Optimal Control of quasi-Newtonian Fluids

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AMFM 2014

Plan

Setting of the control problem

Consider the steady generalized Navier-Stokes equation

$$(SE)\quad \begin{cases} -\nabla\cdot(\tau(Dy)) + (y\cdot\nabla)\,y + \nabla\pi = u & \text{in } \Omega, \\ \nabla\cdot y = 0 & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma, \end{cases}$$

where

y velocity field, π pressure,

τ extra stress tensor

$Dy = \frac{1}{2}\left(\nabla y + (\nabla y)^\top\right)$ symm. part of the velocity gradient

u given body force

$\Omega \subset \mathbb{R}^n$ $(n = 2 \text{ or } n = 3)$ bounded domain, Γ its boundary.

The optimal control problem reads as

$$(P)\begin{cases}\text{Minimize} & J(y,u)=\frac{1}{2}\int_\Omega |y-y_d|^2\,dx+\frac{\lambda}{2}\int_\Omega |u|^2\,dx\\ \text{such that} & (y,u)\text{ satisfies }(SE)\text{ for some }\pi\\ & u\in U_{ad},\end{cases}$$

where

$\lambda \geq 0,\ y_d$ some desired velocity field

U_{ad} nonempty bounded closed subset of $L^2(\Omega)$.

Objective. Establish existence of an optimal control and derive the corresponding necessary optimality conditions.

The extra stress tensor

We assume that $\tau : \mathbb{S}_n \longrightarrow \mathbb{S}_n$ has a potential $\Phi : \mathbb{S}_n \longrightarrow \mathbb{R}^+$, i.e. there exists Φ is of class C^2 with

$$\begin{cases} \tau_{ij}(\eta) = \dfrac{\partial \Phi(|\eta|^2)}{\partial \eta_{ij}} = 2\Phi'(|\eta|^2)\,\eta_{ij} & \forall\, \eta \in \mathbb{S}_n, \\ \Phi(0) = 0, \qquad \dfrac{\partial \Phi(0)}{\partial \eta_{ij}} = 0 & \forall\, i, j. \end{cases}$$

Moreover, we require that for some $\alpha > 1$

A$_1$ - There exists $\gamma > 0$ such that for all $i, j, k, \ell = 1, \cdots, n$

$$\left| \frac{\partial \tau_{k\ell}(\eta)}{\partial \eta_{ij}} \right| \leq \gamma \left(1 + |\eta|^2\right)^{\frac{\alpha-2}{2}} \qquad \forall\, \eta \in \mathbb{S}_n.$$

A$_2$ - There exists $\mu > 0$ such that

$$\begin{aligned} \tau'(\eta) : \zeta : \zeta &= \sum_{ijk\ell} \frac{\partial \tau_{k\ell}(\eta)}{\partial \eta_{ij}} \zeta_{k\ell} \zeta_{ij} \\ &\geq \mu \left(1 + |\eta|^2\right)^{\frac{\alpha-2}{2}} |\zeta|^2 \qquad \forall\, \eta, \zeta \in \mathbb{S}_n. \end{aligned}$$

Example

Assumptions $\mathbf{A}_1$-$\mathbf{A}_2$ are usually used in the literature and cover a wide range of non-Newtonian fluids. A typical prototype of extra tensors used in applications is given

$$\tau(\eta) = 2\mu \left(1 + |\eta|^2\right)^{\frac{\alpha-2}{2}} \eta.$$

Remark. i) A fluid is shear-thickening if $\alpha > 2$ and is shear-thinning if $\alpha < 2$.

ii) For the special case $\tau(\eta) = 2\mu\eta$ $(\alpha = 2)$, we recover the classical Navier-Stokes equation with viscosity coeficient $\mu > 0$.

Assumptions $\mathbf{A}_1$-$\mathbf{A}_2$ imply the following standard properties for τ.

Lemma. Let $\alpha \geq 2$ and τ satisfying $\mathbf{A}_1$-$\mathbf{A}_2$. Then the following properties hold

Continuity.

$$|\tau(\eta)| \leq \frac{n^2\gamma}{\alpha-1}\left(1+|\eta|^2\right)^{\frac{\alpha-2}{2}}|\eta|$$

Coercivity.

$$\tau(\eta) : \eta \geq \mu|\eta|^2$$

$$\tau(\eta) : \eta \geq \frac{\mu}{\alpha-1}|\eta|^\alpha$$

Monotonicity.

$$(\tau(\eta)-\tau(\zeta)) : (\eta-\zeta) \geq \mu\,|\eta-\zeta|^2$$

$$(\tau(\eta)-\tau(\zeta)) : (\eta-\zeta) \geq \frac{\mu}{2^{\alpha-2}(\alpha-1)}\,|\eta-\zeta|^\alpha$$

where γ and μ are the constants appearing in the assumptions $\mathbf{A}_1$-$\mathbf{A}_2$.

