SOLUTIONS

Chapter 3

Section 1

3.1.1 Solution for Img(A): To find Img(A) means to find all vectors $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ for which the system of equations

$$x + 2y + 3z = b_1$$
$$2x + y + 3z = b_2$$
$$x + y + 2z = b_3$$

has a solution. Form the augmented matrix $[A|\mathbf{b}]$, and row reduce it to obtain:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 1 & 3 & b_2 \\ 1 & 1 & 2 & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & -3 & -3 & -2b_1 + b_2 \\ 0 & 1 & 1 & b_1 - b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & 1 & b_1 - b_3 \\ 0 & 0 & 0 & b_1 + b_2 - 3b_3 \end{array}\right] \tag{1}$$

Look at the last row: you see that the system $A\mathbf{x} = \mathbf{b}$ has a solution if and only if

$$b_1 + b_2 - 3b_3 = 0$$

This tells us that Img(A), the subspace spanned by the columns of the matrix A, is the set of vectors $\mathbf{b}$ in R^3 such that

$$\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = 0 \tag{2}$$

We know what this set is! It is the plane in R^3 described by the equation $x + y - 3z = 0$.

3.1.3 Solution for Img(A):

Method 1 You already know how to determine an LU decomposition of a matrix: Row reducing the augmented matrix $[A|I]$ you obtain

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 4 & 2 & 1 & 0 & 0 & 0 \\ 3 & 1 & 7 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & -6 & -8 & 0 & 0 & 1 & 0 \\ 4 & 1 & 11 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 4 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & -5 & -3 & 1 & 0 & 0 \\ 0 & -2 & 10 & 10 & 1 & 0 & -1 & 0 \\ 0 & 1 & -5 & -5 & -4 & 0 & 0 & 1 \end{array}\right]$$

and finally

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 4 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & -5 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \tag{2}$$

Therefore you conclude that

$$U = \begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & -5 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -5 & 2 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$$

Now, the system $A\mathbf{x} = \mathbf{b}$ is equivalent to the system $U\mathbf{x} = R\mathbf{b}$. Since the bottom two rows of U are all zero, this will be solvble if and only if the bottom two entries in $R\mathbf{b}$ are zero – otherwise we would have a pivot in the final column. That means there is a solution if and only if $C\mathbf{b} = 0$ where C consists of the bottom two rows of R – the ones that produce the bottom two entries of $U\mathbf{b}$. Hence with

$$C = \begin{bmatrix} -5 & 2 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$$

and $C\mathbf{x} = 0$ is an equation for Img(A)

Method 2 We row reduce the augmented matrix $[A|\mathbf{b}]$. But do we really need to make all the row operations again? No! Recalling that the matrix R indicates the row operations we had to make in order to row reduce A, we immediately know that we just need to find out how these row operations affect the column vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$, that is we only have to compute:

$$R\mathbf{b} = \begin{bmatrix} b_1 \\ -3b_1 + b_2 \\ -5b + 1 + 2b_2 - b_3 \\ b_1 + b_2 - b_4 \end{bmatrix}$$

Of course this means the row reduction form of $[A|\mathbf{b}]$ is

$$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 2 & b_1 \\ 0 & 1 & -5 & -5 & b_1 + b_2 \\ 0 & 0 & 0 & 0 & b_1 + 2b_2 - b_3 \\ 0 & 0 & 0 & 0 & b_1 + b_2 - b_4 \end{array}\right]$$

and then the equation for Img(A) is, as before,

$$5x - 2y + z = 0$$

$$x + y - w = 0$$

3.1.5 The hypothesis that A a 3 by 3 matrix is invertible implies that Img(A) $= \mathrm{R}^3$. Then the only matrix C such that the solution to $C\mathbf{x} = 0$ coincides with R^3 is the 3 by 3 null matrix.

3.1.7 (The references to Theorem 2 in these problems should be to Theorem 1). Row-reducing the matrix $[A|\mathbf{b}]$ you get

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 4 & 1 & 2b_1 - b_2 \\ 0 & 0 & 0 & b_1 + b_2 - b_3 \end{array}\right]$$

So the equation for Img(A) is

$$x + y - z = 0\ .$$

The second method (without manipulation of variables) is to row reduce $[A|I]$. This leads to

$$[U|R] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 1 \\ 0 & 4 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

since only the bottom row of U is zero, the equation is $[1,1,-1]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x + y - z = 0$ as we found before.

Parameterizing the solution set of $x + y - z = 0$, we get

$$s_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The second part of Theorem 3.1.1 gives us a more direct method: By the row reduction done above, we see that the first two columns of A are the pivotal columns. By Theorem 3.1.1, these provide a one to one parameterization of $\operatorname{Img}(A)$:

$$t_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

Remark: It was more work to get the first parameterization than the second. Was the extra work wasted?

No, because the first parameterization has some significant advantages. To see this, consider $\mathbf{w} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ which satisfies the equation $x + y - z$

For which values of s_1 and s_2 do we have $\mathbf{w} = s_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$? Computing the right hand side we see that

$$s_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -s_1 + s_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

requires $s_1 = 3$ and $s_2 = 5$. The values of the parameters were easy to find because the vectors had a convenient pattern of 1's and 0's in the entries. The first method always produces a parameterization with this feature – the parameter values will always be obvious. Think about why this is so.

By contrast, it takes work to find the values of the parameters t_1 and t_2 such that

$$\mathbf{w} = t_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} .$$

To find them you have to solve a system of equations. So there is no free lunch after all. We had to do two row reductions to find our parameterization with the first method, and only one with the second. But then every single time you want to compute parameters for the second parameterization, you have to solve another equation. If you will be needing parameter values on a frequent basis, the first method is superior. If not, maybe the second one is best.

3.1.9 Row-reducing the matrix $[A|\mathbf{b}]$ we obtain

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 4 & b_1 \\ 0 & 1 & 2 & 1 & b_2 \\ 0 & 0 & 0 & 0 & -4b_1 + 5b_2 + b_3 \\ 0 & 0 & 0 & 0 & 5b_2 - b_3 + 2b_4 \end{array}\right]$$

The Img(A) is the solution set of the equation $C\mathbf{x} = 0$ where

$$C = \begin{bmatrix} -4 & 5 & 1 & 0 \\ 0 & 5 & -1 & 2 \end{bmatrix} .$$

Alternatively, row reducing $[A|I]$, we get

$$[U|R] = \left[\begin{array}{cccc|cccc} 1 & 3 & 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & -1 & 2 \end{array}\right] .$$

Again, we see that Img(A) is the solution set of the equation $C\mathbf{x} = 0$ where

$$C = \begin{bmatrix} -4 & 5 & 1 & 0 \\ 0 & 5 & -1 & 2 \end{bmatrix} .$$

Parameterizing the solution set you get $t_1\mathbf{v}_1 + t_2\mathbf{v}_2$ where

$$\mathbf{v}_1 = \frac{1}{10}\begin{bmatrix} -5 \\ -4 \\ 0 \\ 10 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \frac{1}{10}\begin{bmatrix} 5 \\ 2 \\ 10 \\ 0 \end{bmatrix} .$$

The second method of Theorem 3.1.1 is more direct: By the row reduction done above, we see that the first two columns of A are the pivotal columns. By Theorem 3.1.1, these provide a one to one parameterization of Img(A):

$$t_1\begin{bmatrix} 1 \\ 0 \\ 4 \\ 2 \end{bmatrix} + t_2\begin{bmatrix} 3 \\ 1 \\ 7 \\ 1 \end{bmatrix}$$

3.1.11 The answer to this question as it appears is“No”. There is no connection between A and B as it stands. We could take A to be the $m \times n$ matrix of all zeros, and B to be $I_{n\times n}$ for a counterexample.

Why this wierd question? It should have asked if Img$(A) =$ Img(AB). Now there is a realtion between the two matrices, and now the answer is “Yes, the images must be equal”.

Here is why: Suppose that $\mathbf{b}$ belongs to Img(AB). Then there is some $\mathbf{x}$ in R^n so that $\mathbf{b} = (AB)\mathbf{x} = A(B\mathbf{x})$. Letting $\mathbf{y} = B\mathbf{x}$, we have that $\mathbf{b} = A\mathbf{y}$, and so $\mathbf{b}$ belongs to Img(A). In other words, whenever, $\mathbf{b}$ belongs to Img(AB), it also belongs to Img(A). So far we did not use the invertibility of B, so this much holds true in general.

Now for the other part of the “if and only if”, suppose that $\mathbf{b}$ belongs to Img(A). Then there is some $\mathbf{x}$ in R^n so that $\mathbf{b} = A\mathbf{x} = (ABB^{-1})\mathbf{x} = AB(B^{-1}\mathbf{x})$. Letting $\mathbf{y} = B^{-1}\mathbf{x}$, we have that $\mathbf{b} = AB\mathbf{y}$, and so $\mathbf{b}$ belongs to Img(AB). In other words, whenever, $\mathbf{b}$ belongs to Img(A), it also belongs to Img(AB).

Together, these arguments show that Img(A) and Img(AB) are the same sets of vectors – that is, they have the same members.

Section 2

3.2.1 The matrix A row reduces to $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, and so the first two columns of A are pivotal. Thus, by Theorem 3.2.1, we may take $C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $C^tC = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ so that $(C^tC)^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Also, we easily compute that $C^t\mathbf{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

We can now easily compute **c**, the vector in Img(A) closest to **b**:

$$\mathbf{c} = C(C^tC)^{-1}C^t\mathbf{b} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix} .$$

Now that we have **c**, the distance from **b** to Img(A) is just

$$|\mathbf{b} - \mathbf{c}| .$$

We compute that

$$\mathbf{b} - \mathbf{c} = \frac{1}{3}\left(\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} - \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}\right) = \frac{4}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} .$$

Therefore, $|\mathbf{b} - \mathbf{c}| = 4/\sqrt{3}$ is the distance from **b** to Img(A).

There are two ways to find the least squares solutions: If we just want *some* solution, we can solve the normal equations $C^tC\mathbf{x} = C^t\mathbf{b}$. But if we want all solutions, we need to go back to A.

There are several options, but at this point the easiest is to now solve $A\mathbf{x} = \mathbf{c}$ as usual. Row reducing $[A|\mathbf{c}]$ gives us

$$\begin{bmatrix} 1 & 1 & 2 & | & 7/3 \\ 0 & 1 & 1 & | & 5/3 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} .$$

We can see that indeed, **c** is in Img(A) since there is no pivot in the final column. Using x, y and z for the variables, x is non–pivotal. Solving for x and y in terms of z, we get

$$y = \frac{5}{3} - z \qquad \text{and} \qquad x = \frac{7}{3} - \left(\frac{5}{3} - z\right) - 2z = \frac{2}{3} - z .$$

Hence the solution set is

$$\begin{bmatrix} 2/3 - z \\ 5/3 - z \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} .$$

As z ranges over the real numbers, this parameterization traces out the line of least squares solutions to our equation.

