix} = f(\mathbf{x}) + f(\mathbf{u}) .$$

Hence f is both additive and homogeneous, so it is linear.

Remark: To show that a transformation is *not* additive, or *not* homogeneous, you don't need a general calculation; you just need one example. To see that g and h are not homogenous, consider

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} .$$

Then $g(-\mathbf{x}) \neq g(-\mathbf{x})$ and $h(-\mathbf{x}) \neq h(-\mathbf{x})$.

If we let $\mathbf{u}\begin{bmatrix}-1\\0\end{bmatrix}$, then $\mathbf{x}+\mathbf{u}=0$, and so $g(\mathbf{x}+\mathbf{u})=0$ while $g(\mathbf{x})+g(\mathbf{u})=2\mathbf{x}\neq 0$. Likewise, $h(\mathbf{x}+\mathbf{u})=0$ while $h(\mathbf{x})+h(\mathbf{u})=\begin{bmatrix}2\\2\end{bmatrix}\neq 0$.

You can also see that neither is additive in the sense of (2.3). The problem with g comes from the absolute value and the fact that $|a+b|\leq|a|+|b|$, with $|a+b|\neq|a|+|b|$ if a and b have the opposite sign. The problem with h comes from the squares.

1.2.5 The transformations g and h are both additive and homogeneous, so they are linear. the corresponding matrices are:

$$A_g=\begin{bmatrix}0&1\\0&0\\1&0\end{bmatrix}\quad A_h=\begin{bmatrix}0&0\\0&0\\0&0\end{bmatrix}\ .$$

Note that f is not even additive: $f(\mathbf{u}+\mathbf{v})\neq f(\mathbf{u})+f(\mathbf{v})$.

1.2.7 $A\mathbf{z}=\begin{bmatrix}5\\-5\end{bmatrix}$ $\quad B\mathbf{y}=\begin{bmatrix}5\\3\\13\end{bmatrix}$ $\quad C\mathbf{v}=\begin{bmatrix}3\\2\\-1\\6\end{bmatrix}$ $\quad C\mathbf{x}=\begin{bmatrix}1\\-2\\3\\4\end{bmatrix}$

$D\mathbf{v}=\begin{bmatrix}1\\1\end{bmatrix}$ $\quad D\mathbf{x}=\begin{bmatrix}-1\\1\end{bmatrix}$

1.2.9 Using Theorem 1.2.3:

$$(A\mathbf{x})_3=-1(2)-2(0)+2(0)+1(-1)=-3\ .$$

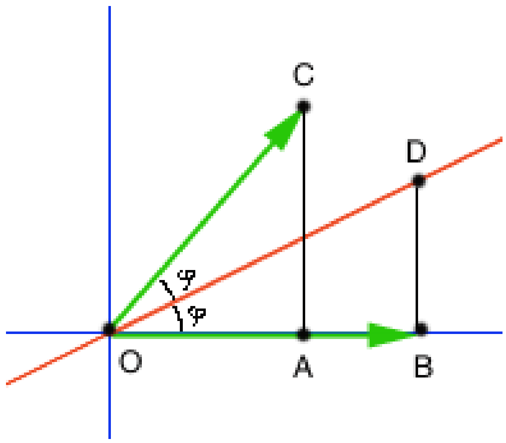
1.2.11 The matrix A_f is given by

$$A_f=[f(\mathbf{e}_1),f(\mathbf{e}_2)]=\begin{bmatrix}1&-3\\3&1\end{bmatrix}\ .$$

1.2.13 Reflecting $\mathbf{e}_1$ about $y=x$ gives us $\mathbf{e}_2$. Reflecting this about $x=0$, the y–axis, doesn't change $\mathbf{e}_2$, so $f(\mathbf{e}_1)=\mathbf{e}_2$. Reflecting $\mathbf{e}_2$ about $y=x$ gives us $\mathbf{e}_1$, and reflecting this about the y–axis gives us $-\mathbf{e}_1$, so $f(\mathbf{e}_2)=\mathbf{e}_1$. Hence, for this transformation,

$$A_f=[f(\mathbf{e}_1),f(\mathbf{e}_2)]=[\mathbf{e}_2,-\mathbf{e}_1]=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\ .$$

1.2.15 Consider the following diagram: The vector $\mathbf{e}_1$ is the vector indicated by the horizontal arrow, so thatthe segment from 0 to B is a unit vector. The reflected vector, $f(\mathbf{e}_1)$ is the other arrow, and the line with slope s is the line running between these vectors.



Since the line has slope s, we see that the segment from B to D has length s since the segment from O to B has unit length. Hence $\tan(\varphi) = s$.