Lemma. Let $1 < \alpha < 2$ and τ satisfying $\mathbf{A}_1$-$\mathbf{A}_2$. Then the following properties hold

Continuity.

$$|\tau(\eta)| \le \frac{n^2\gamma}{\alpha-1}\left(1+|\eta|^2\right)^{\frac{\alpha-2}{2}}|\eta|$$

Coercivity.

$$\tau(\eta):\eta \ge \frac{\mu|\eta|^2}{(1+|\eta|^2)^{\frac{2-\alpha}{2}}}$$

Monotonicity.

$$(\tau(\eta)-\tau(\zeta)):(\eta-\zeta) \ge \frac{\mu|\eta-\zeta|^2}{(1+|\eta|^2+|\zeta|^2)^{\frac{2-\alpha}{2}}}$$

where γ and μ are the constants appearing in the assumptions $\mathbf{A}_1$-$\mathbf{A}_2$.

Definition of weak solutions

Consider

$$\mathcal{V} = \{\varphi \in \mathcal{D}(\Omega) \mid \nabla \cdot \varphi = 0\},$$

and let V_α be the closure of $\mathcal{V}$ with respect to the norm $\|\nabla \cdot\|_\alpha$, i.e.

$$V_\alpha = \{\varphi \in W_0^{1,\alpha}(\Omega) \mid \nabla \cdot \varphi = 0\}.$$

Definition. Let $u \in L^2(\Omega)$. A function $y_u \in V_\alpha$ is a weak solution of the state equation if

$$(\tau(Dy_u), D\varphi) + b(y_u, y_u, \varphi) = (u, \varphi) \quad \forall\, \varphi \in V_\alpha,$$

where $b(y_1, y_2, y_3) = ((y_1 \cdot \nabla)\, y_2, y_3)$.

The formulation is well posed under restrictions on the exponent α. Indeed, if $y_u \in V_\alpha$ then

$$(y_u \cdot \nabla)\, y_u \in L^{\frac{\alpha\alpha^*}{\alpha+\alpha^*}}(\Omega) \subset L^{(\alpha^*)'}(\Omega) \quad \text{if} \quad \alpha \geq \frac{3n}{n+2}.$$

Existence of weak solutions

• Existence of a weak solution for $\alpha \geq \frac{3n}{n+2}$ was first established by Ladyzhenskaya and Lions by using compactness arguments and the methods of monotone operators.

Without any ambition of completeness

• We emphasize the work by Nečas, Frehse, Málek, Steinhauer and Růžička, who extended these results and established existence of weak solutions under the less restrictive condition $\alpha > \frac{2n}{n+2}$.

• Many open problems concerning the global regularity of weak solutions still exist. Related to this aspect, we refer to the works by Kaplický, Málek, Růžička, Stará, Ebmeyer, Diening, Nečas and Beirão da Veiga.

A priori estimates

Proposition. (shear-thickening case) Assume that $\mathbf{A}_1$-$\mathbf{A}_2$ are fulfilled with $\alpha \geq 2$ and that $u \in L^2(\Omega)$. The state equation admits at least a weak solution $y_u \in V_\alpha$ and the following estimates hold

$$\begin{aligned} \|Dy_u\|_2 &\leq \kappa_1 \frac{\|u\|_2}{\mu} \\ \|Dy_u\|_\alpha^\alpha &\leq (\alpha - 1)\left(\kappa_1 \frac{\|u\|_2}{\mu}\right)^2 \end{aligned}$$

with $\kappa_1 = \frac{\sqrt{2}(n-1)}{\sqrt{n}} |\Omega|^{\frac{1}{n}}$.

Proof. Set $\varphi = y_u$ in the weak formulation and use the Poincaré and the Korn inequalities to obtain

$$\begin{aligned}(\tau(Dy_u), Dy_u) = -b(y_u, y_u, y_u) + (u, y_u) = (u, y_u) \\ \leq \kappa_1 \|u\|_2 \|Dy_u\|_2 \qquad (1)\end{aligned}$$

with $\kappa_1 = \sqrt{2}\, C_P$. On the other hand, by taking into account the 2-coercivity condition, we have

$$\mu \|Dy_u\|_2^2 \leq (\tau(Dy_u), Dy_u). \qquad (2)$$

Combining (1) and (2), we obtain the first estimate. Similarly, by using the α-coercivity condition and the first estimate, we get

$$\begin{aligned}\frac{\mu}{\alpha-1} \|Dy_u\|_\alpha^\alpha &\leq (\tau(Dy_u), Dy_u) = (u, y_u) \\ &\leq \kappa_1 \|u\|_2 \|Dy_u\|_2 \leq \frac{(\kappa_1 \|u\|_2)^2}{\mu} \qquad \square\end{aligned}$$