3.2.3 The matrix A row reduces to $\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. As you see, the rank is 3, and therefore, all of the columns of A are pivotal. The unique least squares solution is given by solving the normal equations $A^tA\mathbf{x} = A^t\mathbf{b}$, which work out to

$$\begin{bmatrix} 6 & 7 & -2 \\ 7 & 15 & 13 \\ -2 & 13 & 38 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 16 \\ 7 \end{bmatrix} .$$

The unique solution is $\frac{1}{60}\begin{bmatrix} 293 \\ -38 \\ 73 \end{bmatrix}$.

3.2.5 (a) We form the matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix}$ abd the vector $\mathbf{b} = \begin{bmatrix} 0 \\ 3 \\ 7 \\ 14 \\ 22 \end{bmatrix}$. Then the given points would all lie on the line $y = ax + b$ if and only if

$$A \begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{b} \ .$$

It is clear that the rows are not proportiaonal, so there will be two pivots, so we can use the normal equations to fond the least square solution. We compute $A^t A = \begin{bmatrix} 55 & 15 \\ 15 & 5 \end{bmatrix}$ and $A^t\mathbf{b} = \begin{bmatrix} 193 \\ 46 \end{bmatrix}$. We then solve $A^t A\mathbf{x} = A^t\mathbf{b}$, using the formula for the inverse of a 2×2 matrix, and find the unique solution

$$\mathbf{x}_* = \frac{1}{10} \begin{bmatrix} 55 \\ -73 \end{bmatrix} \ .$$

We next need to compute $|A\mathbf{x}_* - \mathbf{b}|^2$ and see how close we came We find

$$A\mathbf{x}_* - \mathbf{b} = \frac{1}{10} \begin{bmatrix} -18 \\ 7 \\ 22 \\ 7 \\ -18 \end{bmatrix} \ ,$$

and then

$$|A\mathbf{x}_* - \mathbf{b}|^2 = 123 \ .$$

This is a pretty big residual! Maybe we shouldn't be trying to fit these points to a line.

(b) This time we try to fit a parabola to the points, since fitting a line did not work out so well.

We form the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \end{bmatrix}$ and the vector $\mathbf{b} = \begin{bmatrix} 0 \\ 3 \\ 7 \\ 14 \\ 22 \end{bmatrix}$. Then the given points would all lie on the line $y = ax^2 + bx + c$ if and only if

$$A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{b} \ .$$

We compute $A^t A = \begin{bmatrix} 979 & 255 & 55 \\ 255 & 55 & 15 \\ 55 & 15 & 5 \end{bmatrix}$ and $A^t\mathbf{b} = \begin{bmatrix} 849 \\ 193 \\ 46 \end{bmatrix}$. (Notice the realtion with the corresponding results from **(a)**!)

We now solve $A^t A\mathbf{x} = A^t\mathbf{b}$, and find the unique solution

$$\mathbf{x}_* = \frac{1}{70} \begin{bmatrix} 65 \\ -5 \\ -56 \end{bmatrix} \ .$$

We next need to compute $|A\mathbf{x}_* - \mathbf{b}|^2$ and see how close we came We find

$$A\mathbf{x}_* - \mathbf{b} = \frac{1}{35} \begin{bmatrix} 2 \\ -8 \\ 12 \\ -8 \\ 2 \end{bmatrix} \ ,$$

and then

$$|A\mathbf{x}_* - \mathbf{b}|^2 = 8/35 \ .$$

This is much better!

(c) No, we would say it is much more likely that there is a quadratic realtion between y and x than a linear one.

Section 3

3.3.1 We have two methods to check for linear independence:

First Solution, matrix method: Form the matrix $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$, and row reduce to determine the rank of A. The vectors are independent if and only if A has full column rank which is the case exactly when $\text{Ker(A)} = 0$. We have $A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \\ 4 & 1 & 3 \end{bmatrix}$, and so doing the row reduction,

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \\ 4 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 \\ 0 & -5 & -3 \\ 0 & -10 & -2 \\ 0 & -15 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

There is a pivot in each column, and so the rank is 3 and the vectors are linearly independent.

Second Solution, directly using the Definition: In this solution we work directly from the definition. The vectors v_1, v_2 and v_3 are linearly independent if and only if

$$t_1 v_1 + t_2 v_2 + t_3 v_3 = 0 \qquad \Rightarrow t_i = 0 \text{ for each } i \ (1 \leq i \leq 3)$$

What does this mean? Given a set of vectors, that set is linearly independent if and only if the only way to obtain the zero vector as a linear combination of your given vectors is to have all coefficients zero.

Make a linear combination with your vectors v_1, v_2 and v_3 and equal it to zero, that is

$$t_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + t_2 \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} + t_3 \begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now you just need to solve the following system of linear equations on the unknowns t_1, t_2 and t_3:

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \\ 4 & 1 & 3 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

It is an homogeneous system of linear equations and the question became: does it have *ONLY* the zero solution or it has *MORE* solutions besides the zero solution? To decide

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \\ 4 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 \\ 0 & -5 & -3 \\ 0 & -10 & -2 \\ 0 & -5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Now you have the answer! From the last matrix you see that the *ONLY* solution is $t_1 = 0$ $t_2 = 0$ $t_3 = 0$. Therefore the only way to obtain the zero vector as a linear combination of your given vectors is:

$$0 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \end{bmatrix}$$

that is the set $\{v_1, v_2, v_3\}$ is a linearly independent set.

3.3.3 First Solution, matrix method

Form the matrix $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4]$, and row reduce to determine the rank of A. The vectors are independent if and only if A has full column rank which is the case exactly when Ker(A) $= 0$. We have $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$ and doing the row reduction

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 4 & 36 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 40 \end{bmatrix}$$

Yes! There is a pivot in every column, and so the rank is 4 and the vectors are linearly independent.

Second Solution, directly using the definition: Write the linear combination and equal it to zero:

$$t_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} + t_3 \begin{bmatrix} 3 \\ 4 \\ 1 \\ 2 \end{bmatrix} + t_4 \begin{bmatrix} 4 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then you obtain the homogeneous system of linear equations

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

that you have to solve for t_1, t_2, t_3 and t_4. But now you already know what to do. Row reduce your matrix!

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 4 & 36 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 40 \end{bmatrix}$$

Yes! that is exactly it: the only solution to your system of equations is the zero solution $t_1 = 0$ $t_2 = 0$ $t_3 = 0$ and therefore you conclude that the set is a linearly independent set!

3.3.5 Yes, the columns of $C = AB$ must be linearly independent if A and B have linearly independent columns. This is because, when A and B have linearly independent columns, Ker$(A) = 0$ and Ker$(B) = 0$. Hence Ker$(C) = 0$, and hence C has linearly independent columns.

You were not asked about other case, but let's go into one any how. Supose the rows of A are linearly independent, and the columns of B are linearly independent. Then must the columns of $C = AB$ be linearly independent? No, For example, take

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} ,$$

and take B to be the 2×2 identity matrix so that $C = A$.

3.3.7 (a) The vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in the span of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has a solution, where $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$. To check this, row reduce the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 1 & a & 1 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{array}\right] .$$

The result is

$$\left[\begin{array}{rrc|r} 1 & 1 & a & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 2+a & 3 \end{array}\right] .$$

The corresponding system is solvable exactly when there is no pivot in the final column, and clearly this is the case exactly when $a \neq -2$.

The problem doesn't actually ask us to find the values, but let's do this anyway. It will shed some light on the answer to **(b)**, and it is something worth practicing anyhow. Since $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3$, we see from the row reduction above that if

$$x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} ,$$

then

$$\begin{aligned} x + y + az &= 1 \\ -y + z &= 1 \\ (2+a)z &= 3 \end{aligned}$$

so that when $a \neq -2$,

$$z = \frac{3}{2+a} \qquad y = \frac{1-a}{2+a} \qquad \text{and} \qquad x = \frac{1-a}{2+a} .$$

(b) From the row reduction above we see that as long as $a \neq -2$, $\text{rank}(A) = 3$ so that the columns of A are linearly independent, but when $a = -2$, $\text{rank}(A) = 2$ and then the columns of A are linearly dependent.

The independence when $a \neq -2$ corresponds to the fact that x, y and z were *uniquely determined* in our computation following part **(a)**.

3.3.9 The *fastest* route to a basis for either the image or the kernel of a matrix starts with a row reduction. The matrix A row reduces to

$$U = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first three columns are pivotal, so that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\text{Img}(A)$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 6 \\ 1 \\ 0 \end{bmatrix}.$$

Next, for the kernel note that $[A|0]$ row reduces to $[U|0]$, so it suffices to solve $U\mathbf{x} = 0$. Writing $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$, we see that w is non–pivotal, or free. Solving for x, y and z in terms of w, we get that the solution is $\begin{bmatrix} w \\ -2w \\ 0 \\ w \end{bmatrix}$. This is a one to one parameterization of $\text{Ker}(A)$, so

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\text{Ker}(A)$.

We deal with B, C and D the same way.

The matrix B row reduces to

$$U = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The first two columns are pivotal, so that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\text{Img}(B)$, where

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Next, for the kernel note that $[B|0]$ row reduces to $[U|0]$, so it suffices to solve $U\mathbf{x} = 0$. Writing $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we see that z is non–pivotal, or free. Solving for x and y in terms of z, we get that the solution is $\begin{bmatrix} z \\ -z \\ z \end{bmatrix}$. This is a one to one parameterization of $\text{Ker}(B)$, so

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\text{Ker}(B)$.

