Generalized stochastic flows and applications to incompressible viscous fluids

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Introduction: Euler equation
Weighted porous media equation and variational principle
Generalized flows
Existence of generalized flows with prescribed configuration
Existence of generalized flows with prescribed $L^q$ drift
Constructing generalized flows from solutions to finite variation transport equations

joint work with
Alexandra Antoniouk
Ana Bela Cruzeiro
1. Introduction: Euler equation

2. Weighted porous media equation and variational principle

3. Generalized flows
   - Examples

4. Existence of generalized flows with prescribed configuration

5. Existence of generalized flows with prescribed $L^q$ drift

6. Constructing generalized flows from solutions to finite variation transport equations
Euler equation on manifold $M$\begin{align*}
\frac{\partial u}{\partial t} &= -(u \cdot \nabla)u - \nabla p \\
\text{div} u &= 0
\end{align*}

Arnold (66), integral curves $t \mapsto g(t)(x)$ are geodesics on measure preserving diffeomorphisms and they minimize the energy functional
\[ S(g) = \frac{1}{2} \int_0^T \left( \int_M \left\| \frac{dg(t)(x)}{dt} \right\|^2 d\mu(x) \right) dt, \]
for prescribed final configuration.

Ebin-Marsden (70), compact manifold $M$: If $g(\cdot)$ is close to $\text{id}$, assuming some Sobolev regularity, then there exists a unique geodesic in diffeomorphism group from $\text{id}$ to $g(\cdot) = g(T)(\cdot)$. Not true in general

Brenier (89), generalized solutions: random variables $\omega \mapsto (t \mapsto \Theta_t(\omega))$ with values on paths $[0, T] \to M$, such that
- $E[f(\Theta_t)] = \int_M f(x) \, dx$: incompressibility;
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  $\eta(dx, dy) = dx \delta_h(x)(dy)$;
- kinetic energy $S(\Theta) = \frac{1}{2} E \left[ \int_0^T \| \Theta_t' \|^2 dt \right]$. Then $S(\Theta) \leq \lim \inf_{n \to \infty} S(\Theta^n)$ if $\Theta^n \to \Theta$.

With fixed $\eta$, if there exists $\Theta$ with finite $S(\Theta)$ then there exists at least one which minimizes $S(\Theta)$.
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weighted porous media equation, \( M = \mathbb{T}, G, G/H \), compact, \( q \geq 2 \),

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\begin{align*}
\text{(wpmeq)} \quad \frac{\partial u}{\partial t} &= (-u \cdot \nabla + \nu \Delta)(\|u\|^{q-2}u) - \nabla p \\
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g(t)(x) incompressible Brownian flow

\[
dg(t)(x) = \sigma(g(t)(x))dW_t + u(t, g(t)(x)) \, dt.
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q-energy functional

\[
\mathcal{E}_q(g) = \frac{1}{q} \mathbb{E} \left[ \int_0^T dt \int_M dx \|Dg(t, g(t)(x))(\omega)\|^q \right]
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with \( Dg := u \), drift of \( g \).

**Theorem, variational principle**

\( u \) solves (wpmeq) if and only if \( g \) is a critical point of \( \mathcal{E}_q \).
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For $\varphi, \psi \in L^2(M)$, define $\Theta_t(\varphi, \psi) = \Theta^g_t(\varphi, \psi) = \int_M \varphi(x)\psi(g(t)(x)(\omega)) \, dx$. If $\varphi, \psi \in C^\infty(M)$ then by Itô calculus

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So let $\tilde{\Theta}_t(\varphi, \psi) = \Theta_t(\varphi, \psi) - \frac{1}{2} \int_0^t \Theta_s(\varphi, \Delta \psi) \, ds$, define the drift

$D\tilde{\Theta}_t(\varphi, \psi) = \Theta_t(\varphi, \langle d\psi, u \rangle)$ and the $q$-energy for $q \geq 1$

$$\mathcal{E}_q'(\Theta) = \frac{1}{q} \sup \left\{ \mathbb{E} \left[ \int_0^T dt \sum_{j=1}^m \left( \sum_{k=1}^\ell \frac{D\tilde{\Theta}_t(\varphi_j, \psi_k)^2}{(\int_M \varphi_j)^{2(q-1)/q}} \right)^{q/2} \right] \right\},$$

where

$$m, \ell \geq 1, \sum_j \varphi_j = 1, \varphi_j \geq 0, \forall \nu \in TM, \sum_k \langle \nabla \psi_k, \nu \rangle^2 \leq \|\nu\|^2.$$
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$$E'_q(\Theta) = \frac{1}{q} \sup \left\{ \mathbb{E} \left[ \int_0^T dt \sum_{j=1}^m \left( \sum_{k=1}^\ell \frac{D\tilde{\Theta}_t(\varphi_j, \psi_k)^2}{2(q-1)} \right)^{q/2} \right] \right\},$$

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**Proposition**

$$E'_q(\Theta^g) = E_q(g), \quad q > 1$$
Consider a Brownian flow $g(t)(x)$. Denote by $\eta$ the law of $(g(0), g(T))$ on $M \times M$.

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Consider a Brownian flow \( g(t)(x) \). Denote by \( \eta \) the law of \( (g(0), g(T)) \) on \( M \times M \).