Proposition. (shear-thinning case) Assume that $\mathbf{A}_1$-$\mathbf{A}_2$ are fulfilled with $\frac{3n}{n+2} \leq \alpha < 2$ and that $u \in L^2(\Omega)$. The state equation admits at least a weak solution $y_u \in V_\alpha$, and the following estimates hold

$$\|Dy_u\|_\alpha \leq \kappa_1 \left(1 + \frac{\|u\|_2}{\mu}\right)^{\frac{2-\alpha}{\alpha-1}} \frac{\|u\|_2}{\mu},$$

$$\|Dy_u\|_\alpha^\alpha \leq \left(2^{\frac{2-\alpha}{2}} C_\alpha \frac{\|u\|_2}{\mu}\right)^{\frac{\alpha}{\alpha-1}} + |\Omega|,$$

where

$$\begin{cases} \kappa_1 = C_\alpha \left(\frac{|\Omega|}{\alpha-1} + C_\alpha^{\frac{\alpha}{\alpha-1}}\right)^{\frac{2-\alpha}{\alpha}} \\ C_\alpha = \frac{\alpha(n-1)}{2(n-\alpha)\sqrt{n}} \frac{1}{C_{K,\alpha}} |\Omega|^{\frac{(n+2)\alpha-2n}{2\alpha n}} \end{cases}$$

with $C_{K,\alpha}$ being the constant of Korn.

Sketch of the proof of the first estimate. Setting $\varphi = y_u$ in the weak formulation, using the coercivity condition and standard arguments, we obtain

$$\begin{aligned} \frac{\mu \|Dy_u\|_\alpha^2}{\left(|\Omega| + \|Dy_u\|_\alpha^\alpha\right)^{\frac{2-\alpha}{\alpha}}} &\leq \left(\tau\left(Dy_u\right), Dy_u\right) = (u, y_u) \\ &\leq C_\alpha \|u\|_2 \|Dy_u\|_\alpha \end{aligned}$$

Hence

$$\|Dy_u\|_\alpha^{\frac{\alpha}{2-\alpha}} \leq M^{\frac{\alpha}{2-\alpha}} \left(|\Omega| + \|Dy_u\|_\alpha^\alpha\right), \tag{3}$$

where $M = \frac{C_\alpha \|u\|_2}{\mu}$. The Young inequality yields

$$M^{\frac{\alpha}{2-\alpha}} \|Dy_u\|_\alpha^\alpha \leq (2-\alpha) \|Dy_u\|_\alpha^{\frac{\alpha}{2-\alpha}} + (\alpha - 1) M^{\frac{\alpha}{(2-\alpha)(\alpha-1)}} \tag{4}$$

and by combining (3) and (4), we deduce that

$$(\alpha - 1) \|Dy_u\|_\alpha^{\frac{\alpha}{2-\alpha}} \leq M^{\frac{\alpha}{2-\alpha}} |\Omega| + (\alpha - 1) M^{\frac{\alpha}{(2-\alpha)(\alpha-1)}}.$$

This gives the result. □

Uniqueness of weak solutions

Proposition. (shear-thickening case) Assume that $\mathbf{A}_1$-$\mathbf{A}_2$ are fulfilled with $\alpha \geq 2$ and that $u \in L^2(\Omega)$ satisfies

$$\frac{\|u\|_2}{\mu^2} < \frac{\sqrt{n^3}}{4(n-1)^2|\Omega|^{\frac{1}{n-1}}}.$$

Then, the state equation admits a unique weak solution $y_u \in V_\alpha$.

Proof. Let y_1 and y_2 be two weak solutions corresponding to u. Setting $\varphi = y_1 - y_2$ in the weak formulation, and using classical arguments together with the estimate for y_2, we obtain

$$\begin{aligned}(\tau(Dy_1) - \tau(Dy_2), D(y_1 - y_2)) &= -b(y_1 - y_2, y_2, y_1 - y_2) \\ &\leq \kappa_2 \|D(y_1 - y_2)\|_2^2 \|Dy_2\|_2 \\ &\leq \kappa_1\kappa_2 \|D(y_1 - y_2)\|_2^2 \frac{\|u\|_2}{\mu}, \qquad (5)\end{aligned}$$

where $\kappa_2 = \frac{2^{\frac{3}{2}}(n-1)}{n} |\Omega|^{\frac{1}{n(n-1)}}$. On the other hand, due to the 2-monotonicity condition, we have

$$\mu \|D(y_1 - y_2)\|_2^2 \leq (\tau(Dy_1) - \tau(Dy_2), D(y_1 - y_2)). \qquad (6)$$