Next, since the triangle with vertices O, A and C is a right triangle with unit hypotenuse, C is the point

$$(\cos(2\varphi), \sin(2\varphi)) = (\cos^2(\varphi) - \sin^2(\varphi), 2\sin(\varphi)\cos(\varphi)) \ .$$

Now since $\tan(\varphi) = s$,

$$\cos(\varphi) = \frac{1}{\sqrt{1+s^2}} \qquad \text{and} \qquad \cos(\varphi) = \frac{s}{\sqrt{1+s^2}} \ .$$

Hence C is the point

$$\left(\frac{1-s^2}{1+s^2}, \frac{2s}{1+s^2}\right)$$

and so

$$f(\mathbf{e}_1) = \frac{1}{1+s^2}\begin{bmatrix} 1-s^2 \\ 2s \end{bmatrix} \ .$$

In the same way one see,

$$f(\mathbf{e}_2) = \frac{1}{1+s^2}\begin{bmatrix} 2s \\ s^2-1 \end{bmatrix} \ ,$$

and hence

$$A_f = \frac{1}{1+s^2}\begin{bmatrix} 1-s^2 & 2s \\ 2s & s^2-1 \end{bmatrix} \ .$$

It is much simpler to find A_g, since clealry $g(\mathbf{e}_1) = \mathbf{e}_1$ and $g(\mathbf{e}_2) = -\mathbf{e}_2$, and so

$$A_g = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

Hence

$$A_{g\circ f} = A_g A_f = \frac{1}{1+s^2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1-s^2 & 2s \\ 2s & s^2-1 \end{bmatrix} = \frac{1}{1+s^2} \begin{bmatrix} 1-s^2 & 2s \\ -2s & 1-s^2 \end{bmatrix} .$$

If g were reflection about the y axis, we would have

$$A_{g\circ f} = A_g A_f = \frac{1}{1+s^2} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1-s^2 & 2s \\ 2s & s^2-1 \end{bmatrix} = \frac{1}{1+s^2} \begin{bmatrix} s^2-1 & -2s \\ 2s & s^2-1 \end{bmatrix} .$$

Section 3

1.3.1 $AC = \begin{bmatrix} 13 & 3 \\ -3 & 4 \end{bmatrix}$ $\quad BB = \begin{bmatrix} 3 & -3 & 3 \\ 4 & -1 & 2 \\ 10 & -7 & 8 \end{bmatrix}$ $\quad CA = \begin{bmatrix} 3 & 4 & 2 & 3 \\ 2 & 4 & 0 & -2 \\ -1 & -3 & 1 & 4 \\ 6 & 7 & 5 & 9 \end{bmatrix}$

$CD = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ -2 & 1 \\ 1 & 5 \end{bmatrix}$ $\quad DA = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ $\quad DD = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

1.3.3 We just need to compute $A\mathbf{v}_3$ where $\mathbf{v}_3$ is the third columns of B. The result is

$$\begin{bmatrix} 3 \\ 3 \\ 2 \\ 1 \end{bmatrix} .$$

1.3.5 $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

1.3.7

(a) $\begin{bmatrix} a^2 & ab+bc \\ 0 & c^2 \end{bmatrix}$.

(b) Four different sets of values $\{a=1, b=1, c=2\}$,$\{a=1, b=-3, c=-2\}$

$\{a=-1, b=3, c=2\}$ and $\{a=-1, b=-1, c=-2\}$.

1.3.9

(a) $[A, B] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(b) Let $C = [A, B]$. From the previous question you know that

$$C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Now you just compute the commutator the way you did previously:

$$[A, [A, B]] = AC - CA = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

1.3.11

(a) $AB = \begin{bmatrix} 1 & a+u & aw+b+v \\ 0 & 1 & c+w \\ 0 & 0 & 1 \end{bmatrix}$

(b) You can solve this in at least two different ways.

Method 1: The inverse is a matrix B such that $AB = I$. Using (a) you will find that $AB = I$ if and only if the system of equations

$$a + u = 0$$

$$c + w = 0$$

$$aw + v + b = 0$$

is satisfied. The first two equations tell you $u = -a$, and $w = -c$. With this it is easy to solve the third equation for v. This gives you the values of u, v and w for which $B = A^{-1}$, and hence

$$A^{-1} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}.$$

Method 2: Consider a general input vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and define the output vector

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Doing the matrix multiplication

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + ax_2 + bx_3 \\ x_2 + cx_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

you will obtain the system

$$\begin{aligned} x_1 + ax_2 + bx_3 &= y_1 \\ x_2 + cx_3 &= y_2 \\ x_3 &= y_3 \ . \end{aligned}$$

Solving it for x_1, x_2, x_3 you will find

$$\begin{aligned} y_1 - ay_2 + (ac+b)y_3 &= x_1 \\ y_2 - cy_3 &= x_2 \\ y_3 &= x_3 \end{aligned}$$

Writing this in matrix form

$$\begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} .$$

and from it we can see that

$$A^{-1} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} ,$$

as before.

