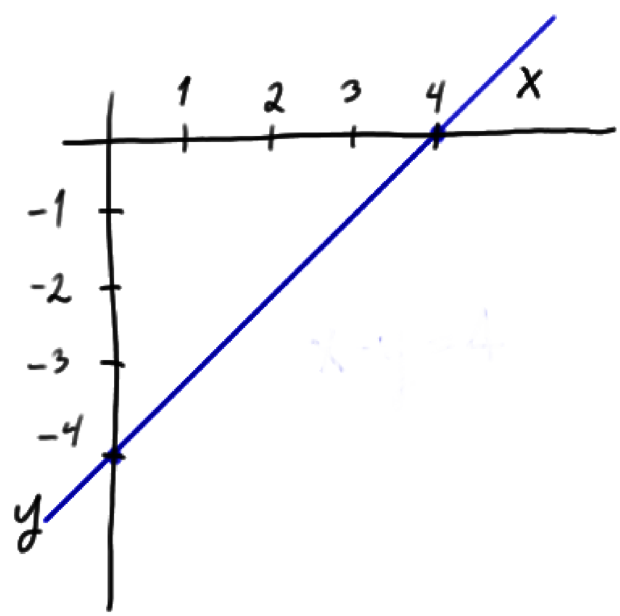
ANSWERS

Chapter 2

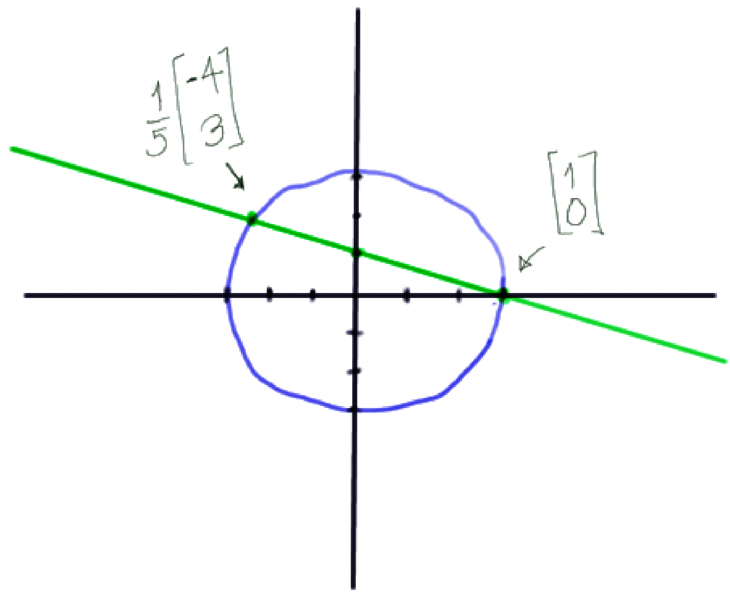
Section 1

2.1.1 The solution set is a line, and by finding two points on it, we can parameterize the line. To find the first point, set $x_2 = 0$. Then the equation becomes $x_1 = 4$, so one point is $\mathbf{x}_0 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$. To find a second point, take $x_1 = 0$. Then the equation becomes $x_2 = -4$, and so a second point is $\mathbf{x}_0 = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$.

We take $\mathbf{x}_0$ as the base point, and $\mathbf{v} = \mathbf{x}_1 - \mathbf{x}_0$ as the direction vector. The parameterization then is $\mathbf{x}_0 + t\mathbf{v}$. Here is a picture:



2.1.3 The line in question passes through $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}$. The points can be found using the method employed in the solution of problem 2.1.1. As you see, one of these points, $\mathbf{x}_0$, is on the unit circle, and the other is inside it. Hence the line intersects the circle in exactly two points, as shown below:



To compute the second point, $\mathbf{x}_3$, we parameterize the line as $\mathbf{x}_0 + t\mathbf{v}$ where

$$\mathbf{v} = \mathbf{x}_1 - \mathbf{x}_0 = \begin{bmatrix} -1 \\ 1/3 \end{bmatrix} .$$

Hence

$$\mathbf{x}_0 + t\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1-t \\ t/3 \end{bmatrix} . \tag{1}$$