Combining (5) and (6), we deduce that

$$\left(\mu - \kappa_1\kappa_2 \frac{\|u\|_2}{\mu}\right) \|D(y_1 - y_2)\|_2^2 \leq 0$$

and thus $y_1 \equiv y_2$ if $\mu^2 > \kappa_1\kappa_2 \|u\|_2$. □

Proposition. (shear-thinning case) Assume that $\mathbf{A}_1$-$\mathbf{A}_2$ are fulfilled with $\frac{3n}{n+2} \leq \alpha < 2$. There exists a constant $\bar{\kappa} \equiv \bar{\kappa}(n, \Omega, \alpha)$ such that if $u \in L^2(\Omega)$ satisfies

$$\bar{\kappa} \left(1 + \frac{\|u\|_2}{\mu}\right)^{\frac{2(2-\alpha)}{\alpha-1}} \frac{\|u\|_2}{\mu^2} < 1,$$

then the state equation admits a unique weak solution $y_u \in V_\alpha$.

The proof is much more complicated and is omitted.

Existence of an optimal control

Theorem. Assume that $\mathbf{A}_1$-$\mathbf{A}_2$ are fulfilled with $\alpha \geq \frac{3n}{n+2}$. Then problem (P) admits at least a solution.

Proof. Assume that $\alpha \geq 2$. Let $(y_k, u_k)_k \subset V_\alpha \times U_{ad}$ be a minimizing sequence. We have

$$\|Dy_k\|_\alpha^\alpha \leq (\alpha - 1)\left(\kappa_1 \frac{\|u_k\|_2}{\mu}\right)^2$$

and by using the continuity property for τ

$$\|\tau(Dy_k)\|_{\alpha'}^{\alpha'} \leq c(\alpha, \gamma, n)\left(|\Omega| + \|Dy_k\|_\alpha^\alpha\right).$$

Since $(u_k)_k$ is uniformly bounded in U_{ad}, it follows that

$$(y_k)_k \;\text{ is uniformly bounded in } V_\alpha,$$

$$(\tau(Dy_k))_k \;\text{ is uniformly bounded in } L^{\alpha'}(\Omega).$$

Then there exists a subsequence, $u \in U_{ad}$, $y \in V_\alpha$ and $\widetilde{\tau} \in L^{\alpha'}(\Omega)$ such that

$$\begin{array}{ccll} (u_k)_k & \longrightarrow & u & \text{weakly in } L^2(\Omega), \\ (y_k)_k & \longrightarrow & y & \text{weakly in } V_\alpha, \\ (\tau(Dy_k))_k & \longrightarrow & \widetilde{\tau} & \text{weakly in } L^{\alpha'}(\Omega). \end{array}$$

Since $\alpha \geq 2 > \frac{2n}{n+1}$, by using compactness results on Sobolev spaces

$$(y_k)_k \longrightarrow y \qquad \text{strongly in } L^{\frac{2\alpha}{\alpha-1}}(\Omega)$$

and for all $\varphi \in V_\alpha$, we have

$$\begin{aligned} & |b(y_k, y_k, \varphi) - b(y, y, \varphi)| \\ & \leq \left(\|\nabla y_k\|_\alpha \|\varphi\|_{\frac{2\alpha}{\alpha-1}} + \|y\|_{\frac{2\alpha}{\alpha-1}} \|\nabla\varphi\|_\alpha \right) \|y_k - y\|_{\frac{2\alpha}{\alpha-1}} \\ & \longrightarrow 0 \qquad \text{when } k \to +\infty. \end{aligned}$$

Passing to the limit in the weak formulation corresponding to y_k, we obtain

$$(\widetilde{\tau}, D\varphi) + b\,(y, y, \varphi) = (u, \varphi) \qquad \text{for all } \varphi \in V_\alpha \tag{7}$$

and by using classical arguments for monotone operators, we can prove that

$$(\widetilde{\tau}, D\varphi) = (\tau\,(Dy)\,, D\varphi) \qquad \text{for all } \varphi \in V_\alpha. \tag{8}$$

Combining (7) and (8), we deduce that

$$(\tau\,(Dy)\,, D\varphi) + b\,(y, y, \varphi) = (u, \varphi) \qquad \text{for all } \varphi \in V_\alpha$$

and thus $y \equiv y_u$. From the convexity and continuity of J, it follows the lower semicontinuity of J in the weak topology and

$$J(y_u, u) \leq \liminf_k J(y_k, u_k) = \inf(P),$$

showing that (y_u, u) is a solution for (P).