1.3.13 No. For example, the transformation given by $f(\mathbf{x}) = 0$ is linear from R^n to R^m for any n and m. As long as $g(0) = 0$, $g \circ f$ will be linear. And many non–linear functions g satisfy $g(0) = 0$ – for example, g and h from Exercise 1.2.1.

One can also give examples where f is not the zero transformation. Here is one with $n = m = p = 2$: $f(x, y) = (0, y)$ and $g(x, y) = (x^2, y)$.

Section 4

1.4.1 $|v| = \sqrt{2} \quad |x| = \sqrt{2} \quad |y| = 3 \quad |z| = \sqrt{15} \quad v \cdot x = 0$

1.4.3 (a)

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot \mathbf{v}_1 = 4$$

$$\mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_3 \cdot \mathbf{v}_1 = 0$$

$$\mathbf{v}_2 \cdot \mathbf{v}_3 = \mathbf{v}_3 \cdot \mathbf{v}_2 = -3$$

$$\mathbf{v}_1 \cdot \mathbf{v}_1 = \mathbf{v}_1 \cdot \mathbf{v}_1 = \mathbf{v}_1 \cdot \mathbf{v}_1 = 5 \ .$$

(b)

$$|\mathbf{v}_1| = |\mathbf{v}_2| = |\mathbf{v}_3| = \sqrt{5} \ .$$

(c) The angle between $\mathbf{v}_1$ and $\mathbf{v}_2$ is $\cos^{-1}(4/5) \approx 0.6435$ radians. The angle between $\mathbf{v}_1$ and $\mathbf{v}_3$ is $\cos^{-1}(0) = \pi 2$. These vectors are orthogonal. The angle between $\mathbf{v}_2$ and $\mathbf{v}_3$ is $\cos^{-1}(-3/5) \approx 0.9272$ radians.

1.4.5

(a) The formula for geometric sums says that for any number a, and any integer n.

$$1 + a + a^2 + \cdots a^{n-1} = \frac{1-a^n}{1-a} \ .$$

(An easy way to derive this is to multiply the left hand side by $1-a$, and notice all the cancellation that results).

Applying this with $a = r^2$, we get that

$$\ell_n = \left(\frac{1-r^{2n}}{1-r^2}\right)^{1/2} \ .$$

(b) Computing the dot product,

$$\begin{bmatrix} 1 \\ r \\ r^2 \\ \vdots \\ r^{n-1} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ s \\ s^2 \\ \vdots \\ s^{n-1} \end{bmatrix} = 1 + rs + (rs)^2 + \cdots + (rs)^{n-1} = \frac{1-(rs)^n}{1-rs} \ .$$

Therefore, the angle α_n between these vectors is

$$\begin{aligned} \alpha_n &= \frac{1-(rs)^n}{1-rs}\left(\frac{1-r^{2n}}{1-r^2}\right)^{-1/2}\left(\frac{1-s^{2n}}{1-s^2}\right)^{-1/2} \\ &= \frac{(1-r^2)^{1/2}(1-s^2)^{1/2}}{1-rs}\left(\frac{1-(rs)^n}{(1-r^{2n})^{1/2}(1-s^{2n})^{1/2}}\right) \ . \end{aligned}$$

(c) Since both $|r| < 1$ and $|s| < 1$, we have that $\lim_{n\to\infty} r^{2n} = 0$, $\lim_{n\to\infty} s^{2n} = 0$ and also, $\lim_{n\to\infty}(rs)^n = 0$. Therefore

$$\lim_{n\to\infty} \ell_n = \frac{1}{\sqrt{1-r^2}} \qquad \text{and} \qquad \lim_{n\to\infty} \alpha_n = \frac{\sqrt{1-r^2}\sqrt{1-s^2}}{1-rs} \ .$$

You just computed the angle between two infinite dimensional vectors! This will turn out to be more than a mere curiosity.

1.4.7 All vectors of the form $\mathbf{x} = \begin{bmatrix} (-bd/a \\ d \end{bmatrix}$.The vectors $\mathbf{a}$ and $\mathbf{x} = \begin{bmatrix} c \\ d \end{bmatrix}$ are orthogonal if and only if $a \neq 0$ and $c = -bd/a$ for any $d \in R$.