The vector on the right is on the unit circle exactly when

$$(1-t)^2 + (t/3)^2 = 1 .$$

Doing the arithmetic, this reduces to

$$\frac{10}{9}t^2 = 2t .$$

One solution is $t = 0$, and the other is $t = 9/5$. The first of these values gives us $\mathbf{x}_0$. Plugging the second of these values into (1), we get that the second point is

$$\mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{9}{5}\begin{bmatrix} -1 \\ 1/3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix} .$$

2.1.5 It will be easiest to write this equation in the form $\mathbf{a} \cdot \mathbf{x} = b$. We know that $\mathbf{a}$ is perpendicular to the direction vector $\mathbf{v}$ of the line. Therefore, we take

$$\mathbf{a} = \mathbf{v}^{\perp} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} .$$

Now that $\mathbf{a}$ is determined, we can find b by plugging the base point $\mathbf{x}_0$ into $\mathbf{a} \cdot \mathbf{x} = b$. This gives us

$$b = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -1 .$$

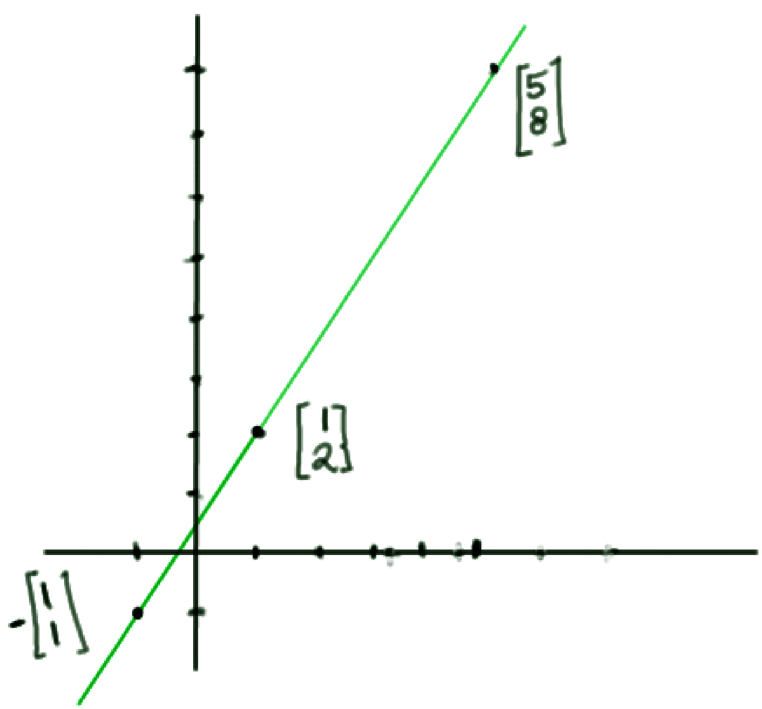
Hence the equation is

$$\begin{bmatrix} 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = -1 ,$$

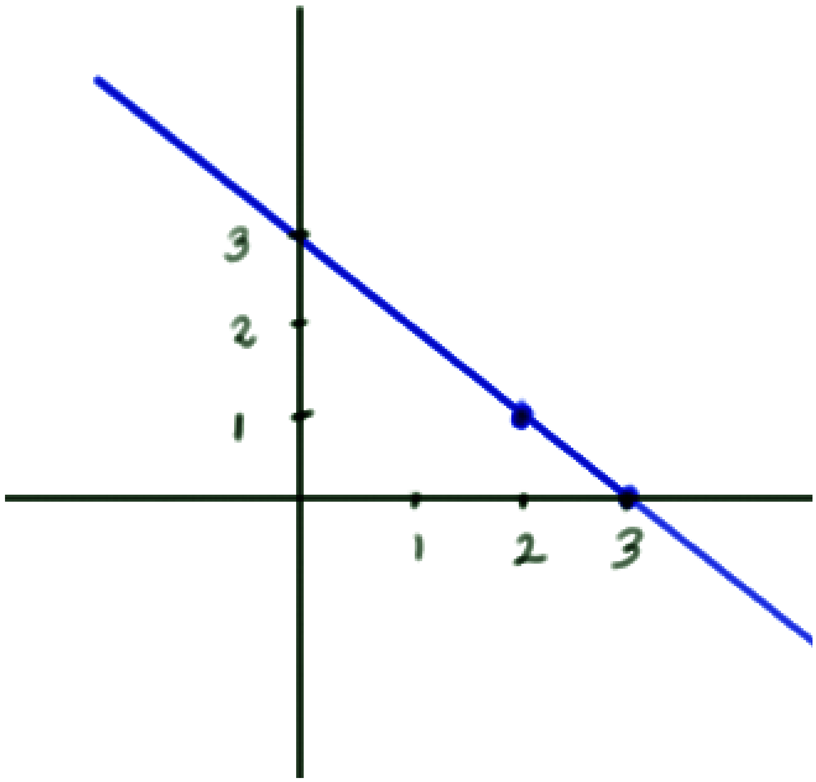
or

$$3x - 2y = -1 .$$

Here is the graph, showing $\mathbf{x}_0 + \mathbf{v}$ in the lower left and $\mathbf{x}_0 - 2\mathbf{v}$ in the upper right:



2.1.7 Call the first point $\mathbf{x}_0$, and the second point $\mathbf{x}_1$. Here is the graph of the line through these two points:



The direction vector is $\mathbf{v} = \mathbf{x}_1 - \mathbf{x}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. A parameterization is then given by

$$\mathbf{x}_0 + t\mathbf{v} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 - t \\ t \end{bmatrix} \ .$$

To find an equation, we proceed as in the solution of problem 2.1.5: We seek the equation in the form $\mathbf{a} \cdot \mathbf{x} = b$, and we take $\mathbf{a}$ to be

$$\mathbf{a} = \mathbf{v}^{\perp} = -\begin{bmatrix} 1 \\ 1 \end{bmatrix} \ .$$

We then plug in $\mathbf{x}_0$ to find

$$b = \mathbf{a} \cdot \mathbf{x}_0 = -3 \ .$$

Hence the equation is

$$-\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = -3 \ ,$$

or

$$x + y = 3 \ .$$

You can easily see that this equation is indeed satisfied by both of the points from which we started.