Necessary optimality conditions

Setting

$$F(u) = J(y_u, u)$$

we can rewrite our problem in a reduced form

$$(P) \quad \begin{cases} \text{Minimize} & F(u) \\ \text{such that} & u \in U_{ad}. \end{cases}$$

For this class of problems, we have the following classical result.

Lemma. If $\bar{u}$ is a local solution of (P) and if F admits directional derivatives at $\bar{u}$, then

$$F'(\bar{u})(v - \bar{u}) \geq 0 \qquad \forall v \in U_{ad}$$

Problem. We need to guarantee that the mapping $u \longmapsto y_u$ admits directional derivatives at $\bar{u}$, defined in an adequate setting.

In practice. For $v \in U_{ad}$ and $\rho > 0$, let $u_\rho = \bar{u} + \rho(v - \bar{u}) \in U_{ad}$, y_ρ a corresponding state and consider

$$z_\rho = \frac{y_\rho - \bar{y}}{\rho}.$$

To derive the optimality conditions:

1. Estimate $(z_\rho)_\rho$ is an adequate setting. This task is directly related with the (local) Lipschitz continuity of the control-to-state mapping.

2. Prove the convergence of $(z_\rho)_\rho$ to some $z_{\bar{u}v}$, solution of an associated linearized equation.

3. Relate the linearized equation with the adjoint equation.

4. Derive the optimality conditions using the state, the adjoint state and the admissible controls.

Control problem governed by the Stokes equation

The main issues are generally related with the steps 1. and 2. However in order to illustrate steps 3. and 4., let us assume that the problem is governed by the Stokes system

$$-\mu\,\Delta y + \nabla\pi = u \qquad \text{in } \Omega.$$

Since the state equation is linear then

$$z_\rho = z_{\bar{u}v} = y_v - \bar{y}.$$

and steps 1. and 2. are obvious. The optimality condition for the control can then be written as

$$0 \le F'(\bar{u})(v-\bar{u}) = (y_v - \bar{y}, \bar{y} - y_d) + \lambda\,(\bar{u}, v-\bar{u})$$

for all $v \in U_{ad}$.

Let the $\bar{p} \in V_2$ be the solution of the adjoint equation

$$-\mu \Delta p + \nabla \pi = \bar{y} - y_d \qquad \text{in } \Omega.$$

Using the Green formula, we obtain

$$\begin{aligned}(y_v - \bar{y}, \bar{y} - y_d) &= (y_v - \bar{y}, -\mu \Delta \bar{p}) \\ &= -(\bar{p}, \Delta (y_v - \bar{y})) = (\bar{p}, v - \bar{u})\end{aligned}$$

implying the next result.

Theorem. Let $(\bar{u}, \bar{y}) \in U_{ad} \times V_2$ be a solution of (P_S). Then there exists a unique $\bar{p} \in V_2$ such that the following holds

Adjoint equation :

$$-\mu \Delta \bar{p} + \nabla \widetilde{\pi} = \bar{y} - y_d \qquad \text{in } \Omega,$$

Optimality condition for $\bar{u}$:

$$(\bar{p} + \lambda \bar{u}, v - \bar{u}) \geq 0 \qquad \text{for all } v \in U_{ad}.$$

Control problem governed by the Navier-Stokes equation

Classical arguments show that $z_\rho \in V_2$ is the solution of

$$-\mu\,\Delta z_\rho + (y_\rho \cdot \nabla)\,z_\rho + (z_\rho \cdot \nabla)\,\bar{y} + \nabla \pi_\rho = v - \bar{u}$$

and satisfies

$$\|Dz_\rho\|_2 \leq \underbrace{\frac{\kappa_1}{2\mu - \kappa_2\|D\bar{y}\|_2}}_{>0} \|v - \bar{u}\|_2\,.$$

If we assume that the linearized equation

$$-\mu\,\Delta z + (\bar{y} \cdot \nabla)\,z + (z \cdot \nabla)\,\bar{y} + \nabla \pi = v - \bar{u}$$

admits a solution $z_{\bar{u}v} \in V_2$, then,

$$\begin{aligned}\|D\,(z_\rho - z_{\bar{u}v})\|_2 \quad &\leq \left(\frac{\kappa_1}{2\mu - \kappa_2\|D\bar{y}\|_2}\right)^2 \|Dz_{\bar{u}v}\|_2\,\|v - \bar{u}\|_2\,\rho \\ &\longrightarrow 0 \quad \text{when } \rho \to 0.\end{aligned}$$

To study the solvability of the linearized equation, consider the associated bilinear form defined in $(V_2)^2$ by

$$B(z_1, z_2) = 2\mu\left(Dz_1, Dz_2\right) + b\left(z_1, \bar{y}, z_2\right) + b\left(\bar{y}, z_1, z_2\right).$$