For \( \varphi, \psi \in L^2(M) \), define \( \Theta_t(\varphi, \psi) = \Theta^g_t(\varphi, \psi) = \int_M \varphi(x) \psi(g(t)(x)(\omega)) \, dx \). If \( \varphi, \psi \in C^\infty(M) \) then by Itô calculus

\[
\Theta_t(\varphi, \psi) = (\varphi, \psi)_{L^2(M)} + \int_0^t \Theta_s(\varphi, \text{div}(\psi \sigma)) \, dW_s + \int_0^t \Theta_s(\varphi, \langle d\psi, u \rangle) \, ds + \frac{1}{2} \int_0^t \Theta_s(\varphi, \Delta \psi) \, ds.
\]

So let \( \tilde{\Theta}_t(\varphi, \psi) = \Theta_t(\varphi, \psi) - \frac{1}{2} \int_0^t \Theta_s(\varphi, \Delta \psi) \, ds \), define the drift

\[
D\tilde{\Theta}_t(\varphi, \psi) = \Theta_t(\varphi, \langle d\psi, u \rangle)
\]

and the \( q \)-energy for \( q \geq 1 \)

\[
\mathcal{E}'_q(\Theta) = \frac{1}{q} \sup \left\{ \mathbb{E} \left[ \int_0^T dt \sum_{j=1}^m \left( \sum_{k=1}^\ell \frac{D\tilde{\Theta}_t(\varphi_j, \psi_k)^2}{(\int_M \varphi_j)^{2(q-1)}} \right)^{q/2} \right] \right\},
\]

\[
m, \ell \geq 1, \sum_j \varphi_j = 1, \varphi_j \geq 0, \forall \nu \in TM, \sum_k \langle \nabla \psi_k, \nu \rangle^2 \leq \|\nu\|^2 \}.
\]

Proposition

\[
\mathcal{E}'_q(\Theta^g) = \mathcal{E}_q(g), \quad q > 1
\]

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Consider a Brownian flow $g(t)(x)$. Denote by $\eta$ the law of $(g(0), g(T))$ on $M \times M$.

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- $E [\Theta_T(\varphi, \psi)] = \int_{M \times M} \varphi(x) \psi(y) \eta(dx, dy)$;
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A generalized flow with diffusion coefficient $\sigma$ and final configuration $\eta$ is a bilinear map $\Theta$ which to $\varphi, \psi \in L^2(M)$ associates a continuous semimartingale $\Theta_t(\varphi, \psi)$ satisfying all the properties above. Its kinetic $q$-energy is $E'_q(\Theta)$. The set of laws of generalized flows with finite kinetic energy is denoted by $\mathcal{H}'_q = \mathcal{H}'_q(\sigma, \eta, T)$. 

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Examples

- $\Theta^g$ with $g(t)(x)$ an incompressible Brownian flow:

$$dg(t)(x) = \sigma(g(t)(x))dW_t + u(t, g(t)(x)) dt;$$

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$$\begin{cases}
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\end{cases}$$

(Fang-Luo, Stoch. Analysis Appl., 2007)

- If $\Theta_1^i(\omega_1)$ and $\Theta_2^i(\omega_2)$ are two generalized flows defined on $\Omega_1$ and $\Omega_2$, then on $\Omega_1 \times \Omega_2 \times \{1, 2\}$ with product probability and any law on $\{1, 2\}$,

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Define
\[ \mathcal{E}_q'(\sigma, \eta, T) = \inf \{ \mathcal{E}_q'(\Theta), \ Law(\Theta) \in \mathcal{H}_q'(\sigma, \eta, T) \} \]
with convention \( \mathcal{E}_q'(\sigma, \eta, T) = \infty \) if the set is empty.

**Theorem**
If \( \mathcal{E}_q'(\sigma, \eta, T) < \infty \) then there exists a generalized flow \( \Theta \) with law belonging to \( \mathcal{H}_q'(\sigma, \eta, T) \), and such that
\[ \mathcal{E}_q'(\Theta) = \mathcal{E}_q'(\sigma, \eta, T). \]

**Sketch of proof**
- Let \( (\Theta^n)_{n \geq 1} \) satisfy \( \mathcal{E}_q'(\Theta^n) \to \mathcal{E}_q'(\sigma, \eta, T) \);
- for \( (\tilde{\phi}^j)_{j \geq 1} \) and \( (\tilde{\psi}^k)_{k \geq 1} \) smooth and dense for uniform convergence, there exists a subsequence \( (\Theta^{n\ell})_{\ell \geq 1} \) such that all \( \Theta^{n\ell}(\tilde{\phi}^j, \tilde{\psi}^k) \) converge together in law as \( \ell \to \infty \) (W.A. Zheng 85). Limit \( \Theta(\phi^j, \psi^k) \) extends to \( \Theta(\varphi, \psi) \).

\[ \mathcal{E}_q'(\Theta) \leq \liminf_{\ell \to \infty} \mathcal{E}_q'(\Theta^{n\ell}) \quad \text{(P.A. Meyer, W.A. Zheng 85)} \]

**Proposition**
The set of laws of generalized flows minimizing \( \mathcal{E}_q' \) is convex.

**Sketch of proof:** use construction \( \Theta_t(\omega_1, \omega_2, i) = \Theta^i_t(\omega_j) \).
Define

\[ \mathcal{E}'_q(\sigma, \eta, T) = \inf \{ \mathcal{E}'_q(\Theta), \ \text{Law}(\Theta) \in \mathcal{H}'_q(\sigma, \eta, T) \} \]

with convention \( \mathcal{E}'_q(\sigma, \eta, T) = \infty \) if the set is empty.

**Theorem**

If \( \mathcal{E}'_q(\sigma, \eta, T) < \infty \) then there exists a generalized flow \( \Theta \) with law belonging to \( \mathcal{H}'_q(\sigma, \eta, T) \), and such that

\[ \mathcal{E}'_q(\Theta) = \mathcal{E}'_q(\sigma, \eta, T). \]

**Sketch of proof**

- Let \( (\Theta^n)_{n \geq 1} \) satisfy \( \mathcal{E}'_q(\Theta^n) \rightarrow \mathcal{E}'_q(\sigma, \eta, T) \);
- for \( (\tilde{\varphi}^j)_{j \geq 1} \) and \( (\tilde{\psi}^k)_{k \geq 1} \) smooth and dense for uniform convergence, there exists a subsequence \( (\Theta^{n_\ell})_{\ell \geq 1} \) such that all \( \Theta^{n_\ell}(\tilde{\varphi}^j, \tilde{\psi}^k) \) converge together in law as \( \ell \rightarrow \infty \) (W.A. Zheng 85). Limit \( \Theta(\tilde{\varphi}^j, \tilde{\psi}^k) \) extends to \( \Theta(\varphi, \psi) \).

\[ \mathcal{E}'_q(\Theta) \leq \lim \inf_{\ell \rightarrow \infty} \mathcal{E}'_q(\Theta^{n_\ell}) \]  
(P.A. Meyer, W.A. Zheng 85)

**Proposition**

The set of laws of generalized flows minimizing \( \mathcal{E}'_q \) is convex.