2.1.9 An easy solution is to let the two given vectors "take turns" at being the base point: Letting $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, the direction vector $\mathbf{v}$ is

$$\mathbf{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \end{bmatrix} .$$

This gives the parameterization

$$\mathbf{x}_0 + t\mathbf{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + t \begin{bmatrix} -2 \\ -6 \end{bmatrix} .$$

Using the other given point as the base point, we get $\mathbf{x}_0 = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$, and

$$\mathbf{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} .$$

This gives the parameterization

$$\mathbf{x}_0 + t\mathbf{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 6 \end{bmatrix} .$$

2.1.11 If you consider x_2 and x_3 the independent variables, and write x_1, the dependent variable, in terms of the other two, you will have

$$x_1 = \frac{1}{2} 1 - 3x_2 + x_3$$

Now take $x_2 = t_2$ and $x_3 = t_3$, where t_2 and t_3 are real parameters, and you find

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1 - 3x_2 + x_3) \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3/2 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} .$$

If you prefer to work with a more simple expression, you can consider x_1 and x_2 the independent variables and, in that case, you will obtain

$$x_3 = -1 + 2x_1 + 3x_2$$

The parameterization will be:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + t_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} .$$

where $x_2 = t_2$ and $x_3 = t_3$.

2.2.13 If you consider x_2, x_3 and x_4 the independent variables you will find the one-to-one parameterization:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t_1 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where t_1, t_2 and t_3 are real numbers.

2.2.15 If the lines are the same they must have at least be parallel, which means that the two direction vectors must be proportional. Indeed,

$$\begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} .$$

Hence the lines are at least parallel.

To see if they are the same, we check to see if the second base point lies on the first line. That is, we ask if there is an s so that

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix} .$$

As you can check, $s = -1$ works. so the first line passes through the base point of the second one, and in the same direction, so the two lines are the same.