For every $z \in V_2$, we have

$$B(z, z) = 2\mu\left\|Dz\right\|_2^2 + b\left(z, \bar{y}, z\right) \geq \left(2\mu - \kappa_2\left\|D\bar{y}\right\|_2\right)\left\|Dz\right\|_2^2$$

and thus B is coercive on V_2 if $\bar{y}$ satisfies

$$2\mu - \kappa_2\left\|D\bar{y}\right\|_2 > 0.$$

Similarly, for every $z_1, z_2 \in V_2$

$$\left|B(z_1, z_2)\right| \leq 2\left(\mu + \kappa_2\left\|D\bar{y}\right\|_2\right)\left\|Dz_1\right\|_2\left\|Dz_2\right\|_2$$

and B is continuous on V_2. Applying the Lax-Milgram theorem, we deduce that the linearized equation admits a unique solution in V_2.

Lemma. Assume that $\bar{y}$ satisfies the following condition

$$\|D\bar{y}\|_2 < \frac{2\mu}{\kappa_2} \tag{9}$$

then $u \longmapsto y_u$ is Gâteaux-differentiable at $\bar{u}$ and the derivative $z_{\bar{u}v}$ in the direction $v - \bar{u}$ is the unique solution in V_2 of the linearized equation

$$-\mu\,\Delta z + (z\cdot\nabla)\,\bar{y} + (\bar{y}\cdot\nabla)\,z + \nabla\pi = v - \bar{u}.$$

Remark. The estimate in V_2 for $\bar{y}$ implies that condition (9) holds if the optimal control satisfies the restriction

$$\|\bar{u}\|_2 < \frac{\sqrt{n^3}\,\mu^2}{(n-1)^2|\Omega|^{\frac{1}{n-1}}} \tag{10}$$

Condition (10) guarantees then existence and uniqueness of the corresponding state, linearized state (and adjoint state).

Statement of the necessary optimality conditions

Theorem. Let $(\bar{u}, \bar{y})$ be a solution of (P_{NS}) with $\bar{u}$ satisfying

$$\|\bar{u}\|_2 < \frac{\sqrt{n^3}\mu^2}{(n-1)^2|\Omega|^{\frac{1}{n-1}}}$$

Then there exists a unique $\bar{p} \in V_2$ such that the following holds

Adjoint equation :

$$-\mu\,\Delta\bar{p} - (\bar{y}\cdot\nabla)\,\bar{p} + (\nabla\bar{y})^\top\bar{p} + \nabla\widetilde{\pi} = \bar{y} - y_d \quad \text{in } \Omega,$$

Optimality condition for $\bar{u}$:

$$(\bar{p} + \lambda\bar{u}, v - \bar{u}) \geq 0 \qquad\qquad \text{for all } v \in U_{ad}.$$

Optimal control of shear-thickening fluids

Using the monotonicity conditions, we can prove that z_ρ is the solution of the linearized equation

$$-\nabla \cdot (\sigma_\rho : Dz_\rho) + (y_\rho \cdot \nabla) z_\rho + (z_\rho \cdot \nabla) \bar{y} + \nabla \pi_\rho = v - \bar{u}$$

with

$$\sigma_\rho = \int_0^1 \tau' (D\bar{y} + sD(y_\rho - \bar{y})) \, ds.$$

If $\|D\bar{y}\|_2 < \frac{\mu}{\kappa_2}$, then

$$\|Dz_\rho\|_2 \leq \frac{\kappa_1}{\mu - \kappa_2 \|D\bar{y}\|_2} \|v - \bar{u}\|_2$$

$$c_\alpha \|Dz_\rho\|_\alpha^\alpha \leq \left(\frac{\kappa_1}{\mu - \kappa_2 \|D\bar{y}\|_2} \|v - \bar{u}\|_2 \right)^2 \rho^{2-\alpha}.$$

Problem. The first estimate is not sufficient to pass to the limit unless σ_ρ can be "controlled" in L^∞, the second estimate is useless.

In order to find an adequate Hilbert setting, let us now carry out a quick analysis of the "informal" linearized equation

$$-\nabla \cdot (\tau'(D\bar{y}) : Dz) + (\bar{y} \cdot \nabla)\, z + (z \cdot \nabla)\, \bar{y} + \nabla \pi = v - \bar{u}.$$

The corresponding bilinear form can be written

$$B(z_1, z_2) = (\tau'(D\bar{y}) : Dz_1, Dz_2) + b(\bar{y}, z_1, z_2) + b(z_1, \bar{y}, z_2).$$

Using $\mathbf{A}_2$, for every $z \in V_2$ we have

$$(\tau'(D\bar{y}) : Dz, Dz) \geq \mu \left\| (1 + |D\bar{y}|)^{\frac{\alpha-2}{4}}\, Dz \right\|_2^2 \geq \mu \|Dz\|_2^2$$

and B may be coercive in V_2. However, B is not continuous in V_2 unless $D\bar{y} \in L^\infty(\Omega)$.