The matrix C row reduces to

$$U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the first two columns are pivotal, so that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\text{Img}(C)$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Next, for the kernel note that $[C|0]$ row reduces to $[U|0]$, so it suffices to solve $U\mathbf{x} = 0$. Writing $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$, we see that w and z are non–pivotal, or free. Solving, by back substitution, for x and y in terms of w and z, we get that the solution is $\begin{bmatrix} -z-2w \\ -z-w \\ z \\ w \end{bmatrix} = z\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + w\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. This is a one to one parameterization of $\text{Ker}(C)$, so

$$\{\mathbf{w}_1, \mathbf{w}_2\}$$

is a basis for $\text{Ker}(C)$ where

$$\mathbf{w}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w}_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

The matrix D row reduces to

$$U = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 2 & 5 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the first two columns are pivotal, so that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\text{Img}(D)$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

Next, for the kernel note that $[D|0]$ row reduces to $[U|0]$, so it suffices to solve $U\mathbf{x} = 0$. Writing $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$, we see that w and z are non–pivotal, or free. Solving, by back substitution, for x and y in terms of w and z, we get that the solution is $\begin{bmatrix} -(1/2)z-(1/2)w \\ -(5/2)z-(1/2)w \\ z \\ w \end{bmatrix} = z\begin{bmatrix} -1/2 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + w\begin{bmatrix} -1/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}$. This is a one to one parameterization of $\text{Ker}(D)$, so

$$\{\mathbf{w}_1, \mathbf{w}_2\}$$

is a basis for $\text{Ker}(D)$ where

$$\mathbf{w}_1 = \begin{bmatrix} -1 \\ -5 \\ 2 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w}_2 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \end{bmatrix}.$$

3.3.10, parts C and D

The matrix C row reduces to

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

Since the rank is 3, $\mathrm{Img}(C) = R^3$, and $\mathrm{Ker}(C) = 0$. We can use any basis for R^3 that we want, say the standard basis, and there is no basis for the zero subspace.

Finally, the matrix D row reduces to

$$U = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

The first two columns are pivotal, so that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathrm{Img}(D)$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 21 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} .$$

Next, for the kernel note that $[D|0]$ row reduces to $[U|0]$, so it suffices to solve $U\mathbf{x} = 0$. Writing $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$, we see that z and w are non–pivotal, or free. Solving for x, y in terms of z and w, we get that the solution is

$$\begin{bmatrix} -z \\ -w \\ z \\ w \end{bmatrix} = z \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} .$$

This is a one to one parameterization of $\mathrm{Ker}(D)$, so $\{\mathbf{w}_1, \mathbf{w}_2\}$ is a basis for $\mathrm{Ker}(D)$ where

$$\mathbf{w}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} .$$

3.3.11 No. For example the standard basis vector $\mathbf{e}_1$ belongs to S, but $2\mathbf{e}_1$ does not.

3.3.13 We just have to show that $S + \tilde{S}$ is closed under scalar multiplication and addition.

First, scalar multiplication: Suppose that $\mathbf{v}$ belongs to $S + \tilde{S}$, and a is any number. Then, by the definition of $S + \tilde{S}$, $\mathbf{v} = \mathbf{x} + \mathbf{y}$ for some $\mathbf{x}$ in S and some $\mathbf{y}$ in $\tilde{S}$. Hence

$$a\mathbf{v} = a(\mathbf{x} + \mathbf{y}) = (a\mathbf{x}) + (a\mathbf{y}) .$$

This does indeed belong to $S + \tilde{S}$: $a\mathbf{x}$ belongs to S since S is a subspace, and $a\mathbf{y}$ belongs to $\tilde{S}$ since $\tilde{S}$ is a subspace.

Next, addition. Suppose $\mathbf{v}$ and $\mathbf{w}$ are two vectors in $S + \tilde{S}$. Then $\mathbf{v} = \mathbf{x} + \mathbf{y}$ and $\mathbf{w} = \mathbf{b} + \mathbf{c}$ where $\mathbf{x}$ and $\mathbf{b}$ are in S, and $\mathbf{y}$ and $\mathbf{c}$ are in $\tilde{S}$. Then

$$\mathbf{v} + \mathbf{w} = (\mathbf{x} + \mathbf{y}) + (\mathbf{b} + \mathbf{c}) = (\mathbf{x} + \mathbf{b}) + (\mathbf{y} + \mathbf{c})$$

which is in $S + \tilde{S}$ since S and $\tilde{S}$ are closed under addition.

3.3.15 The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ satisfy the equation for the plane, so they both belong to it. They are not proportional, so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent subset contained in the plane. To compute the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$, we compute the equation for the image of $A = [\mathbf{v}_1, \mathbf{v}_2]$ by row reducing $[A|\mathbf{x}]$. The result is the equation of the plane. This shows that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set of vectors that spans the plane.

The vector $\mathbf{w}$ also satsifies the equation of the plane, so it has a unique expression as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{w} = s\mathbf{v}_1 + t\mathbf{v}_2$$

for some uniquely determined coordinates s and t. To find them, solve

$$A \begin{bmatrix} s \\ t \end{bmatrix} = \mathbf{w}$$

where $A = [\mathbf{v}_1, \mathbf{v}_2]$ as above. To do this, rowreduce $[A|\mathbf{w}]$ as usual. The result is

$$\begin{bmatrix} 1 & 0 & | & 1/2 \\ 0 & 1 & | & 1/2 \\ 0 & 0 & | & 0 \end{bmatrix} .$$

¿From here we see that $s = t = 1/2$. And indeed, you can easily check that

$$\frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 = \mathbf{w} .$$

3.3.17 The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ satisfy the equation for the plane, so they both belong to it. Next,

$$\mathbf{v}_1 \cdot \mathbf{v}_1 = \mathbf{v}_2 \cdot \mathbf{v}_2 = 1$$

and

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0 ,$$

so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal, and hence linearly independent, subset contained in the plane.

To compute the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$, we compute the equation for the image of $A = [\mathbf{v}_1, \mathbf{v}_2]$ by row reducing $[A|\mathbf{x}]$. The result is the equation of the plane. This shows that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set of vectors that spans the plane.

The vector $\mathbf{w}$ also satsifies the equation of the plane, so it has a unique expression as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{w} = s\mathbf{v}_1 + t\mathbf{v}_2$$

for some uniquely determined coordinates s and t. To find them, we do not need to solve anything – we just take dot products. This is a big advantage of orthonormal bases!

$$s = \mathbf{v}_1 \cdot \mathbf{w} = -3 \qquad \text{and} \qquad t = \mathbf{v}_2 \cdot \mathbf{w} = 6 .$$

Indeed, you can easily check that

$$\mathbf{w} = 6\mathbf{v}_2 - 3\mathbf{v}_1 .$$

Section 4

3.4.1 Form the matrix $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4]$, and row reduce to determine its rank:

$$A = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & 3 & 1 & 4 \\ 3 & 2 & 4 & 1 \\ 4 & 1 & 3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & -5 & -3 & -2 \\ 0 & -10 & -2 & -8 \\ 0 & -15 & -5 & -10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & -5 & -3 & -2 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Evidently,

$$\text{rank}(A) = 3 \ .$$

This means that $A\mathbf{x} = 0$ has a non–trivial solution. Since there is no pivot in the fourth column, x_4 is a non-pivotal variable, and we can freely assign it. Hence there are numbers t_1, t_2 and t_3 so that

$$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3 + \mathbf{v}_4 = 0 \ . \tag{1}$$

Hence $\mathbf{v}_4$ can be written as a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and so

$$\text{Sp}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) = \text{Sp}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}) \ .$$

Now $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent since

$$\text{rank}([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]) = 3 \ .$$

We don't need to do any additional row reduction to see this; just supress the fourth column in our last row reduction. Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent spanning set – the spanning part is for free – it is a basis, and so the dimension is 3.

Discussion: We were only asked for the *dimension* of $\text{Sp}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\})$ But can we get a more detailed picture of what this subspace is?

By definition, this subspace consists of all vectors $\mathbf{b} \in R^4$ that can be written as a linear combination of the vectors in the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. In other words, this is the set of all vectors $\mathbf{b}$ that satisfy, for some $t_1,\ t_2,\ t_3,\ t_4 \in R$,

$$t_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + t_2 \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} + t_3 \begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \end{bmatrix} + t_4 \begin{bmatrix} 3 \\ 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \tag{2}$$

Let's see when (2) is solvable by row reducing as in Example 1 of Section 1:

$$\left[\begin{array}{cccc|c} 1 & 4 & 2 & 3 & b_1 \\ 2 & 3 & 1 & 4 & b_2 \\ 3 & 2 & 4 & 1 & b_3 \\ 4 & 1 & 3 & 2 & b_4 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 4 & 2 & 3 & b_1 \\ 0 & -5 & -3 & -2 & -2b_1 + b_2 \\ 0 & -10 & -2 & -8 & -3b_1 + b_3 \\ 0 & -15 & -5 & -10 & -4b_1 + b_4 \end{array}\right]$$

Continuing to perform row operations we get

$$\left[\begin{array}{cccc|c} 1 & 4 & 2 & 3 & b_1 \\ 0 & -5 & -3 & -2 & -2b_1 + b_2 \\ 0 & 0 & 4 & -4 & b_1 - 2b_2 + b_3 \\ 0 & 0 & 4 & -4 & 2b_1 - 3b_2 + b_4 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 4 & 2 & 3 & b_1 \\ 0 & -5 & -3 & -2 & -2b_1 + b_2 \\ 0 & 0 & 4 & -4 & b_1 - 2b_2 + b_3 \\ 0 & 0 & 0 & 0 & b_1 - b_2 - b_3 + b_4 \end{array}\right]$$

There is a pivot in the final column of the augmented matrix unless

$$b_1 - b_2 - b_3 + b_4 = 0 \ . \tag{3}$$

We can also write

$$\mathrm{Sp}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}) = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in R^4 : \ x - y - z + w = 0 \right\}$$

We can write equation (3) in the form

$$\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \cdot \mathbf{b} = 0 \ . \tag{4}$$

The conclusion is that $\mathbf{b}$ belongs to $\mathrm{Sp}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\})$ if and only if $\mathbf{b}$ is orthogonal to $\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$. Note that equation (4) looks just like the equation that describes a plane in R^3, except that in the problem we are working in R^4. The solution set of (4), which is our subspace of R^4 is the equation of a 3 dimensional *hyperplane* in R^4.

You could also work out a one–to–one parameterization of it, since you know how to parameterize the solution set of any single linear equation. As you know from Chapter 2, you will need three parameters, and this corresponds to the fact that the dimension is 3.

3.4.3 All of these questions are easy to answer if we know the ranks of the matrices. So first we row reduce to find

$$\mathrm{rank}(A) = 3 \qquad \mathrm{rank}(B) = 2 \qquad \mathrm{rank}(C) = 3 \qquad \mathrm{rank}(D) = 3 \ .$$

(a) The columns are linearly independent exactly when the rank equals the number of columns. This is the case only for A and D.

(b) The rows are linearly independent exactly when the rank equals the number of rows. This is the case only for C and D.

(c) This is another way of phrasing **(a)**: The kernel is zero exactly when the columns are linearly independent, and so exactly when the rank equals the number of columns. This is the case only for A and D.