2.2.17 Eliminating x, we have $x = (1 + y - z)/2$, and so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 + y - z \\ 2y \\ 2z \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\frac{1}{2} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + z\frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} .$$

Eliminating y, we have $y = 2x + z - 1$, and so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 2x + z - 1 \\ z \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} .$$

Eliminating z, we have $z = 1 + y - 2x$, and so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 + y - 2x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} .$$

In each case, you could change the names of the two "leftover" variables to s and t, say, to emphasize their new found status as parameters, but this isn't really necessary.

Section 2

2.2.1 First of all the equation of the plane $2x_1 + 3x_2 - x_3 = 1$ can be written in the form **a.x**=b with

$$a = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = 1.$$

From **Theorem 1** on distances to Solution Sets you know that the distance you have been asked is given by

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{14}\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 8 \\ 5 \\ 4 \end{bmatrix}.$$

Could any entry of the final result be negative? No! A distance has to be always positive.

2.2.3 (a) You know that three points determine a plane. Let $\mathbf{x}_0$, $\mathbf{x}_1$ and $\mathbf{x}_2$ be the three points in question. To get a parametrization, take one of these points as a base point. We will choose $\mathbf{x}_0$. Then the two direction vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are given by $\mathbf{v}_1 = \mathbf{x}_1 - \mathbf{x}_0$ and $\mathbf{v}_2 = \mathbf{x}_2 - \mathbf{x}_0$, and the parameterization is

$$\mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2 \ .$$

In the case at hand,

$$\mathbf{v}_1 = \mathbf{x}_1 - \mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \quad \mathbf{v}_2 = \mathbf{x}_2 - \mathbf{x}_0 = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$$

you obtain the parameterization:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + s\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + t\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} \quad t_1, t_2 \in R$$

(b) From the previous part, you have the parametric description of the plane:

$$x = 1 + s + 2t$$

$$y = -2 + 2s + 5t$$

$$z = 3 - 2s$$

You want to find an equation relating x, y and z, you just need to eliminate the parameters s and t. To eliminate t, subtract twice the second equation from 5 times the first. The result is

$$5x - 2y = 9 + s \ .$$

Multiplying this through by 2, and adding the third equation, we eliminate s too, with the result

$$10x - 4y + z = 21 \ .$$

This is the equation we seek. (Later, we will have a more systematic way of doing this, but as a general rule progress in algebra means the elimination of variables, and proceeding on that simple principle, we get our answer).

(c) The point $(-1, 1, 3)$ is in the plane if and only if its coordinates satisfy the equation of the plane. Since

$$10(-1) - 4(1) + 1(3) = -11 \neq 21$$

the point is not in the plane.

Section 3

2.3.1 You see that you have one independent variable and two dependent variables. You can choose x_3 as the independent variable. Considering $x_3 = t$ and expressing x_1 and x_2 in terms of x_3 you have

$$\begin{aligned} x_3 &= t \\ x_2 &= 2 - 2t \qquad\qquad t \in R \\ x_1 &= -2x_2 - x_1 = -4 + 3t \end{aligned}$$

That is, in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \quad t \in R$$

2.3.3 The difference between this system and the one considered in problem 2.3.1 is that now $x_3 = 3$. This means we just evaluate the previous answer at $t = 3$: You find

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

There is no parameter left!

Section 4

2.4.1 To find the intersection of the two lines just means to solve the indicated system. Row reducing the augmented matrix you will find that it has a unique solution, given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 5/3 \end{bmatrix}$$

If you think that each equation represents a line in R^2, the point $(5/3, 5/3)$ is exactly the only point where the two lines intersect.

2.4.3 First of all, you should row-reduce the augmented matrix. You will find

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & b \\ 0 & 3 & 0 & 2b-2 \\ 0 & 0 & a-3 & 1-b \end{array}\right]$$

This is the aumented-matrix of an equivalent system, that is, a system that has exactly the same solution set as the initial one. Just for complete clarity write, for once, that new system:

$$x + 2y + z = b$$

$$3y = 2b - 2$$

$$(a-3)z = 1 - b$$

Looking to the last equation you see immediately the answer to most part of the questions!

(a) The system has a unique solution for all values $a \neq 3$, and b can be any real number!