The estimates for $(z_\rho)_\rho$ and the analysis of the linearized equation suggest that if $D\bar{y}$ is bounded, then the optimality conditions may be obtained as in the Navier-Stokes case.

This fact has been already observed and used by Slawig (in the steady case) and Wachsmuth and Roubíček (in the unsteady case) to study similar problems. Neverthless, they had to restrict their studies to the case $n = 2$ and to a smaller class of tensor to exploit the existing regularity results.

Problem. What happens if no global regularity result is available?

The solution is related with the fact that $\mathbf{A}_1$-$\mathbf{A}_2$ imply

$$|(\tau'(D\bar{y}) : Dz_1, Dz_2)| \leq \gamma \, \|z_1\|_{\bar{y}} \, \|z_2\|_{\bar{y}}$$

$$(\tau'(D\bar{y}) : Dz, Dz) \geq \mu \, \|z\|_{\bar{y}}^2$$

where

$$\|z\|_{\bar{y}} = \left\| (1 + |D\bar{y}|)^{\frac{\alpha-2}{4}} \, Dz \right\|_2 .$$

Following Casas and Fernandez, we consider the weighted Sobolev space $H_\alpha^{\bar{y}}$ defined as the completion of $\mathcal{V}$ with respect to the norm $\|\cdot\|_{\bar{y}}$. It may be verified that $H_\alpha^{\bar{y}}$ is a Hilbert space and that, since $\alpha \geq 2$, we have

$$V_\alpha \subset H_\alpha^{\bar{y}} \subset V_2$$

with continuous injections.

In this new setting, we prove that if $\mu - \kappa_2 \|D\bar{y}\|_2 > 0$, then:

- The sequence $(z_\rho)_\rho$ is uniformly bounded in $H_\alpha^{\bar{y}}$ with

$$\|z_\rho\|_{\bar{y}} \leq \frac{\kappa_\alpha}{\mu - \kappa_2 \|D\bar{y}\|_2} \|v - \bar{u}\|_2$$

- The linearized equation

$$-\nabla \cdot (\tau'(D\bar{y}) : Dz) + (\bar{y} \cdot \nabla) z + (z \cdot \nabla) \bar{y} + \nabla \pi = v - \bar{u}$$

is well posed on $H_\alpha^{\bar{y}}$ and admits a unique solution $z_{\bar{u}v}$ in this space. (The same result holds for the associated adjoint equation.)

- By carrying out a very careful analysis, we can prove that $(z_\rho)_\rho$ converges to $z_{\bar{u}v}$ weakly in $H_\alpha^{\bar{y}}$ and strongly in V_2. This step is, technically, the most difficult part.

Statement of the necessary optimality conditions

Theorem. Assume that $\mathbf{A}_1$-$\mathbf{A}_2$ are fulfilled with $\alpha \geq 2$. Let $(\bar{u}, \bar{y})$ be a solution of (P) with $\bar{u}$ satisfying condition

$$\frac{\|u\|_2}{\mu^2} < \frac{\sqrt{n^3}}{4(n-1)^2|\Omega|^{\frac{1}{n-1}}}$$

Then there exists a unique $\bar{p} \in H_{\alpha}^{\bar{y}}$ such that the following holds

Adjoint equation :

$$-\nabla \cdot \left(\tau'(D\bar{y})^{\top} : D\bar{p}\right) - (\bar{y} \cdot \nabla)\,\bar{p} + (\nabla\bar{y})^{\top}\bar{p} + \nabla\widetilde{\pi} = \bar{y} - y_d,$$

Optimality condition for $\bar{u}$:

$$(\bar{p} + \lambda\bar{u}, v - \bar{u}) \geq 0 \qquad \text{for all } v \in U_{ad}.$$

Optimal control of shear-thinning fluids

The weighted Sobolev space setting used in the shear-thickening case cannot be applied. Indeed, we can only prove that

$$(z_\rho)_\rho \text{ is uniformly bounded in } V_\alpha$$

but since $\alpha < 2$, we have

$$H_\alpha^{\bar{y}} \subset V_\alpha$$

and no uniform bound in $H_\alpha^{\bar{y}}$ is guaranteed.

To overcome this difficulty, we consider a family of approximate problems governed by a regularized equation and falling into the case $\alpha = 2$.