(d) The dimension of the image of the transpose of a matrix is the rank of the transposed matrix, which is the same as the rank of the original matrix. So this question is just another way of asking which if these matrices have rank 3. Hence the answer is A, C and D.

(e) The dimension of the image of a matrix is the dimension of its image. So this question is just another way of asking which if these matrices have rank 3. Hence the answer is A, C and D.

3.4.5 All of these questions are easy to answer if we know the ranks of the matrices. So first we row reduce to find

$$\mathrm{rank}(A) = 2 \qquad \mathrm{rank}(B) = 3 \qquad \mathrm{rank}(C) = 3 \qquad \mathrm{rank}(D) = 2 \ .$$

(a) The columns are linearly independent exactly when the rank equals the number of columns. This is the case only for C.

(b) The rows are linearly independent exactly when the rank equals the number of rows. This is the case only for B and C.

(c) This is another way of phrasing **(a)**: The kernel is zero exactly when the columns are linearly independent, and so exactly when the rank equals the number of columns. This is the case only for C.

(d) The dimension of the image of the transpose of a matrix is the rank of the transposed matrix, which is the same as the rank of the original matrix. So this question is just another way of asking which of these matrices have rank 3. Hence the answer is B and C.

(e) The dimension of the image of a matrix is the dimension of its image. So this question is just another way of asking which of these matrices have rank 3. Hence the answer is B and C.

3.4.7 You have

$$t_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + t_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Then you have to solve the system

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

where $A = [\mathbf{v_1}, \mathbf{v}_2, \mathbf{v}_3]$ Row-reducing the corresponding augmented matrix you obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 6 \end{array}\right] \tag{1}$$

You see that your original system of equations has a unique solution. Solving it, by back substitution, you obtain the unique solution

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

and then you have

$$-1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

To answer the basis question we need to decide two issues:

- *Does* $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ *span* R^3*?*
- *Is* $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ *linearly independent?*

The answer to the first question is "yes" if and only if A has full row rank, and the answer to the second question is "yes" if and only if A has full column rank. But you see from (1) that $\text{rank}(A) = 3$, and so A has full column rank and full row rank, and so $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a basis.

3.4.9 Write the linear combination of the vectors you are given:

$$t_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + t_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \tag{1}$$

which is the same as

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}.$$

The row-reduction of the corresponding augmented matrix gives:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ -1 & 1 & -2 & 2 \\ 0 & -1 & 1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & -1 & 1 & -3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

You see right away from the row reduction that

$$\text{rank}(A) = 2$$

and so A has neither full row rank nor full column rank. This means that $\text{Ker}(A) \neq 0$, and so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not linearly independent , and it does not span R^3, and so it definitely is not a basis.

Now let's parameterize the solution set of (1), and thus find all ways of writing $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ as a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

The variable t_3 is non-pivotal, so we make it a parameter: $t_3 = s$. Back substitution gives $t_2 = 3 + s$ and $t_1 = 1 - s$. Therefore, as s ranges over R,

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 1-s \\ 3+s \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

gives us all of the coefficients we are looking for.

3.4.11 Finding a basis for $\text{Img}(A)$**:** To find a basis for the column space of the matrix A is equivalent to find a basis for the row space of the matrix A^t and in order to do it you can row-reduce A^t

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 2 & 3 & 1 \\ 4 & 2 & 7 & 3 \\ 1 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1}$$

A basis for Img(A) is the set of vectors:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\}$$

Finding a basis for $\text{Ker}(A)$**:** In order to find a basis for Ker(A) you just find a one–to–one parameterization of the solution set of $A\mathbf{x} = 0$, and the basis is the set of vectors used in the parameterization. (In computational terms, this is a familiar problem form Chapter 2, and we are just talking about it in a different way now).

You know what to do – row-reduce A:

$$\begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 2 & 2 & 0 \\ 2 & 3 & 7 & 1 \\ 1 & 1 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2}$$

Then

$$A \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

is equivalent to the system

$$\begin{aligned} x + 2y + 4z + w &= 0 \\ y + z &= 0 \\ w &= 0 \ . \end{aligned}$$

Only z is non–pivotal. Make z into a parameters, and put $z = t$. Then the system becomes

$$\begin{aligned} x + 2y + w &= -4s \\ y &= -s \\ w &= \ \ 0 \ . \end{aligned}$$

We see right away what w and y are, and we then see that $x = -2s$.

Therefore, $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ belongs to Ker(A) exactly when it is of the form

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

for some number s and we have

$$\text{Ker(A)} = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in R^4 : \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } s \in R \right\}$$

Hence the set

$$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

spans Ker(A), is plainly linearly independent, and so it is the basis we seek.

Now let's find bases for $\text{Img}(A^t)$ and $\text{Ker}(A^t)$.

Finding a basis for $\mathrm{Img}(A^t)$**:** We can do just what we did for A itself. We're off to a head start though since all the computations we need for finding a basis for $\mathrm{Img}(A^t)$ have already been performed since $(A^t)^t = A$ and (2) is the row-reduced form of A. Hence

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\mathrm{Img}(A^t)$.

Finding a basis for $\mathrm{Ker}(A^t)$**:** In this case too, almost all calculations are already done! We've already worked out the row reduction of A^t in (1), and so $A^t \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$ exactly when

$$\begin{aligned} x + 2z + w &= 0 \\ 2y + z - w &= 0 \\ -2w &= 0\ . \end{aligned}$$

Only the variable z is non-pivotal, and we make it a parameter, putting $z = s$. Then our system becomes

$$\begin{aligned} x + w &= -2s \\ 2y - w &= -s \\ w &= \ \ 0\ . \end{aligned}$$

You see that

$$\mathrm{Ker}(A^t) = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} : \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \frac{s}{2} \begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}\ .$$

Therefore, a basis for Ker(A^{t}) is:

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}\ .$$

Discussion: *How can you check your results?* Theorem 5 of Section 3 says that

$$\mathrm{Img}(A) = \left(\mathrm{Ker}(A^t)\right)^{\perp}\ , \tag{3}$$

and then by Theorem 3 of the same section, and the fact that A is a 4×4 matrix,

$$\dim(\mathrm{Img}(A)) + \dim(\mathrm{Ker}(A)) = 4 \tag{4}$$

We found in this case that $\dim(\mathrm{Img}(A)) = 3$ you know that $\dim(\mathrm{Ker}(A)) = 1$, and these add up to 4, and they must. Likewise with $\dim(\mathrm{Img}(A^t)) = 3$ you know that $\dim(\mathrm{Ker}(A^t)) = 1$.

We can get more confirmation from (3). This says that $\begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix}$, our basis vector for $\mathrm{Ker}(A^t)$, must be orthogonal to each of the basis vectors we found for $\mathrm{Img}(A)$. Computing the three dot products, you do see that this is indeed the case! Likewise, the basis vector we found for Ker(A), must be, and is, orthogonal to each of the basis vectors we found for Img(A^t).

3.4.13 Solution for a basis for Img(A)**:** Form the transpose of A and row-reduce it. You will find

$$A^t = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore a basis for Img(A) is the set of vectors $\{\mathbf{v_1}, \mathbf{v_2}\}$ where

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{v_2} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

Finding a basis for Ker(A)**:** Row-reducing A you have

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{1}$$

Now you have to solve the system of equations

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since its solution set is

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in R^3 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \text{ and } s \in R \right\}$$

a basis is the set $\{\mathbf{v_1}\}$ where

$$\mathbf{v_1} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Solution for a basis for Img(A^t)**:** From (1) you see that a basis for Img(A^t) is the set of vectors $\{\mathbf{v_1}, \mathbf{v_2}\}$ where

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \qquad \mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Solution for a basis for Ker(A^t)**:** Using the row-reduced form of A^t you see that you have to solve the system

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

You obtain for its solution set

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \qquad \text{for all} \quad s \in R$$

and therefore a basis for this subspace of R^3 is the set $\{\mathbf{v_1}\}$ with

$$\mathbf{v_1} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

3.4.15 Let $A = \begin{bmatrix} -1 & 1 \\ -2 & 1 \\ 2 & -1 \end{bmatrix}$ and let $\mathbf{a} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Then as in Example 3.4.7, the vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ satisfies $A^t A\mathbf{x} = \mathbf{a}$. In this case $A^t A = \begin{bmatrix} 9 & -5 \\ -5 & 3 \end{bmatrix}$ so that

$$\mathbf{x} = (A^t A)^{-1}\mathbf{a} = \frac{1}{2}\begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 13 \\ 23 \end{bmatrix} .$$

Hence $x = 13$ and $y = 23$.

You can easily check the result:

$$\mathbf{w} = 13\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} + 23\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \\ 3 \end{bmatrix} ,$$

so indeed $\mathbf{w} \cdot \mathbf{v}_1 = 2$ and $\mathbf{w} \cdot \mathbf{v}_2 = 4$.

3.4.17 Let $A = \begin{bmatrix} -1 & -2 & 2 \\ -2 & 1 & 0 \\ -2 & -1 & 1 \\ 1 & 2 & -3 \end{bmatrix}$ and let $\mathbf{a} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$. Then as in Example 3.4.7, the vector $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfies $A^t A\mathbf{x} = \mathbf{a}$. In this case $A^t A = \begin{bmatrix} 10 & 0 & -3 \\ 0 & 10 & -11 \\ -3 & -11 & 14 \end{bmatrix}$ so that to solve $A^t A\mathbf{x} = \mathbf{a}$ we row reduce

$$\begin{bmatrix} 10 & 4 & -3 & | & 2 \\ 4 & 10 & -11 & | & 4 \\ -3 & -11 & 14 & | & 1 \end{bmatrix}$$

This row reduces to

$$\begin{bmatrix} 10 & 4 & -3 & | & 2 \\ 0 & 10 & -11 & | & 4 \\ 0 & 0 & 1 & | & 6 \end{bmatrix}$$

By back substitution, we find that the solution is

$$\begin{bmatrix} 2 \\ 7 \\ 6 \end{bmatrix} .$$

You can easily check the result:

$$\mathbf{w} = A\begin{bmatrix} 2 \\ 7 \\ 6 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 3 \\ -2 \end{bmatrix} ,$$

so indeed $\mathbf{w} \cdot \mathbf{v}_1 = 2$, $\mathbf{w} \cdot \mathbf{v}_2 = 4$ and $\mathbf{w} \cdot \mathbf{v}_3 = 1$.

3.4.19 (a) The columns of A are clearly not proportional, so they are linearly independent. Hence $\text{rank}(A) = 2$. Likewise, $\text{rank}(B) = 2$.