**Sketch of proof:** use construction \( \Theta_t(\omega_1, \omega_2, i) = \Theta^i_t(\omega_j) \).
Define
\[ \mathcal{E}'_q(\sigma, \eta, T) = \inf \{ \mathcal{E}'_q(\Theta), \text{ Law}(\Theta) \in \mathcal{H}'_q(\sigma, \eta, T) \} \]
with convention \( \mathcal{E}'_q(\sigma, \eta, T) = \infty \) if the set is empty.

**Theorem**

If \( \mathcal{E}'_q(\sigma, \eta, T) < \infty \) then there exists a generalized flow \( \Theta \) with law belonging to \( \mathcal{H}'_q(\sigma, \eta, T) \), and such that
\[ \mathcal{E}'_q(\Theta) = \mathcal{E}'_q(\sigma, \eta, T). \]

**Sketch of proof**

- Let \( (\Theta^n)_{n \geq 1} \) satisfy \( \mathcal{E}'_q(\Theta^n) \rightarrow \mathcal{E}'_q(\sigma, \eta, T) \);
- for \( (\tilde{\varphi}_j)_{j \geq 1} \) and \( (\tilde{\psi}_k)_{k \geq 1} \) smooth and dense for uniform convergence, there exists a subsequence \( (\Theta^{n_\ell})_{\ell \geq 1} \) such that all \( \Theta^{n_\ell}(\tilde{\varphi}_j, \tilde{\psi}_k) \) converge together in law as \( \ell \rightarrow \infty \) (W.A. Zheng 85). Limit \( \Theta(\tilde{\varphi}_j, \tilde{\psi}_k) \) extends to \( \Theta(\varphi, \psi) \).

\[ \mathcal{E}'_q(\Theta) \leq \lim \inf_{\ell \rightarrow \infty} \mathcal{E}'_q(\Theta^{n_\ell}) \quad (P.A. \ Meyer, \ W.A. \ Zheng \ 85) \]

**Proposition**

The set of laws of generalized flows minimizing \( \mathcal{E}'_q \) is convex.

**Sketch of proof:** use construction \( \Theta_t(\omega_1, \omega_2, i) = \Theta^i_t(\omega_j) \).
Define
\[ E'_q(\sigma, \eta, T) = \inf \{ E'_q(\Theta), \ Law(\Theta) \in \mathcal{H}'_q(\sigma, \eta, T) \} \]
with convention \( E'_q(\sigma, \eta, T) = \infty \) if the set is empty

**Theorem**
If \( E'_q(\sigma, \eta, T) < \infty \) then there exists a generalized flow \( \Theta \) with law belonging to \( \mathcal{H}'_q(\sigma, \eta, T) \), and such that
\[ E'_q(\Theta) = E'_q(\sigma, \eta, T). \]

**Sketch of proof**
- Let \( (\Theta^n)_{n \geq 1} \) satisfy \( E'_q(\Theta^n) \rightarrow E'_q(\sigma, \eta, T) \);
- for \( (\tilde{\varphi}^j)_{j \geq 1} \) and \( (\tilde{\psi}^k)_{k \geq 1} \) smooth and dense for uniform convergence, there exists a subsequence \( (\Theta^{n\ell})_{\ell \geq 1} \) such that all \( \Theta^{n\ell}(\tilde{\varphi}^j, \tilde{\psi}^k) \) converge together in law as \( \ell \rightarrow \infty \) (W.A. Zheng 85). Limit \( \Theta(\tilde{\varphi}^j, \tilde{\psi}^k) \) extends to \( \Theta(\varphi, \psi) \).

\[ E'_q(\Theta) \leq \lim inf_{\ell \rightarrow \infty} E'_q(\Theta^{n\ell}) \quad (\text{P.A. Meyer, W.A. Zheng 85}) \]

**Proposition**
The set of laws of generalized flows minimizing \( E'_q \) is convex.

**Sketch of proof:** use construction \( \Theta_t(\omega_1, \omega_2, i) = \Theta^i_t(\omega_i) \).
Define

\[ E'_q(\sigma, \eta, T) = \inf \{ E'_q(\Theta), \ \text{Law}(\Theta) \in \mathcal{H}'_q(\sigma, \eta, T) \} \]

with convention \( E'_q(\sigma, \eta, T) = \infty \) if the set is empty.

**Theorem**

If \( E'_q(\sigma, \eta, T) < \infty \) then there exists a generalized flow \( \Theta \) with law belonging to \( \mathcal{H}'_q(\sigma, \eta, T) \), and such that

\[ E'_q(\Theta) = E'_q(\sigma, \eta, T). \]

**Sketch of proof**

- Let \( (\Theta^n)_{n \geq 1} \) satisfy \( E'_q(\Theta^n) \rightarrow E'_q(\sigma, \eta, T) \);
- for \( (\varphi^j)_{j \geq 1} \) and \( (\tilde{\psi}^k)_{k \geq 1} \) smooth and dense for uniform convergence, there exists a subsequence \( \Theta^{n_\ell} \) such that all \( \Theta^{n_\ell}(\varphi^j, \tilde{\psi}^k) \) converge together in law as \( \ell \rightarrow \infty \) (W.A. Zheng 85). Limit \( \Theta(\varphi, \psi) \) extends to \( \Theta(\varphi, \psi) \).