Setting of the approximate problem

Let $(\bar{u}, \bar{y})$ be a fixed solution of (P) and introduce the cost functional

$$I(u, y) = J(u, y) + \frac{1}{2} \int_\Omega |u - \bar{u}|^2 \, dx.$$

For $\varepsilon > 0$, consider the following approximate control problem

$$(P^\varepsilon) \quad \begin{cases} \text{Minimize} \quad I(u, y^\varepsilon) \\ \text{such that} \quad (u, y^\varepsilon) \in U_{ad} \times V_2 \text{ satisfies} \\ -\varepsilon \, \Delta y - \nabla \cdot (\tau(Dy)) + (y \cdot \nabla)\, y + \nabla \pi = u \qquad \text{in } \; \Omega. \end{cases}$$

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Difficulties. Even though (P^{ε}) falls in the case $\alpha = 2$, several issues related with the combined effect of the nonlinear shear stress and the convective term have to be managed.

1. In the case of the generalized Stokes systems, the differentiability of the mapping $u \mapsto y_u^{\varepsilon}$ in V_2 can be established by using standard arguments. The approximate optimality conditions are derived and the optimality conditions for (P) are obtained by passing to the limit.

2. Unlike this case where no restriction on the admissible controls is needed and the case of Navier-Stokes equations and shear-thickening fluids where only the optimal control is restrained, obtaining uniform estimates for $(z_{\rho}^{\varepsilon})_{\rho}$ in V_2 without restraining all the admissible controls is not an easy issue.

By carrying out a careful analysis, we prove that uniform estimates for $(z_\rho^\varepsilon)_\rho$ can be established by imposing a restriction only on $\bar{u}^\varepsilon$, if

$$\rho < \frac{\mu}{U}\left(\frac{\varepsilon}{\mu}\right)^{\frac{\alpha-1}{2-\alpha}} \qquad \text{where } U = \sup_{u \in U_{ad}} \|u\|_2.$$

This allows the derivation of the optimality conditions for (P_ε)

Statement of the approximate necessary optimality conditions

Theorem. Assume that $\mathbf{A}_1$-$\mathbf{A}_2$ are fulfilled with $\frac{3n}{n+2} \leq \alpha < 2$. For each $\varepsilon > 0$, problem (P^{ε}) admits at least one solution $(\bar{u}^{\varepsilon}, \bar{y}^{\varepsilon})$. Moreover, if $\bar{u}^{\varepsilon}$ satisfies

$$\bar{\kappa}\left(1 + \frac{\|\bar{u}^{\varepsilon}\|_2}{\mu}\right)^{\frac{2(2-\alpha)}{\alpha-1}} \frac{\|\bar{u}^{\varepsilon}\|_2}{\mu^2} < 1$$

then there exists $\bar{p}^{\varepsilon} \in V_2$ such that

Adjoint equation :

$$-\varepsilon\Delta\bar{p}^{\varepsilon} - \nabla\cdot(\tau'(D\bar{y}^{\varepsilon}) : D\bar{p}^{\varepsilon}) + (\nabla\bar{y}^{\varepsilon})^T\bar{p}^{\varepsilon} - \bar{y}^{\varepsilon}\cdot\nabla\bar{p}^{\varepsilon} + \nabla\tilde{\pi}^{\varepsilon} = \bar{y}^{\varepsilon} - y_d$$

Optimality condition for $\bar{u}^{\varepsilon}$:

$$(\bar{p}^{\varepsilon} + (\lambda+1)\bar{u}^{\varepsilon} - \bar{u}, v - \bar{u}^{\varepsilon}) \geq 0 \qquad \text{for all } v \in U_{ad}.$$

Statement of the necessary optimality conditions

Theorem. Assume that $\mathbf{A}_1$-$\mathbf{A}_2$ are fulfilled with $\frac{3n}{n+2} \leq \alpha < 2$. Let $(\bar{u}, \bar{y})$ be a solution of (P) with $\bar{u}$ satisfying condition

$$\bar{\kappa} \left(1 + \frac{\|\bar{u}\|_2}{\mu}\right)^{\frac{2(2-\alpha)}{\alpha-1}} \frac{\|\bar{u}\|_2}{\mu^2} < 1.$$

Then there exists $\bar{p} \in V_\alpha$ such that the following holds

Adjoint equation :

$$-\nabla \cdot \left(\tau'(D\bar{y})^\top : D\bar{p}\right) - (\bar{y} \cdot \nabla)\,\bar{p} + (\nabla \bar{y})^\top \bar{p} + \nabla \widetilde{\pi} = \bar{y} - y_d,$$

Optimality condition for $\bar{u}$:

$$(\bar{p} + \lambda \bar{u}, v - \bar{u}) \geq 0 \qquad \text{for all } v \in U_{ad}.$$

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