By back substitution you obtain the solution:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{15+a(b-4)-6b}{9-3a} \\ \frac{2}{3}(b-1) \\ \frac{1-b}{a-3} \end{bmatrix}$$

(b) The system has no solution for $a = 3$ and $b \neq 1$.

(c) The system has infinitely many solutions for $a = 3$ and $b = 1$.

2.4.5 First of all, there is no b in this system, just a. so forget about b.

(a) After row-reducing the augmented matrix of the system you obtain

$$\left[\begin{array}{ccc|c} 1 & -2 & a & 2 \\ 0 & 3 & 1-a & -2 \\ 0 & 0 & a & 4 \end{array}\right]$$

There is a pivot in each of the rows and each of the first three columns if and only if $a \neq 0$. So there is a unique solution if and only if $a \neq 0$. We can find it using back substitution: Clearly, $z = 4/a$. From the second equation, $3y + (1-a)z = -2$, so

$$y = -\frac{2}{3} - \frac{1-a}{3}z = -\frac{2}{3} - \frac{4-4a}{3a} = \frac{2}{3} - \frac{4}{3a} \ .$$

From the first equation,

$$x = 2 + 2y - az = -\frac{2}{3} - \frac{8}{3a} \ .$$

(b) From the last row, which corresponds to the equation,

$$0x + 0y + az = 4 \ ,$$

you see that if $a = 0$ the system has no solution because, in that case, the last equation will read $0 = 4$! So the system has no solution for $a = 0$.

(c) From the row-reduced matrix you obtained in part (a) you see that there are no values for a such that the system has infinitely many solutions.

2.4.7 After two elementary operations you will have the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & 5 & 5 & a-6 \\ 0 & a+6 & 2 & -4 \end{array}\right]$$

It would be more convenient to have the entry $a+6$ in position with indexes $3,3$. Just interchange the columns 2 and 3 and you will have it, but when you solve the system do not forget that you interchanged the columns relative to the variables y and z! Lets go back to the original problem. You have

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 3 \\ 0 & 5 & 5 & a-6 \\ 0 & 2 & a+6 & -4 \end{array}\right]$$

and then

$$\begin{bmatrix} 1 & -1 & -2 & | & 3 \\ 0 & 1 & 1 & | & (a-6)/5 \\ 0 & 2 & a+4 & | & (-2a-8)/5 \end{bmatrix}$$

Therefore the system has a unique solution if and only if $a \neq -4$.

Recalling that you interchanged columns 2 and 3, the row-reduced matrix corresponds to the system:

$$x - z - 2y = 3$$

$$z + y = (a-6)/5$$

$$(a+4)y = (-2a-8)/5$$

2.4.11 The augmented matrix $[A|0]$ row reduces to

$$\begin{bmatrix} 1 & 2 & 3 & | & 0 \\ 0 & 4 & 8 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} .$$

Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ belongs to Ker(A) is and only if

$$x + 2y + 3z = 0$$

$$y + 2z = 0 \ .$$

Hence $y = -2z$ and $x = -2y - 3z = 4z - 3z = z$. That is,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -2z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} .$$

Hence Ker(A) is the line through the origin and $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

2.4.13 Suppose that $\mathbf{x}$ belongs to Ker(B). Then

$$(AB)\mathbf{x} = A(B\mathbf{x}) = A0 = 0$$

and so $\mathbf{x}$ belongs to Ker(AB). This shows that

$$\text{Ker}(B) \subset \text{Ker}(AB) \ . \qquad (1)$$

Now suppose that $\mathbf{x}$ belongs to $\mathrm{Ker}(AB)$. This means that

$$0 = (AB)\mathbf{x} = A(B\mathbf{x}) \ .$$

Therefore, $B\mathbf{x}$ belongs to $\mathrm{Ker}(A)$. By hypothesis, $\mathrm{Ker}(A) = 0$, and so $B\mathbf{x} = 0$. But this means that $\mathbf{x}$ belongs to $\mathrm{Ker}(B)$. Hence,

$$\mathrm{Ker}(AB) \subset \mathrm{Ker}(B) \ . \tag{2}$$

Now if two sets contain one another, they are the same, so indeed, from (1) and (2) we have

$$\mathrm{Ker}(AB) = \mathrm{Ker}(B) \ .$$