**Proposition**

The set of laws of generalized flows minimizing \( E'_q \) is convex.

**Sketch of proof:** use construction \( \Theta_t(\omega_1, \omega_2, i) = \Theta^i_t(\omega_j) \).
Define
\[ \mathcal{E}_q'(\sigma, \eta, T) = \inf \{ \mathcal{E}_q'(\Theta), \ \text{Law}(\Theta) \in \mathcal{H}_q'(\sigma, \eta, T) \} \]
with convention \( \mathcal{E}_q'(\sigma, \eta, T) = \infty \) if the set is empty

**Theorem**

If \( \mathcal{E}_q'(\sigma, \eta, T) < \infty \) then there exists a generalized flow \( \Theta \) with law belonging to \( \mathcal{H}_q'(\sigma, \eta, T) \), and such that
\[ \mathcal{E}_q'(\Theta) = \mathcal{E}_q'(\sigma, \eta, T). \]

**Sketch of proof**

- Let \( (\Theta^n)_{n \geq 1} \) satisfy \( \mathcal{E}_q'(\Theta^n) \to \mathcal{E}_q'(\sigma, \eta, T) \);
- for \( (\tilde{\varphi}^j)_{j \geq 1} \) and \( (\tilde{\psi}^k)_{k \geq 1} \) smooth and dense for uniform convergence, there exists a subsequence \( (\Theta^{n\ell})_{\ell \geq 1} \) such that all \( \Theta^{n\ell}(\tilde{\varphi}^j, \tilde{\psi}^k) \) converge together in law as \( \ell \to \infty \) (W.A. Zheng 85). Limit \( \Theta(\tilde{\varphi}^j, \tilde{\psi}^k) \) extends to \( \Theta(\varphi, \psi) \).

\[ \mathcal{E}_q'(\Theta) \leq \liminf_{\ell \to \infty} \mathcal{E}_q'(\Theta^{n\ell}) \] (P.A. Meyer, W.A. Zheng 85)

**Proposition**

The set of laws of generalized flows minimizing \( \mathcal{E}_q' \) is convex.

**Sketch of proof:** use construction \( \Theta_t(\omega_1, \omega_2, i) = \Theta^i_t(\omega_j) \).
Define
\[ E'_q(\sigma, \eta, T) = \inf \{ E'_q(\Theta), \text{Law}(\Theta) \in \mathcal{H}'_q(\sigma, \eta, T) \} \]
with convention \( E'_q(\sigma, \eta, T) = \infty \) if the set is empty.

**Theorem**

If \( E'_q(\sigma, \eta, T) < \infty \) then there exists a generalized flow \( \Theta \) with law belonging to \( \mathcal{H}'_q(\sigma, \eta, T) \), and such that
\[ E'_q(\Theta) = E'_q(\sigma, \eta, T). \]

**Sketch of proof**

- Let \( (\Theta^n)_{n \geq 1} \) satisfy \( E'_q(\Theta^n) \rightarrow E'_q(\sigma, \eta, T) \);
- for \( (\tilde{\varphi}^j)_{j \geq 1} \) and \( (\tilde{\psi}^k)_{k \geq 1} \) smooth and dense for uniform convergence, there exists a subsequence \( (\Theta^{n\ell})_{\ell \geq 1} \) such that all \( \Theta^{n\ell}(\tilde{\varphi}^j, \tilde{\psi}^k) \) converge together in law as \( \ell \rightarrow \infty \) (W.A. Zheng 85). Limit \( \Theta(\tilde{\varphi}^j, \tilde{\psi}^k) \) extends to \( \Theta(\varphi, \psi) \).
\[ E'_q(\Theta) \leq \liminf_{\ell \rightarrow \infty} E'_q(\Theta^{n\ell}) \] (P.A. Meyer, W.A. Zheng 85)

**Proposition**

The set of laws of generalized flows minimizing \( E'_q \) is convex.

**Sketch of proof:** use construction \( \Theta_t(\omega_1, \omega_2, i) = \Theta^i_t(\omega_i) \).
Define

$$\mathcal{E}_q'(\sigma, \eta, T) = \inf \{ \mathcal{E}_q'(\Theta), \text{Law}(\Theta) \in \mathcal{H}_q'(\sigma, \eta, T) \}$$

with convention $\mathcal{E}_q'(\sigma, \eta, T) = \infty$ if the set is empty.

**Theorem**

If $\mathcal{E}_q'(\sigma, \eta, T) < \infty$ then there exists a generalized flow $\Theta$ with law belonging to $\mathcal{H}_q'(\sigma, \eta, T)$, and such that

$$\mathcal{E}_q'(\Theta) = \mathcal{E}_q'(\sigma, \eta, T).$$

**Sketch of proof**

- Let $(\Theta^n)_{n \geq 1}$ satisfy $\mathcal{E}_q'(\Theta^n) \to \mathcal{E}_q'(\sigma, \eta, T)$;
- for $(\tilde{\varphi}^j)_{j \geq 1}$ and $(\tilde{\psi}^k)_{k \geq 1}$ smooth and dense for uniform convergence, there exists a subsequence $(\Theta^{n_\ell})_{\ell \geq 1}$ such that all $\Theta^{n_\ell}(\tilde{\varphi}^j, \tilde{\psi}^k)$ converge together in law as $\ell \to \infty$ (W.A. Zheng 85). Limit $\Theta(\tilde{\varphi}^j, \tilde{\psi}^k)$ extends to $\Theta(\varphi, \psi)$.

$$\mathcal{E}_q'(\Theta) \leq \liminf_{\ell \to \infty} \mathcal{E}_q'(\Theta^{n_\ell}) \quad \text{(P.A. Meyer, W.A. Zheng 85)}$$

**Proposition**

The set of laws of generalized flows minimizing $\mathcal{E}_q'$ is convex.