(b) The matrix $[A|\mathbf{x}]$ row reduces to $\begin{bmatrix} 1 & 2 & | & x \\ 0 & 3 & | & 2x - y \\ 0 & 0 & | & z - x - y \end{bmatrix}$. Hence $\text{Img}(A)$ is the plane with the equation

$$z - x - y = 0 .$$

Each of the columns of B does satisfy this equation.

(c) Yes: Since both columns of B satisfy $z - x - y = 0$, so does any linear combination of the columns of B. Hence every vector in $\mathrm{Img}(B)$ satisfies $z - x - y = 0$, the equation for $\mathrm{Img}(A)$. Hence $\mathrm{Img}(B) \subset \mathrm{Img}(A)$.

(d) Yes, by the dimension principle and **(a)** which says that the images of A and B have the same dimension, namely 2.

Section 5

3.5.1 By the Fredholm criterion, a solution exists exactly when $\mathbf{b}$ is orthogonal to every vector in $\mathrm{Ker}(A^t)$. Given that $\mathrm{Ker}(A^t)$ is the line through $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$, we just need to check that

$$\mathbf{b} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} = 0 \ .$$

doing the computation, we find that this is the case. So $\mathbf{b} \in \mathrm{Img}(A)$, and there is at least one solution.

On the other hand, there is never exactly one, because A has more columns than rows. So the solution exists, but is not unique.

Discussion: Suppose you were asked the same question but with $b = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ What could you say about the existence of solutions of $A\mathbf{x} = \mathbf{b}$?

You could see, without any computation, that the system had no solutions because b is in $\mathrm{Ker}(A^t)$, not its orthogonal complement.

3.5.3 Form the matrix $A = [\mathbf{v}_1, \mathbf{v}_2]$ so that

$$S = \mathrm{Img}(A) \ .$$

By Theorem 3.5.2,

$$S^\perp = \mathrm{Ker}(A^t) \ .$$

Let P be the orthogonal projection onto S, and let $P^\perp$ be the orthogonal projection onto $S^\perp$. We know that

$$P = I - P^\perp \qquad P^\perp = I - P \tag{$*$}$$

so as soon as we have computed one of these projections, we know the other.

- *Since you only need to compute one of them directly, you may as well choose the simplest one.*

Which one is that? Well

$$\dim(S) = 2 \qquad \text{and} \qquad \dim\left(S^\perp\right) = 1$$

by the dimension formulas, Theorem 3.4.6. (Since $\mathbf{v}_2$ and $\mathbf{v}_2$ are not proportional, they are linearly independent so that $\dim(S) = \text{rank}(A) = 2$. So it is going to be easier to directly compute $P^\perp$ since it is a projection onto a one dimensional space.

To compute a basis for $\text{Ker}(A^t)$, you just need to row reduce A^t:

$$A^t = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

So far so good! In the system

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

you have only one non–pivotal variable, namely z. Set the value of z to be 1, and solving for x and y, you see that $y = -2$ and $x = 1$. Hence $\{\mathbf{v}\}$ where $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is a basis for $\text{S}^\perp$. Normalizing it, we get

$$\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} ,$$

and $\{\mathbf{u}\}$ is an orthonormal basis for the same space.

Now proceed as in the Example 3.5.3, and use Theorem 1 which says, since $\mathbf{u}$ is a unit vector,

$$P_{i,j} = \mathbf{u}_i \mathbf{u}_j \ .$$

¿From here you see that

$$P^\perp = \frac{1}{6} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

Next, using the relation (5.14) between P and $P^\perp$ you have

$$P = I - P^\perp = \frac{1}{6} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$$

(Pulling out a factor of 1/6 from the identity makes it very easy to do the subtraction. Highly recommended by four out of five mathematicians!)

Finally, the one–to–one parametric representation for $S^\perp$ is a piece of cake! Since we have a basis for $S^\perp$, the one–to–one parameterization is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad t_1 \in R$$

Discussion: Instead of using Theorem 3.5.2 to get a handle on $S^\perp$, you could have started with the definition. Recall that, by definition, $S^\perp$ is the subspace of R^3 defined by

$$S^\perp = \{\mathbf{u} : \mathbf{u} \cdot \mathbf{x} = 0 \text{ for all } \mathbf{x} \in S\}$$

But you also know that a necessary and sufficient condition for $\mathbf{u}$ to be orthogonal to all elements of S is to be orthogonal to a basis of S. Therefore to find $S^\perp$, is equivalent to find all vectors of R^3 orthogonal to the set $\{\mathbf{v_1}, \mathbf{v_2}\}$ where

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \mathbf{v_2} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

To find all such vectors, find the kernel of $\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}$. That would just be the space $\mathrm{Ker}(\mathrm{A}^\mathrm{t})$, and now you would proceed as above.

3.5.5 (a) You just need to use Theorem 3.5.2. Form A^t and row-reduce it:

$$A^t = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Then the subspace of R^3, $\mathrm{Ker}(\mathrm{A}^\mathrm{t})$, consists of all elements $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ of R^3 such that

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

that is

$$\mathrm{Ker}(A^t) = \left\{ t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \qquad \text{for all } t \right\}$$

and the set $\{\mathbf{u}_1\}$ where

$$\mathbf{u}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

is an orthonormal basis for $\mathrm{Ker}(A^t)$. Now we can compute the orthogonal projection onto $\mathrm{Ker}(A^t)$ using (5.15) which says that

$$P_{i,j} = \mathbf{u}_i \mathbf{u}_j$$

By Theorem 3.5.2, the matrix you are computing is the orthogonal projection onto $\mathrm{S}^\perp$ where $\mathrm{S} = \mathrm{Img(A)}$, and so you have

$$P^\perp = \frac{1}{6} \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix}$$

(b) You get the answer right away now using $P = I - P^\perp$:

$$P = I - P^\perp = \frac{1}{6} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix}$$

(c) The sum of the matrices you found in parts (a) and (b) gives the identity matrix. Remember that Theorem 3.5.2 tells you that $\mathrm{Img(A)} = \mathrm{Ker}(\mathrm{A}^\mathrm{t})^\perp$, and that $P + P^\perp = I$!

(d) You know that $\text{rank}(A) = 2$, so

$$\dim(\text{Img}(A^t)) = 2 \qquad \text{and} \qquad \dim(\text{Ker}(A)) = 1 \ .$$

It is therefore easier to compute the orthogonal projection onto $\text{Img}(A^t)^\perp = \text{Ker}(A)$ since this space has dimension 1. (We are using Theorem 3.5.2 again).

After row-reducing A you obtain

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $\text{rank}(A) = 2$ and

To find a basis for the kernel, you have to solve

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} ,$$

and you can do this, by now for sure, with ease. You obtain the orthonormal basis $\{\mathbf{u_1}\}$ where

$$u_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

and the orthogonal projection matrix onto $\text{Img}(\text{A}^\text{t})$ is

$$P = \frac{1}{3} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Section 6

3.6.1 Before applying the Gramm-Schmidt method we have to find a basis for Img(A), which you now know how to do after solving the problems for section 4. We *could* use the basis you found in problem **3.4.11**. But if instead we row reduce A^t all the way to echelon form, so that we "clean out" as many entries as possible, all those zeros will make the Gramm–Schmidt computations go much more smoothly.

So let's row reduce A^t, all the way to echelon form. We find:

$$A^t = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 2 & 3 & 1 \\ 4 & 2 & 7 & 3 \\ 1 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore a basis for Img(A) is the set of vectors $\{\mathbf{v}_1, \mathbf{v_2}, \mathbf{v_3}\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

This is the basis to which we apply the Gramm-Schmidt process. The resulting orthonormal basis is the set $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ where

$$\mathbf{u_1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{u_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{u_3} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

3.6.3 To find the QR decomposition means that we have to find an orthogonal matrix Q and an upper triangular matrix R such that $QR = A$. In fact Q is a matrix whose columns form an orthonormal basis for Img(A) obtained by the Gramm–Schmidt algorithm, and R is just the change of basis matrix.

So consider the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \mathbf{v_2} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \quad \mathbf{v_3} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} ,$$

which are the columns of A.

We find $|\mathbf{v}_1| = \sqrt{5}$, so $\mathbf{u}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

Next,

$$\mathbf{u}_1 \cdot \mathbf{v}_2 = \frac{2}{\sqrt{5}} .$$

Hence

$$\begin{aligned} \mathbf{w}_2 &= \mathbf{v}_2 - (\mathbf{u}_1 \cdot \mathbf{v}_2)\mathbf{u}_1 \\ &= \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \\ &= \frac{1}{5}\left(\begin{bmatrix} 10 \\ 15 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}\right) \\ &= \frac{1}{5}\begin{bmatrix} 8 \\ 15 \\ -4 \end{bmatrix} . \end{aligned}$$

to find $\mathbf{w}_2$, we just normalize this. Now let's not do extra work: When computing

$$\frac{1}{|\mathbf{w}_2|}\mathbf{w}_2 ,$$

we can ignore the factor of $1/5$ *since it will cancel out, top and bottom.* We find

$$\mathbf{u}_2 = \frac{1}{|\mathbf{w}_2|}\mathbf{w}_2 = \frac{1}{\sqrt{305}}\begin{bmatrix} 8 \\ 15 \\ -4 \end{bmatrix} .$$

Two down, one to go!