1.4.9 You just need to see that from the definition of length

$$|x + y|^2 = |x|^2 + |y|^2 + x \cdot y + y \cdot x$$

and then condition

$$|x + y|^2 = |x|^2 + |y|^2$$

implies that the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$ is $\frac{\pi}{2}$.

1.4.11 Computing the dot product between the vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ you will obtain the equation $x + 2y - z = 0$. That is the equation for a plane in R^3.

Section 5

1.5.1 $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 4$

1.5.3

(a)

$$(AB)_{2,3} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = [1 \quad 3 \quad 1 \quad 2]^t \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = 10$$

(b) $a = 1, b = 0, c = 0, d = 2$ These are the corresponding coefficients of the second column of the matrix B. That is $a = b_{1,2}, b = b_{2,2}, c = b_{3,2}$ and $d = b_{4,2}$ where $b_{i,j} 1 \leq i, j \leq 3$ are the entries of the matrix B.

(c) Think that the second row of AB is, using the definition of transpose, the second column of $(AB)^t$. But you know that

$$(AB)^t = B^t A^t$$

and then the second row of AB is a linear combination of the rows of the matrix B where the coefficients are the corresponding coefficients from the second row of the matrix A. You have

$$1\begin{bmatrix}1 & 1 & 1\end{bmatrix} + 3\begin{bmatrix}2 & 0 & 2\end{bmatrix} + 1\begin{bmatrix}0 & 0 & 1\end{bmatrix} + 2\begin{bmatrix}3 & 2 & 1\end{bmatrix}$$

1.5.5 Let $B = [b_{i,j}]$ with $i, j = 1, 2, 3$. Then

$$v_2 + v_3 = b_{1,1}v_1 + b_{2,1}v_2 + b_{3,1}v_3,$$

so $b_{1,1} = 0, b_{2,1} = 1, b_{3,1} = 1$. In the same way we find

$$b_{1,2} = 0 \quad b_{2,2} = 1 \quad b_{3,2} = 0 \quad b_{1,3} = 1 \quad b_{2,3} = 1 \quad b_{3,3} = 0.$$

1.5.7

(a) The rows of the matrix AB are the rows of the matrix B and the columns of BA are the columns of B. In fact, the matrix A is the Identity matrix, and then $AB = IB = B$.

(b) Yes. Take, for example, the matrix $A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ and find all matrices B such that $AB = BA$. You will find an infinite number that satisfy the condition, that is, all matrices of the form $B = \begin{bmatrix} a & 0 \\ d-a & d \end{bmatrix}$ for all $a, c, d \in R$. So take, for example, the matrix

$$B = \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}$$

1.5.9 No. Each column $j, 1 \le j \le p$, of AB is a linear combination of columns of A with coefficients coming from the corresponding column j of B. So if B has at least, one zero column, AB will have, at least, one zero column.

1.5.11

(a) The third and fourth rows of AB, since all their entries will always be 0.

(b) None of the columns can be freely modified.

1.5.13 Let

$$C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then $CA = B$. Multiplying on the right by A^{-1} you will find

$$C = \begin{bmatrix} -\frac{4}{3} & \frac{5}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

1.5.15 Yes. Consider

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \\ 1 & 0 \end{bmatrix}$$

1.5.17 The matrices A and C.

Section 6

1.6.1

(a) All points of the line y-axis are of the form $(0, y)$. The images of these points are given by

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 2y \\ 3y \end{bmatrix}$$

Then $\begin{bmatrix} u \\ v \end{bmatrix}$ belongs to the image if and only if

$$u = 2y$$

$$v = 3y$$

Solving the system for u and v you obtain the line with equation $v = \frac{3}{2}u$.

(b) The line with equation $v = 3u - 9$.

(c) An ellipse with equation $9u^2 + 5v^2 - 6uv = 9$.

1.6.3

(a) The area of the image is 3.

(b) $3 \times 3 = 9$ since the first triangle has an area equal to 3 that is magnified by the factor $|1 \times 3 - 2 \times 0| = 3$.

1.6.5 Neither – both are the same, and the condition $|ad - bc| = 1$ is not satisfied. If we had instead, as was intended,

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

then B would have been area preserving.

1.6.5 Computing $|ab - cd|$ for A we find 9, so A is not area preserving. As written, B is the same, so it is not area preserving. If however, we had

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

, then we would have $|ab - cd| = 1$, so this would be an example of an area preserving matrix.

1.6.7 By the results of the previous problem, the eqaution of the ellipse is

$$25u^2 + 5v^2 - 22uv = 4 \ .$$