**Sketch of proof**

use construction $\Theta_t(\omega_1, \omega_2, i) = \Theta^i_t(\omega_j)$.
Define

\[ \mathcal{E}'_q(\sigma, \eta, T) = \inf \{ \mathcal{E}'_q(\Theta), \ Law(\Theta) \in \mathcal{H}'_q(\sigma, \eta, T) \} \]

with convention \( \mathcal{E}'_q(\sigma, \eta, T) = \infty \) if the set is empty

**Theorem**

If \( \mathcal{E}'_q(\sigma, \eta, T) < \infty \) then there exists a generalized flow \( \Theta \) with law belonging to \( \mathcal{H}'_q(\sigma, \eta, T) \), and such that

\[ \mathcal{E}'_q(\Theta) = \mathcal{E}'_q(\sigma, \eta, T). \]

**Sketch of proof**

- Let \( (\Theta^n)_{n \geq 1} \) satisfy \( \mathcal{E}'_q(\Theta^n) \to \mathcal{E}'_q(\sigma, \eta, T) \);
- for \( (\tilde{\varphi}^j)_{j \geq 1} \) and \( (\tilde{\psi}^k)_{k \geq 1} \) smooth and dense for uniform convergence, there exists a subsequence \( (\Theta^{n\ell})_{\ell \geq 1} \) such that all \( \Theta^{n\ell}(\tilde{\varphi}^j, \tilde{\psi}^k) \) converge together in law as \( \ell \to \infty \) (W.A. Zheng 85). Limit \( \Theta(\varphi, \psi) \) extends to \( \Theta(\varphi, \psi) \).

\[ \mathcal{E}'_q(\Theta) \leq \lim \inf_{\ell \to \infty} \mathcal{E}'_q(\Theta^{n\ell}) \quad (P.A. Meyer, W.A. Zheng 85) \]

**Proposition**

The set of laws of generalized flows minimizing \( \mathcal{E}'_q \) is convex.

**Sketch of proof:** use construction \( \Theta_t(\omega_1, \omega_2, i) = \Theta^j_i(\omega_i) \).
Let $u(t, x) \in L^1([0, T], L^q(\Gamma(TM))) = L^q([0, T] \times M, TM)$ a divergence-free drift, $q > 1$.

**Theorem**

There exists a generalized flow $\Theta$ such that

$$D\tilde{\Theta}_t(\varphi, \psi) = \Theta_t(\varphi, \langle d\psi, u(t, \cdot) \rangle)$$

and

$$\mathcal{E}_q'(\Theta) \leq \frac{1}{q} \|u\|_q^q.$$

**Sketch of proof**

- Let $u^n$ smooth $\to u$ in $L^q$, such that $\text{div} u^n = 0$;
- Construct $g^n$ associated flow;
- As before $\Theta g^{n, \ell} \to \Theta$ as $\ell \to \infty$;
- $\mathcal{E}_q'(\Theta) \leq \liminf_{\ell \to \infty} \mathcal{E}_q'(\Theta g^{n, \ell})$;
- $\mathcal{E}_q'(\Theta g^{n, \ell}) = \mathcal{E}_q(g^{n, \ell}) = \frac{1}{q} \|u^{n, \ell}\|_q^q$. 
Let \( u(t, x) \in L^1([0, T], L^q(\Gamma(TM))) = L^q([0, T] \times M, TM) \) a divergence-free drift, \( q > 1 \).

**Theorem**

There exists a generalized flow \( \Theta \) such that

\[
D\tilde{\Theta}_t(\varphi, \psi) = \Theta_t(\varphi, \langle d\psi, u(t, \cdot) \rangle)
\]

and

\[
\mathcal{E}_q'(\Theta) \leq \frac{1}{q} ||u||^q.
\]

**Sketch of proof**

- Let \( u^n \) smooth \( \to u \) in \( L^q \), such that \( \text{div} u^n = 0 \);
- Construct \( g^n \) associated flow;
- As before \( \Theta g^n \to \Theta \) as \( \ell \to \infty \);
- \( \mathcal{E}_q'(\Theta) \leq \liminf_{\ell \to \infty} \mathcal{E}_q'(\Theta g^n) \);
- \( \mathcal{E}_q'(\Theta g^n) = \mathcal{E}_q(g^n) = \frac{1}{q} ||u^n||^q \).
Let \( u(t, x) \in L^1([0, T], L^q(\Gamma(TM))) = L^q([0, T] \times M, TM) \) a divergence-free drift, \( q > 1 \).

**Theorem**

There exists a generalized flow \( \Theta \) such that

\[
D\tilde{\Theta}_t(\varphi, \psi) = \Theta_t(\varphi, \langle d\psi, u(t, \cdot) \rangle)
\]

and

\[
\mathcal{E}'_q(\Theta) \leq \frac{1}{q} \|u\|_q^q.
\]

**Sketch of proof**

- Let \( u^n \) smooth \( \to u \) in \( L^q \), such that \( \text{div} u^n = 0 \);
- Construct \( g^n \) associated flow;
- As before \( \Theta g^{n\ell} \to \Theta \) as \( \ell \to \infty \);
- \( \mathcal{E}'_q(\Theta) \leq \liminf_{\ell \to \infty} \mathcal{E}'_q(\Theta g^{n\ell}) \);
- \( \mathcal{E}'_q(\Theta g^{n\ell}) = \mathcal{E}_q(g^{n\ell}) = \frac{1}{q} \|u^{n\ell}\|_q^q \).
Let \( u(t, x) \in L^1([0, T], L^q(\Gamma(TM))) = L^q([0, T] \times M, TM) \) a divergence-free drift, \( q > 1 \).