To compute $\mathbf{w}_3$, we first compute

$$\mathbf{u}_1 \cdot \mathbf{v}_3 = \sqrt{5} \qquad \text{and} \qquad \mathbf{u}_2 \cdot \mathbf{v}_3 = \frac{50}{\sqrt{305}} .$$

This gives us

$$\begin{aligned}
\mathbf{w}_3 &= \mathbf{v}_3 - \sqrt{5}\mathbf{u}_1 - \frac{50}{\sqrt{305}}\mathbf{u}_2 \\
&= \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{10}{61}\begin{bmatrix} 8 \\ 15 \\ -4 \end{bmatrix} \\
&= \frac{1}{61}\left(\begin{bmatrix} 183 \\ 122 \\ 61 \end{bmatrix} - \begin{bmatrix} 61 \\ 0 \\ 122 \end{bmatrix} - \begin{bmatrix} 80 \\ 150 \\ -40 \end{bmatrix} \right) \\
&= \frac{1}{61}\begin{bmatrix} 42 \\ -28 \\ -21 \end{bmatrix} \\
&= \frac{7}{61}\begin{bmatrix} 6 \\ -4 \\ -3 \end{bmatrix} .
\end{aligned}$$

It is a good idea to check at this point that $\mathbf{w}_3 \cdot \mathbf{v}_1 = 0$ and $\mathbf{w}_3 \cdot \mathbf{v}_2 = 0$. *This checks out, which gives us confidence in all those hairy calculations!*

Again, when we compute

$$\frac{1}{|\mathbf{w}_3|}\mathbf{w}_3 ,$$

we can ignore the factor of $7/61$ *since it will cancel out, top and bottom.* This isn't so bad as it looked a short while ago. We get

$$\mathbf{u}_3 = \frac{1}{|\mathbf{w}_3|}\mathbf{w}_3 = \frac{1}{\sqrt{61}}\begin{bmatrix} 6 \\ -4 \\ -3 \end{bmatrix} .$$

We now have

$$Q = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{8}{\sqrt{305}} & \frac{6}{\sqrt{61}} \\ 0 & \frac{15}{\sqrt{305}} & \frac{-4}{\sqrt{61}} \\ \frac{2}{\sqrt{5}} & \frac{-4}{\sqrt{305}} & \frac{-3}{\sqrt{61}} \end{bmatrix} .$$

Now let's find R. According to formula (7.9) in Theorem 3.7.2,

$$R = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{v}_1 & \mathbf{u}_1 \cdot \mathbf{v}_2 & \mathbf{u}_1 \cdot \mathbf{v}_3 \\ \mathbf{u}_2 \cdot \mathbf{v}_1 & \mathbf{u}_2 \cdot \mathbf{v}_2 & \mathbf{u}_2 \cdot \mathbf{v}_3 \\ \mathbf{u}_3 \cdot \mathbf{v}_1 & \mathbf{u}_3 \cdot \mathbf{v}_2 & \mathbf{u}_3 \cdot \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{v}_1 & \mathbf{u}_1 \cdot \mathbf{v}_2 & \mathbf{u}_1 \cdot \mathbf{v}_3 \\ 0 & \mathbf{u}_2 \cdot \mathbf{v}_2 & \mathbf{u}_2 \cdot \mathbf{v}_3 \\ 0 & 0 & \mathbf{u}_3 \cdot \mathbf{v}_3 \end{bmatrix} ,$$

since, according to Theorem 3.7.2, R is guaranteed to be upper–triangular.

We've already computed all of the entries above the diagonal, so all we need to do now is to work out the diagonal entries. These are:

$$\mathbf{u}_1 \cdot \mathbf{v}_1 = \sqrt{5} \quad , \quad \mathbf{u}_2 \cdot \mathbf{v}_2 = \sqrt{61/5} \quad \text{and} \quad \mathbf{u}_3 \cdot \mathbf{v}_3 = 7/\sqrt{61} .$$

Therefore,

$$R = \begin{bmatrix} \sqrt{5} & \frac{2}{\sqrt{5}} & \sqrt{5} \\ 0 & \sqrt{\frac{61}{5}} & \frac{50}{\sqrt{305}} \\ 0 & 0 & \frac{7}{\sqrt{61}} \end{bmatrix} .$$

If you now do the multiplication as a check, you find that indeed, $A = QR$.

Discussion Weren't those numbers disgusting? Well, things like this do take some getting used to. The fact is, that you are not going to compute very many of these by hand, even if your career involves a lot of linear algebra. All computer programs for doing linear algebra have "routines" or "methods" built in for computing QR decompositions. So you won't really have to deal with all the messy numbers yourself.

But second, they aren't so bad after all. Make sure you understand this one well. We won't give this much detail every time we compute Q and R, so if you get stuck later, come back and review this one. It's all here!

Going through a few like this is also good practice in efficient computation. They provide a good check on how systematic you are in your computation. If you aren't systematic, you will not succeed.

3.6.5 (a) Form the matrix A^t and row-reduce it. You will obtain:

$$A^t = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

You see that rank(A) = 2 so the dim(Img(A)) = 2 and a basis for Img(A), a subspace of R^3, is the set $\{\mathbf{v}_1, v_2\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

After applying the Gramm-Schmidt process to the basis $\{\mathbf{v}_1, v_2\}$, we obtain the orthonormal basis $\{\mathbf{u_1}, \mathbf{u_2}\}$ forImg(A) with

$$\mathbf{u_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{u_2} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Great! You have computed the matrix Q:

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \sqrt{\frac{2}{3}} \end{bmatrix}$$

As you saw in the previous problem to determine the change of basis matrix R you just need to express the vectors $\mathbf{v_i}$ $(i = 1, 2)$ as linear combinations of the vectors in the Gramm–Schmidt basis.

First solution: We have

$$R = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{v}_1 & \mathbf{u}_1 \cdot \mathbf{v}_2 & \mathbf{u}_1 \cdot \mathbf{v}_3 \\ \mathbf{u}_2 \cdot \mathbf{v}_1 & \mathbf{u}_2 \cdot \mathbf{v}_2 & \mathbf{u}_2 \cdot \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{v}_1 & \mathbf{u}_1 \cdot \mathbf{v}_2 \\ 0 & \mathbf{u}_2 \cdot \mathbf{v}_2 \end{bmatrix} ,$$

since, according to Theorem 3.7.2, R is guaranteed to be upper–triangular.

In the course of computing $\mathbf{u}_2$, you already found that $\mathbf{u}_1 \cdot \mathbf{v}_2 = 1/sqrt2$. Computing the other dot products, we find

$$R = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} & \frac{3}{\sqrt{6}} \end{bmatrix}$$

Second solution: Theorem 3.6.4 tells us what is really going on here. We want to express the $\mathbf{v_j}$'s as linear combinations of the $\mathbf{u_i}$'s. That's a piece of cake – You just need to compute dot products!

If you write

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = x_1 \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

you determine that

$$x_1 = \mathbf{v}_1 \cdot \mathbf{u_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$x_2 = \mathbf{v}_1 \cdot \mathbf{u_2} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Analogously:

$$\mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = y_1 \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y_2 \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

with

$$y_1 = \mathbf{v_2} \cdot \mathbf{u_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad y_2 = \mathbf{v_2} \cdot \mathbf{u_2} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

To finalize you have to calculate the coordinates of the third column of A, the "redundant" vector, with respect to the Gramm-Schmidt basis $\{\mathbf{u_1}, \mathbf{u_2}\}$:

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = z_1 \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z_2 \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

with

$$z_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad z_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

You have the matrix R:

$$R = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} & \frac{3}{\sqrt{6}} \end{bmatrix}$$

At this point you should check your result. The dimensions are very easy to check: the product of a 3 by 2 matrix by a 2 by 3 matrix, gives a 3 by 3 matrix. The rest is matrix multiplication:

$$QR = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \sqrt{\frac{2}{3}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} & \frac{3}{\sqrt{6}} \end{bmatrix} = A$$

(b) The row-reduced form of the matrix A is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore you conclude that $\text{rank}(\text{A}^{\text{t}}) = 2$, and a basis for $\text{Img}(\text{A}^{\text{t}})$, a subspace of R^3 is the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Of course you can also use the first two columns of the matrix A^t *but since you are not asked any question that would involve the original columns you can choose which matrix you like best.* (We assume you like simple vectors best, and the ones you get by row–reducing A^t are simple).

After applying Gramm-Schmidt to the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ you obtain the orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ for $\mathrm{Img}(\mathrm{A}^t)$ where

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix} \qquad \text{and} \qquad \mathbf{u}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\2\\1\end{bmatrix}$$

(c) Solution for $\mathrm{Img}(A)$**:** From **(a)** you know that the set$\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for Img(A) with

$$\mathbf{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix} \qquad \mathbf{v_2} = \begin{bmatrix}0\\1\\-1\end{bmatrix}$$

The Img(A) is a plane in R^3 with the one-to-one parameterization

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = t_1\begin{bmatrix}1\\1\\0\end{bmatrix} + t_2\begin{bmatrix}0\\1\\-1\end{bmatrix} \qquad t_1, t_2 \in R$$

The equation for the plane is

$$x - y - z = 0$$

Solution for Ker(A)**:** Using the dimension formula (4.7) you know that dim(Ker(A)) = 1, that is, Ker(A), a subspace of R^3, is a line in R^3. In order to write a one-to-one parameterization we need to have a basis. Solving the system $A\mathbf{x} = \mathbf{0}$ you find that the set $\{\mathbf{v}_1\}$ with $\mathbf{v}_1 = \begin{bmatrix}-1\\-1\\1\end{bmatrix}$ is a basis for Ker(A). Therefore a one-to-one parameterization for Ker(A) is

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = t\begin{bmatrix}-1\\-1\\1\end{bmatrix} \qquad t \in R$$

(d) You have $\dim(\mathrm{Img}(\mathrm{A})) = 2$ and $\dim(\mathrm{Img}(\mathrm{A}^t)) = 2$.

(e) Since you conclude in part (a) that rank(A) = 2, you know that the columns and the rows of A are not linearly independent.

3.6.7 (a) The columns of A are linearly independent so you can take as your basis for Img(A) the set of vectors $\{\mathbf{v}_1, v_2\}$ where

$$\mathbf{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix} \qquad \mathbf{v_2} = \begin{bmatrix}0\\1\\1\end{bmatrix}$$

We apply the Gramm– Schmidt process (in fact this is the same basis we used in the problem **3.7.5**) and we obtain the orthonormal basis $\{\mathbf{u_1}, \mathbf{u_2}\}$ where

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} \qquad \mathbf{u}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

You should compare the problems **2.5.7** and **2.5.7 (a)**: the only and very important difference is the size of the matrices. So Q is the same matrix Q you found in **2.5.5 (a)**:

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \sqrt{\frac{2}{3}} \end{bmatrix}$$

and R is the upper triangular matrix

$$R = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} \end{bmatrix}$$

(b) The one-to-one parameterization of Img(A), a plane in R^3, is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad t_1, t_2 \in R$$

Eliminating the parameters you have

$$x + y - z = 0$$

3.6.9 (a) Form the matrix

$$A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4]$$

The span of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is Img(A), and we know how to find orthonormal bases for images. Row-reducing A you get

$$A = \begin{bmatrix} -1 & 2 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

You see that rank(A) = 3 and therefore the span of $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ has dimension 3. If we ignore the last column in the row reduction above, we would still get rank 3, and so the first three columns are linearly independent. Hence the set $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a basis for the span of $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ where

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

(We could also have found the basis by row reducing A^t, which would actually simplify some of the computations that follow, but we've got a basis, so let's go on).