**Theorem**

There exists a generalized flow \( \Theta \) such that

\[
D\tilde{\Theta}_t(\varphi, \psi) = \Theta_t(\varphi, \langle d\psi, u(t, \cdot) \rangle)
\]

and

\[
E'_q(\Theta) \leq \frac{1}{q} \|u\|^q_q.
\]

**Sketch of proof**

- Let \( u^n \) smooth \( \to u \) in \( L^q \), such that \( \text{div} u^n = 0 \);
- Construct \( g^n \) associated flow;
- As before \( \Theta g^{n\ell} \to \Theta \) as \( \ell \to \infty \);
- \( E'_q(\Theta) \leq \lim \inf_{\ell \to \infty} E'_q(\Theta g^{n\ell}) \);
- \( E'_q(\Theta g^{n\ell}) = E_q(g^{n\ell}) = \frac{1}{q} \|u^{n\ell}\|^q_q \).
Let \( u(t, x) \in L^1([0, T], L^q(\Gamma(TM))) = L^q([0, T] \times M, TM) \) a divergence-free drift, \( q > 1 \).

**Theorem**

There exists a generalized flow \( \Theta \) such that

\[
D\tilde{\Theta}_t(\varphi, \psi) = \Theta_t(\varphi, \langle d\psi, u(t, \cdot) \rangle)
\]

and

\[
\mathcal{E}'_q(\Theta) \leq \frac{1}{q} \|u\|_q^q.
\]

**Sketch of proof**

- Let \( u^n \) smooth \( \to u \) in \( L^q \), such that \( \text{div} u^n = 0 \);
- Construct \( g^n \) associated flow;
  - As before \( \Theta^{g^n} \to \Theta \) as \( \ell \to \infty \);
  - \( \mathcal{E}'(\Theta) \leq \lim \inf_{\ell \to \infty} \mathcal{E}'(\Theta^{g^n}) \);
  - \( \mathcal{E}'(\Theta^{g^n}) = \mathcal{E}_q(\Theta^{g^n}) = \frac{1}{q} \|u^n\|_q^q. \)
Let \( u(t, x) \in L^1([0, T], L^q(\Gamma(TM))) = L^q([0, T] \times M, TM) \) a divergence-free drift, \( q > 1 \).

**Theorem**

There exists a generalized flow \( \Theta \) such that

\[
D\bar{\Theta}_t(\varphi, \psi) = \Theta_t(\varphi, \langle d\psi, u(t, \cdot) \rangle)
\]

and

\[
\mathcal{E}'_q(\Theta) \leq \frac{1}{q} \|u\|_q^q.
\]

**Sketch of proof**

- Let \( u^n \) smooth \( \rightarrow u \) in \( L^q \), such that \( \text{div} u^n = 0 \);
- Construct \( g^n \) associated flow;
- As before \( \Theta g^{n\ell} \rightarrow \Theta \) as \( \ell \rightarrow \infty \);
- \( \mathcal{E}'_q(\Theta) \leq \lim \inf_{\ell \rightarrow \infty} \mathcal{E}'_q(\Theta g^{n\ell}) \)
- \( \mathcal{E}'_q(\Theta g^{n\ell}) = \mathcal{E}_q(g^{n\ell}) = \frac{1}{q} \|u^{n\ell}\|_q^q \).
Let $u(t, x) \in L^1([0, T], L^q(\Gamma(TM))) = L^q([0, T] \times M, TM)$ a divergence-free drift, $q > 1$.

**Theorem**

There exists a generalized flow $\Theta$ such that

$$D\tilde{\Theta}_t(\varphi, \psi) = \Theta_t(\varphi, \langle d\psi, u(t, \cdot) \rangle)$$

and

$$\mathcal{E}'_q(\Theta) \leq \frac{1}{q} \|u\|_q^q.$$

**Sketch of proof**

- Let $u^n$ smooth $\rightarrow u$ in $L^q$, such that $\text{div} u^n = 0$;
- Construct $g^n$ associated flow;
- As before $\Theta g^{n_\ell} \rightarrow \Theta$ as $\ell \rightarrow \infty$;
- $\mathcal{E}'_q(\Theta) \leq \liminf_{\ell \rightarrow \infty} \mathcal{E}'_q(\Theta g^{n_\ell})$

$$\mathcal{E}'_q(\Theta g^{n_\ell}) = \mathcal{E}_q(g^{n_\ell}) = \frac{1}{q} \|u^{n_\ell}\|_q^q.$$
Let $u(t, x) \in L^1([0, T], L^q(P_1(TM))) = L^q([0, T] \times M, TM)$ a divergence-free drift, $q > 1$.

**Theorem**

There exists a generalized flow $\Theta$ such that

$$D\tilde{\Theta}_t(\varphi, \psi) = \Theta_t(\varphi, \langle d\psi, u(t, \cdot) \rangle)$$

and

$$\mathcal{E}'_q(\Theta) \leq \frac{1}{q} \|u\|_q^q.$$

**Sketch of proof**

- Let $u^n$ smooth $\rightarrow u$ in $L^q$, such that $\text{div} u^n = 0$;
- Construct $g^n$ associated flow;
- As before $\Theta g^{n\ell} \rightarrow \Theta$ as $\ell \rightarrow \infty$;
- $\mathcal{E}'_q(\Theta) \leq \liminf_{\ell \rightarrow \infty} \mathcal{E}'_q(\Theta g^{n\ell})$
- $\mathcal{E}'_q(\Theta g^{n\ell}) = \mathcal{E}_q(g^{n\ell}) = \frac{1}{q} \|u^n\|_q^q$. 
If (1) \( dg(t) = \sigma(g(t)) \circ dW_t + u(t, g(t)) \, dt \), \( g(0)(x) = x \)
consider \( d\tilde{g}(t) = \sigma(\tilde{g}(t)) \circ dW_t \), \( \tilde{g}(0)(x) = x \)
Then \( g(t)(x) = \tilde{g}(t)(\beta(t)(x)) \) with
\[ d\beta(t)(x) = \tilde{u}(t, \beta(t)(x)) \, dt, \quad \tilde{u}(t, y, \omega) = (T_y \tilde{g}(t)(\cdot))^{-1} u(t, \tilde{g}(t)(y)) \]
(Ocone-Pardoux 89).
Get \( \Theta^g_t(\varphi, \psi) = \Theta^\tilde{g}_t(\theta^\beta, \varphi, \psi) \) where \( \theta^\beta, \varphi \) solves transport equation
\[ (2) \quad \frac{\partial \theta^\beta, \varphi}{\partial t} = - (\tilde{u} \cdot \nabla) \theta^\beta, \varphi, \theta^0, \varphi = \varphi. \]
Conversely, to solve (1), first solve (2) (DiPerna Lions 89)) then let
\( \Theta^\sigma_{\sigma, u}(\varphi, \psi) = \Theta^\tilde{g}_t(\theta^\beta, \varphi, \psi). \)

**Proposition**

We have
\[
D\Theta^\sigma_{\sigma, u}(\varphi, \psi) = \Theta^\sigma_{\sigma, u}(\varphi, \langle d\psi, u(t, \cdot) \rangle) \quad \text{and} \quad \mathcal{E}'(\Theta^\sigma_{\sigma, u}) \leq \frac{1}{q} \| u \|_q^q.
\]
If (1) \( dg(t) = \sigma(g(t)) \circ dW_t + u(t, g(t)) \, dt, \quad g(0)(x) = x \)
consider \( d\tilde{g}(t) = \sigma(\tilde{g}(t)) \circ dW_t, \quad \tilde{g}(0)(x) = x \)
Then \( g(t)(x) = \tilde{g}(t)(\beta(t)(x)) \) with
\[
d\beta(t)(x) = \ddot{u}(t, \beta(t)(x)) \, dt, \quad \ddot{u}(t, y, \omega) = (T_y\tilde{g}(t)(\cdot))^{-1} \, u(t, \tilde{g}(t)(y)) \quad \text{(Ocone-Pardoux 89)}. \]
Get \( \Theta^g_t(\varphi, \psi) = \Theta^\tilde{g}_{t} \left( \theta^\beta, \varphi, \psi \right) \) where \( \theta^\beta, \varphi \) solves transport equation
\[
(2) \quad \frac{\partial \theta^\beta, \varphi}{\partial t} = - (\ddot{u} \cdot \nabla) \theta^\beta, \varphi, \quad \theta^\beta, \varphi_0 = \varphi. \]
Conversely, to solve (1), first solve (2) (DiPerna Lions 89)) then let
\( \Theta^{\sigma, u}_t(\varphi, \psi) = \Theta^\tilde{g}_{t} \left( \theta^\beta, \varphi, \psi \right). \)

**Proposition**
We have
\[
D\tilde{\Theta}^{\sigma, u}_t(\varphi, \psi) = \Theta^{\sigma, u}_t(\varphi, \langle d\psi, u(t, \cdot) \rangle) \quad \text{and} \quad \mathcal{E}'(\Theta^{\sigma, u}) \leq \frac{1}{q} \| u \|^q. \]
If (1) \[ dg(t) = \sigma(g(t)) \circ dW_t + u(t, g(t)) \, dt, \quad g(0)(x) = x \]
consider \[ d\tilde{g}(t) = \sigma(\tilde{g}(t)) \circ dW_t, \quad \tilde{g}(0)(x) = x \]
Then \[ g(t)(x) = \tilde{g}(t)(\beta(t)(x)) \]
\[ d\beta(t)(x) = \tilde{u}(t, \beta(t)(x)) \, dt, \quad \tilde{u}(t, y, \omega) = (T_y \tilde{g}(t)(\cdot))^{-1} u(t, \tilde{g}(t)(y)) \] (Ocone-Pardoux 89).

Get \[ \Theta^g_t(\varphi, \psi) = \Theta^{\tilde{g}}_t(\theta^\beta, \varphi, \psi) \]
where \( \theta^\beta, \varphi \) solves transport equation
\[ \frac{\partial \theta^\beta, \varphi}{\partial t} = - (\tilde{u} \cdot \nabla) \theta^\beta, \varphi, \theta^\beta, \varphi_0 = \varphi. \]

Conversely, to solve (1), first solve (2) (DiPerna Lions 89)) then let \[ \Theta^{\sigma, u}_t(\varphi, \psi) = \Theta^{\tilde{g}}_t(\theta^\beta, \varphi, \psi). \]

**Proposition**

We have
\[ D\Theta^{\sigma, u}_t(\varphi, \psi) = \Theta^{\sigma, u}_t(\varphi, \langle d\psi, u(t, \cdot) \rangle) \quad \text{and} \quad \mathcal{E}'(\Theta^{\sigma, u}) \leq \frac{1}{q} \| u \|^q_q. \]
If \( \text{dg}(t) = \sigma(g(t)) \circ dW_t + u(t, g(t)) \, dt, \quad g(0)(x) = x \)

consider \( \text{d}\tilde{g}(t) = \sigma(\tilde{g}(t)) \circ dW_t, \quad \tilde{g}(0)(x) = x \)

Then \( g(t)(x) = \tilde{g}(t)(\beta(t)(x)) \) with

\[
d\beta(t)(x) = \tilde{u}(t, \beta(t)(x)) \, dt, \quad \tilde{u}(t, y, \omega) = (T_y \tilde{g}(t)(\cdot))^{-1} u(t, \tilde{g}(t)(y)) \text{ (Ocone-Pardoux 89).}
\]

Get \( \Theta^g_t(\varphi, \psi) = \tilde{\Theta}^\tilde{g}_t(\theta^\beta, \varphi, \psi) \) where \( \theta^\beta, \varphi \) solves transport equation

\[
\frac{\partial \theta^\beta, \varphi}{\partial t} = -\left(\tilde{u} \cdot \nabla\right) \theta^\beta, \varphi, \quad \theta^\beta, \varphi_0 = \varphi.
\]

Conversely, to solve (1), first solve (2) (DiPerna Lions 89)) then let

\( \Theta^\sigma, u_t(\varphi, \psi) = \tilde{\Theta}^\tilde{g}_t(\theta^\beta, \varphi, \psi) \).