Using Gramm-Schmidt you obtain the orthonormal basis $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ with

$$\mathbf{u_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{u_2} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ \frac{1}{3} \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{u_3} = \frac{1}{2\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 3 \\ -1 \end{bmatrix}$$

(b) You know that

$$S^{\perp} = \{\mathbf{x} \in R^3 : \mathbf{x} \cdot \mathbf{y} = 0 \text{ for all } \mathbf{y} \in S\}$$

and from Theorem 3.5.1

$$\dim(\mathrm{S}) + \dim(\mathrm{S}^\perp) = 4\ .$$

You can immediately conclude that $\dim(S^\perp) = 1$, and that $S^\perp = \mathrm{Ker}(A)$ where A is the matrix constructed in part (a).

To find the kernel, solve the system

$$\begin{bmatrix} -1 & 2 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

The result is that $S^\perp$ is the set of vectors of the form $t \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ for any number t. Hence $\{\mathbf{u}\}$ is an orthonormal basis for $S^\perp$ where

$$\mathbf{u} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} .$$

(c) Good news: You have done most part of the work in the previous parts! Think again of Theorem 3.5.4 and determine the orthogonal projection P onto $S^\perp$ first:

Solution for the orthogonal projection $P^\perp$ onto $S^\perp$: Using $P^\perp_{i,j} = \mathbf{u}_i \mathbf{u}_j$,

$$P^\perp = \frac{1}{4} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} .$$

Solution for the orthogonal projection P onto S:

Method 1: Use $P = I - P^\perp$ so that

$$P = \frac{1}{4} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 & 1 \\ -1 & 3 & 1 & 1 \\ 1 & 1 & 3 & -1 \\ 1 & 1 & -1 & 3 \end{bmatrix}$$

Method 2: Of course you could also use the formula $P = QQ^t$! The matrix Q is $Q = [\mathbf{u}_1, \mathbf{u}_1, \mathbf{u}_3]$ where the unit vectors are the ones found in part (a). Therefore,

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{2\sqrt{3}} \\ 0 & 0 & \frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} \end{bmatrix}$$

and then the orthogonal projection P onto S is given by

$$P = QQ^\perp = \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 & 1 \\ -1 & 3 & 1 & 1 \\ 1 & 1 & 3 & -1 \\ 1 & 1 & -1 & 3 \end{bmatrix}$$

(d) From section 5 you know that

$$\mathbf{x} = P\mathbf{x} + P^{\perp}\mathbf{x}$$

where P is the projection matrix onto S and $P^{\perp}$ is the projection matrix onto $S^{\perp}$. Using the matrices P and $P^{\perp}$ you have determined in part (c) you find

$$\mathbf{w} = P\mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = P^{\perp}\mathbf{x} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

3.6.11 In this problem $S = \text{Img}(C)$, and so $S^{\perp} = \text{Ker}(C^t)$.

(a) From considerations about the dimensions of S and $S^{\perp}$ you see that is much more eficient to compute the orthogonal projection onto $S^{\perp}$ since $\dim(\text{S}^{\perp}) = 1$.

Orthogonal projection onto $S^{\perp}$: To find vector $\mathbf{v}$ means you need to determine the matrix $P^{\perp}$ that represents the orthogonal projection onto $S^{\perp}$ and apply it to the vector $\mathbf{x}$. But those are the steps you have made in **3.6.9 (d)**. First you need to find a basis for $S^{\perp}$. Of course you already know what to do! You have to solve the system

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and you find out that the set $\{\mathbf{v_1}\}$ where $\{\mathbf{v_1}\} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ is a basis for $S^{\perp}$. But remember: the matrix Q must have orthonormal columns, therefore the basis you just determined has to be transformed into an orthonormal basis. That's easy: normalize the basis you just found! The set $\{\mathbf{u_1}\}$ where $\mathbf{u_1} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ is an orthonormal basis for $S^{\perp}$.

¿From $P = QQ^t$ you obtain the matrix P that represents orthogonal projection onto $S^{\perp}$:

$$P^{\perp} = \frac{1}{6} \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix}$$

and consequently the vector $\mathbf{v}$ you are looking for is:

$$\mathbf{v} = P^{\perp}x = \frac{1}{6} \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -7 \\ 7 \\ 14 \end{bmatrix},$$

and

$$\mathbf{w} = \mathbf{b} - \mathbf{v} = \frac{1}{3} \begin{bmatrix} 7 \\ 4 \\ 5 \end{bmatrix}.$$

(b) The distance from $\mathbf{b}$ to $S = \text{Img}(C)$ is

$$|\mathbf{b} - P\mathbf{b}| = |\mathbf{b} - \mathbf{w}| = |\mathbf{v}| = \frac{7}{6}\sqrt{6}.$$

(c) The distance from $\mathbf{b}$ to $S^\perp = (\mathrm{Img}(C))^\perp$ is

$$|\mathbf{b} - P^\perp\mathbf{b}| = |\mathbf{b} - \mathbf{v}| = |\mathbf{w}| = \sqrt{10}\ .$$

3.6.13 Subtract $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ from each of the three vectors. This is a translation, and does not affect relative distances. So it is the same to find the distance from the vector $\mathbf{c}$ defined by

$$\mathbf{c} = \mathbf{b} - \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

to the line through the origin and

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 2 \end{bmatrix}\ .$$

Now, after translation, the line is a subspace! This is the point of the translation: We have methods for dealing with subspaces – very effective methods.

Call this subspace S. Hence the distance is $|\mathbf{c} - P\mathbf{c}|$ where P is the orthogonal projection onto S. Since $\{\mathbf{u}\}$ is an orthonormal basis for S, where

$$\mathbf{u} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 2 \end{bmatrix}\ ,$$

we have from

$$P_{i,j} = (\mathbf{e}_i \cdot \mathbf{u})(\mathbf{e}_i \cdot \mathbf{u})\ ,$$

the convenient formula for one dimensional projections, that

$$P = \frac{1}{5} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix}\ .$$

Now you find

$$\begin{aligned} \mathbf{c} - P\mathbf{c} &= \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 0 \\ 0 \\ -3 \\ 6 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 5 \\ -5 \\ -5 \\ 5 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 0 \\ 0 \\ -3 \\ 6 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 5 \\ -5 \\ -2 \\ -1 \end{bmatrix}\ . \end{aligned}$$

Hence

$$|\mathbf{c} - P\mathbf{c}| = \frac{\sqrt{55}}{5} \ ,$$

and this is the distance from the point to the line.

Section 7

3.7.1 Solution for (a): You find the QR decomposition of A using the method from section 3.7 with the result that

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \qquad \text{and} \qquad R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

It fact, this is pretty easy to do since the first two columns of A are already orthogonal.

Solution for (b): You compute $Q^t\mathbf{b}$, and then solve $R\mathbf{x} = Q^t\mathbf{b}$, which is easy since R is upper triangular:

$$Q^t\mathbf{b} = \begin{bmatrix} 2 \\ 2\sqrt{2} \\ 1 \end{bmatrix} \ .$$

Now $R\mathbf{x} = Q^t\mathbf{b}$ just reduces to the system

$$\begin{aligned} x_1 + x_3 &= 2 \\ \sqrt{2}x_2 + \sqrt{2}x_3 &= 2\sqrt{2} \\ x_3 &= 1 \end{aligned}$$

and you see that $x_3 = 1$, $x_2 = 1$ and $x_1 = 1$ by back substitution. Hence the (unique) least squares solution is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \ .$$

Discussion for (b): Let's try to check your answer and see if it makes sense. Let's see what $A\mathbf{x}$ is. Computing the product, you find

$$A\mathbf{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix} \ ,$$

and hence

$$\mathbf{b} - A\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \ .$$

You can easily see that this vector is orthogonal to each of the columns of A, and hence to every vector in $\mathrm{Img}(A)$. Since $A\mathbf{x}$ is in $\mathrm{Img}(A)$, by definition, it follows that $\mathbf{b} - A\mathbf{x}$ is the projection of $\mathbf{b}$ onto $\mathrm{Img}(A)$ since

$$\mathbf{b} = (\mathrm{Img}(A)) + (\mathbf{b} - A\mathbf{x}) \ ,$$

where the first vector in the sum is in Img(A), and the second vector in the sum is in $(\text{Img}(A))^{\perp}$, and there is only one way to split $\mathbf{b}$ up like this, according to Theorem 3.5.3. So what you've done, implicitly, was to solve $A\mathbf{x} = P_c\mathbf{b}$, where P_c is the orthogonal projection onto Img(A), the span of the columns of A:

$$P_c\mathbf{b} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

Now when you solve $A\mathbf{x} = P_c\mathbf{b}$ you obtain the solution you have obtained before and since this is what is supposed to be going on in a least squares solution you can be sure you have done it right!!

Solution for (c): This will be easy if you did the check on the solution to part (b): You will recogonize the vector $\mathbf{c}$!

In any case, you find $Q^t\mathbf{c} = 0$, and so the least squares solution $\mathbf{x}$ solves $R\mathbf{x} = 0$, and the only solution of this is $\mathbf{x} = 0$.

3.7.3 Let $\mathbf{b}$ be the vector

$$\mathbf{b} = \begin{bmatrix} 1 \\ e^{-1/4} \\ e^{-1} \\ e^{-9/4} \\ e^{-4} \\ e^{-25/4} \end{bmatrix} ,$$

and let A be the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{64} & \frac{1}{16} & \frac{1}{4} & 1 \\ 1 & 1 & 1 & 1 \\ \frac{729}{64} & \frac{81}{16} & \frac{9}{4} & 1 \\ 64 & 16 & 4 & 1 \\ \frac{15625}{64} & \frac{625}{16} & \frac{25}{4} & 1 \end{bmatrix} .$$

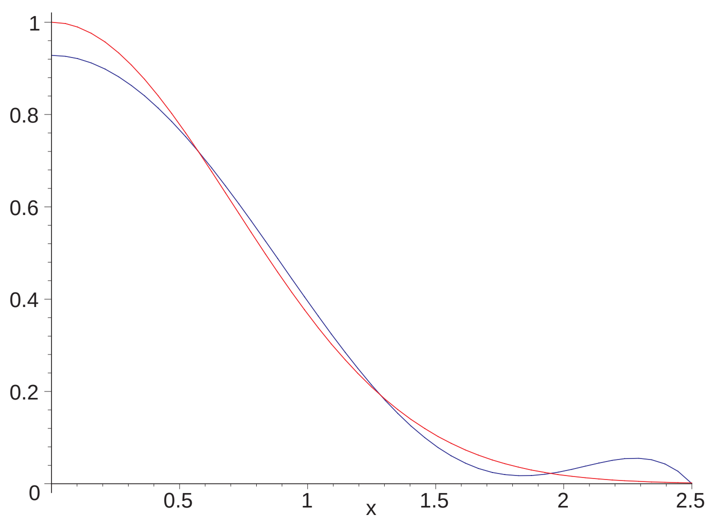
What you are looking for is a least squares solution to $A\mathbf{x} = \mathbf{b}$. All you need to do is to work out the QR decomposition of A, find $Q^t\mathbf{b}$ and then solve $R\mathbf{x} = Q^t\mathbf{b}$. Easier said than done you say? Not with Maple, MATLAB, Mathematica or some such program. For instance with Maple, after entering A and $\mathbf{b}$, with $\mathbf{b}$ entered as a single column matrix, you would use the following sequence of commands, in which `QA` and `RA` are just the names you have chosen to give to the Q and R of the matrix A:

```
RA := QRdecomp(A,Q='QA',fullspan=false):

c := evalm(transpose(QA) &* b):

linsolve(RA,c):

evalf(%);
```

The result is $\mathbf{x} = \begin{bmatrix} -.01303121502 \\ .1676767898 \\ -.6873939029 \\ .9282766757 \end{bmatrix}$. Thus the polynomial $y(x)$ you are looking for is

$$y(x) = -.01303121502x^6 + .1676767898x^4 - .6873939029x^2 + .9282766757 \ .$$

Here is a plot of $f(x) = e^{-x^2}$ together with $y(x)$ on the interval $[0, 5, 2]$ in which you see there is a pretty good fit. (The polynomial $y(x)$ is the non—monotonic curve).