**Proposition**

We have

\[
D\Theta^\sigma, u_t(\varphi, \psi) = \Theta^\sigma, u_t(\varphi, \langle d\psi, u(t, \cdot) \rangle) \quad \text{and} \quad E'(\Theta^\sigma, u) \leq \frac{1}{q} \| u \|^q.
\]
If (1) \( dg(t) = \sigma(g(t)) \circ dW_t + u(t, g(t)) \, dt, \quad g(0)(x) = x \)
consider \( d\tilde{g}(t) = \sigma(\tilde{g}(t)) \circ dW_t, \quad \tilde{g}(0)(x) = x \)
Then \( g(t)(x) = \tilde{g}(t)(\beta(t)(x)) \) with
\[
d\beta(t)(x) = \tilde{u}(t, \beta(t)(x)) \, dt, \quad \tilde{u}(t, y, \omega) = (T_y \tilde{g}(t)(\cdot))^{-1} u(t, \tilde{g}(t)(y)) \) (Ocone-Pardoux 89).

Get \( \Theta_t^g(\phi, \psi) = \Theta_t^\tilde{g} \left( \theta^\beta, \phi, \psi \right) \) where \( \theta^\beta, \phi \) solves transport equation
\[
(2) \quad \frac{\partial \theta^\beta, \phi}{\partial t} = - (\tilde{u} \cdot \nabla) \theta^\beta, \phi, \quad \theta^\beta, \phi_0 = \phi.
\]
Conversely, to solve (1), first solve (2) (DiPerna Lions 89)) then let
\( \Theta_t^{\sigma, u}(\phi, \psi) = \Theta_t^\tilde{g} \left( \theta^\beta, \phi, \psi \right) \).

**Proposition**

We have
\[
D\Theta_t^{\sigma, u}(\phi, \psi) = \Theta_t^{\sigma, u}(\phi, \langle d\psi, u(t, \cdot) \rangle) \quad \text{and} \quad \mathcal{E}'(\Theta_t^{\sigma, u}) \leq \frac{1}{q} \|u\|_q^q.
\]
If (1) \( \frac{d}{dt} g(t) = \sigma(g(t)) \circ dW_t + u(t, g(t)) \, dt, \quad g(0)(x) = x \)
consider \( \frac{d}{dt} \tilde{g}(t) = \sigma(\tilde{g}(t)) \circ dW_t, \quad \tilde{g}(0)(x) = x \)
Then \( g(t)(x) = \tilde{g}(t)(\beta(t)(x)) \) with
\[
d\beta(t)(x) = \tilde{u}(t, \beta(t)(x)) \, dt, \quad \tilde{u}(t, y, \omega) = \left( T_y \tilde{g}(t)(\cdot) \right)^{-1} u(t, \tilde{g}(t)(y)) \text{ (Ocone-Pardoux 89).}
\]
Get \( \Theta^g_t(\varphi, \psi) = \Theta^\tilde{g}_t \left( \theta^\beta, \varphi, \psi \right) \) where \( \theta^\beta, \varphi \) solves transport equation
\[
\frac{\partial \theta^\beta, \varphi}{\partial t} = - (\tilde{u} \cdot \nabla) \theta^\beta, \varphi, \quad \theta^\beta, \varphi_0 = \varphi.
\]
Conversely, to solve (1), first solve (2) (DiPerna Lions 89)) then let
\( \Theta^\sigma, u_T(\varphi, \psi) = \Theta^\tilde{g}_t \left( \theta^\beta, \varphi, \psi \right) \).

**Proposition**

We have
\[
D\Theta^\sigma, u_T(\varphi, \psi) = \Theta^\sigma, u_T(\varphi, \langle d\psi, u(t, \cdot) \rangle) \quad \text{and} \quad \mathcal{E}'(\Theta^\sigma, u) \leq \frac{1}{q} \| u \|_q^q.
\]
If (1) \( dg(t) = \sigma(g(t)) \circ dW_t + u(t, g(t)) \, dt \), \( g(0)(x) = x \)
consider \( d\tilde{g}(t) = \sigma(\tilde{g}(t)) \circ dW_t \), \( \tilde{g}(0)(x) = x \)
Then \( g(t)(x) = \tilde{g}(t)(\beta(t)(x)) \) with
\[
d\beta(t)(x) = \tilde{u}(t, \beta(t)(x)) \, dt, \quad \tilde{u}(t, y, \omega) = (T_y \tilde{g}(t)(\cdot))^{-1} u(t, \tilde{g}(t)(y)) \quad \text{(Ocone-Pardoux 89)}.
\]
Get \( \Theta^g_t(\varphi, \psi) = \Theta^\tilde{g}_t(\theta^\beta, \varphi, \psi) \) where \( \theta^\beta, \varphi \) solves transport equation
\[
(2) \quad \frac{\partial \theta^\beta, \varphi}{\partial t} = -(\tilde{u} \cdot \nabla) \theta^\beta, \varphi, \quad \theta^\beta, \varphi_0 = \varphi.
\]
Conversely, to solve (1), first solve (2) (DiPerna Lions 89)) then let
\( \Theta^{\sigma, u}_t(\varphi, \psi) = \Theta^\tilde{g}_t(\theta^\beta, \varphi, \psi) \).

**Proposition**

We have
\[
D\Theta^{\sigma, u}_t(\varphi, \psi) = \Theta^{\sigma, u}_t(\varphi, \langle d\psi, u(t, \cdot) \rangle) \quad \text{and} \quad \mathcal{E}'(\Theta^{\sigma, u}_t) \leq \frac{1}{q} \|u\|^q.
\]