In fact, it does *much* better than the sixth degree Taylor approximation

$$P_6(x) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} \ .$$

Try graphing it it and you'll see!

Doing the integral, you find

$$\int_0^{5/2} y(x)\mathrm{d}x = .8792206340 \ .$$

From the graph, you can get a good idea of the accuracy of this results, which is not bad. The correct value, with 16 correct digits shown, is

$$.8858662736175309$$

and so the approximation is correct to within one percent.

The method used in this problem provides a good approach to numerical integration, and it is possible to build on what you know to estimate the error, but you won't do that here.

3.7.5 You have seen this matrix before in problem (3.5.4). Notice that the third column is the sum of the first two, so the first two columns of A already span $\mathrm{Img}(A)$. Computing the QR decomposition of A you find:

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \\ 0 & 1/\sqrt{3} \end{bmatrix} \qquad \text{and} \qquad R = \begin{bmatrix} \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{3} & \sqrt{3} \end{bmatrix} .$$

Next, solve, by back substitution, $R\mathbf{x} = Q^t\mathbf{b}$ where

$$Q^t\mathbf{b} = \begin{bmatrix} 3/\sqrt{2} \\ 2/\sqrt{3} \end{bmatrix}$$

You find the one parameter family of solutions

$$\begin{bmatrix} 3/2 \\ 2/3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} .$$

If you apply A to any of these vectors, you get the vector in Img(A) that is closest to $\mathbf{b}$. (The vector that is multiplied by t is in the kernel of A, so the result does not depend on t). The result is

$$A \begin{bmatrix} 3/2 \\ 2/3 \\ 0 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 13 \\ 5 \\ 4 \end{bmatrix} .$$

3.7.7 Computing the QR decomposition of A you find:

$$Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{bmatrix} \qquad \text{and} \qquad R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3/2} \end{bmatrix} .$$

Next, solving $R\mathbf{x} = Q^t\mathbf{b}$ by back substitution, where

$$Q^t\mathbf{b} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2/3} \end{bmatrix}$$

you find the unique solution

$$\begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix} .$$

If you apply A to this vector, you get the vector in Img(A) that is closest to $\mathbf{b}$. The result is

$$A \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} .$$

Taking the least squares solution for $\mathbf{x}$, you have (by definition) the vector that makes $|A\mathbf{x} - \mathbf{b}|^2$ as small as possible. Since you now know both $A\mathbf{x}$ and $\mathbf{b}$ you can compute this number:

$$A\mathbf{x} - \mathbf{b} = \frac{1}{3} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} .$$

Hence the minimum value of $|A\mathbf{x} - \mathbf{b}|^2$ is

$$\frac{1}{9}3 = \frac{1}{3} .$$

3.7.9 Solution for (a) You may notice right away that the third column is the sum of the first two, so the first two columns span Img(A). Since they are not multiples of one another, they are linearly independent, and so they are a basis for Img(A). (You could row reduce A^t, but for simple matrices like this, you may as you will proceed "by inspection").

Solution for (b): First Method: Computing the QR decomposition of A you find

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$

Then

$$P_c = QQ^t = \frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} .$$

Second Method: The orthogonal complement to Img(A) is one dimensional, so it will be easier to compute the orthogonal projection onto it, $P_c^\perp$. For this you can use the formula (6.1) in Theorem 1 of Section 6.

By Theorem 3.5.2, $P_c^\perp$ is the orthogonal projection onto Ker(A^t). A^t row reduces to

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Choosing the value 1 for the non-pivotal variable x_3, you then find $x_2 = 1$ and $x_1 = -1$, so the vector $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is a basis for Ker(A^t). Normalizing this, the single vector

$$\frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

is an orthonormal basis for Ker(A^t). That was easy – no Gramm-Schmidt needed for a one–dimensional space! It is also easy to apply (6.1) since there is just one vector – no sum is needed! You find

$$P_c^\perp = \frac{1}{3}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} .$$

Then since $P_c + P_c^\perp = I$, you have $P_c = I - P_c^\perp$ or

$$P_c = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} .$$

this is the same result you got before. It may look a little longer, but you did every single step explicitly instead of jumping to the results of a QR decomposition.

Solution for (c) Since you have P_c, you just compute

$$P_c\mathbf{b} = \frac{1}{3}\begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} .$$

Now solve

$$A\mathbf{x} = \frac{1}{3}\begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} .$$

You know how to do this in a straightforward manner, but notice that the sum of the columns of A is $\begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$. This means that a least squares solution is

$$\frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} .$$

Solution for (d) You have seen above that Img(A) is two dimensional, and so it is a plane. Since Img$(A) = (\text{Ker}(A^t))^\perp$ by Theorem 3.5.2 once again, you see that a vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ belongs to Img(A) if and only if it is orthogonal to $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ since this vector is a basis for Ker(A^t). Hence the equation of the plane is

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

or

$$-x + y + z = 0 .$$

Section 8

3.8.1 We row reduce the matrix

$$[W|V] = \begin{bmatrix} 1 & 0 & | & 2 & 1 \\ -1 & 1 & | & -1 & 1 \\ 0 & -1 & | & -1 & -2 \end{bmatrix}$$

"all the way" as if we are looking for an inverse, and we find

$$\begin{bmatrix} 1 & 0 & | & 2 & 1 \\ 0 & 1 & | & 1 & 2 \\ 0 & 0 & | & 0 & 0 \end{bmatrix}$$

The change of basis matrix $[\gamma|\beta]$ is therefore

$$[\gamma, \beta] = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} .$$

Next, consider the vector $2\mathbf{v}_1 - 3\mathbf{v}_2$, which has the coordinate vactor $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ with respect to the basis $\beta = \{\mathbf{v}_1, \mathbf{v}_2\}$. The coordinates of this same vector with respect to the basis $\beta = \{\mathbf{w}_1, \mathbf{w}_2\}$ are given by

$$[\gamma|\beta]\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

Hence the vector is

$$\mathbf{w}_1 - 4\mathbf{w}_2 \ .$$

Let's check this answer: Computing we find

$$2\mathbf{v}_1 - 3\mathbf{v}_2 = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$$

and

$$\mathbf{w}_1 - 4\mathbf{w}_2 = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix} \ .$$

This checks out: we get the same vector, as we should.

3.8.3 First, let's compute the "nice" basis for Img(A). To do this, we row reduce A^t "all the way" to the form

$$\begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

We can take the first two rows as our basis, but let's multiply each by 3 to avoid fractions. We get that $\gamma = \{\mathbf{w}_1, \mathbf{w}_2\}$ where

$$\mathbf{w}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \ .$$

Next, as we have, the rank of A is two, so the first two columns of A must be pivotal, as they are not proportional. (That is, we don't need to row reduce A to find out which columns are pivotal, since we have already row reduced A, and that tells us what we need to know). Hence, $\beta = \{\mathbf{v}_1, \mathbf{v}_2\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \ .$$

Now we row reduce $[W|V] = \begin{bmatrix} 3 & 0 & | & 1 & 2 \\ 0 & 3 & | & 2 & 1 \\ 1 & 1 & | & 1 & 1 \end{bmatrix}$ "all the way", and find

$$\begin{bmatrix} 1 & 0 & | & 1/3 & 2/3 \\ 0 & 1 & | & 2/3 & 1/3 \\ 0 & 0 & | & 0 & 0 \end{bmatrix} \ .$$

Therefore, the change of basis matrix is

$$[\gamma, \beta] = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \ .$$

3.8.5 First, let's compute the "nice" basis for Img(A). To do this, we row reduce A^t "all the way" to the form

$$\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & -3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We can take the first three rows as our basis, but let's multiply each by appropriate integers to avoid fractions.

Then we get that $\gamma = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ where

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 6 \end{bmatrix} \qquad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w}_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -3 \end{bmatrix} .$$

Next, we know from our row reduction of A^t that the rank of A is three, but this doesn't tell which columns are pivotal – it *could in principle* be the case that the third column is a linear combination of the first two, in which case columns 1, 2 and 4 would be pivotal. But doing the row reduction, we see this didn't happen. The first three columns are the pivotal ones, and so

$\beta = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} .$$

Now we row reduce $[W|V] = \begin{bmatrix} 1 & 0 & 0 & | & 1 & 1 & 0 \\ 0 & 4 & 0 & | & 4 & 2 & 6 \\ 0 & 0 & 2 & | & 4 & 3 & 1 \\ 6 & 1 & -3 & | & 1 & 2 & 0 \end{bmatrix}$ "all the way", and find

$$\begin{bmatrix} 1 & 0 & 0 & | & 1 & 1 & 0 \\ 0 & 1 & 0 & | & 1 & 1/2 & 3/2 \\ 0 & 0 & 1 & | & 2 & 3/2 & 1/2 \\ 0 & 0 & 0 & | & 0 & 0 & 0 \end{bmatrix} .$$

Therefore, the change of basis matrix is

$$[\gamma, \beta] = \frac{1}{2} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 3 \\ 4 & 3 & 1 \end{bmatrix} .$$

3.8.7 The identity matrix is the simplest example that there is.