Generalized stochastic flows and applications to incompressible viscous fluids

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joint work with

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Euler equation on manifold M $\begin{cases} \frac{\partial u}{\partial t} &= -(u\cdot\nabla)u - \nabla p \\ \mathrm{div} u &= 0 \end{cases}$

Arnold (66), integral curves $t \mapsto g(t)(x)$ are geodesics on measure preserving diffeomorphisms and they minimize the energy functional

$$S(g) = \frac{1}{2}\int_0^T \left(\int_M \left\| \frac{dg(t)(x)}{dt} \right\|^2 d\mu(x) \right) dt,$$

for prescribed final configuration.

Ebin-Marsden (70), compact manifold M: If $g(\cdot)$ is close to id, assuming some Sobolev regularity, then there exists a unique geodesic in diffeomorphism group from id to $g(\cdot) = g(T)(\cdot)$. Not true in general

Brenier (89), generalized solutions: random variables $\omega \mapsto (t \mapsto \Theta_t(\omega))$ with values on paths $[0,T] \to M$, such that

- $\mathbb{E}[f(\Theta_t)] = \int_M f(x)\, dx$: incompressibility;
- $\mathbb{E}[f(\Theta_0,\Theta_T)] = \int_M f(x,y)\,\eta(dx,dy)$: prescribed final configuration $\eta(dx,dy) = dx\delta_{h(x)}(dy)$;
- kinetic energy $S(\Theta) = \frac{1}{2}\mathbb{E}\left[\int_0^T \|\Theta_t'\|^2\, dt\right]$.

Then $S(\Theta) \le \liminf_{n\to\infty} S(\Theta^n)$ if $\Theta^n \to \Theta$.

With fixed η, if there exists Θ with finite $S(\Theta)$ then there exists at least one which minimizes $S(\Theta)$.

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$g(t)(x)$ incompressible Brownian flow

$$dg(t)(x) = \sigma(g(t)(x))dW_t + u(t, g(t)(x))\,dt.$$

q-energy functional

$$\mathcal{E}_q(g) = \frac{1}{q}\mathbb{E}\left[\int_0^T dt \int_M dx\, \|Dg(t, g(t)(x))(\omega))\|^q\right]$$

with $Dg := u$, drift of g.

Theorem, variational principle

u solves (wpmeq) if and only if g is a critical point of $\mathcal{E}_q$.

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So let $\tilde\Theta_t(\varphi, \psi) = \Theta_t(\varphi, \psi) - \frac{1}{2}\int_0^t \Theta_s(\varphi, \Delta\psi)\, ds$, define the drift $D\tilde\Theta_t(\varphi, \psi) = \Theta_t(\varphi, \langle d\psi, u\rangle)$ and the q-energy for $q \geq 1$

$$\mathcal{E}_q'(\Theta) = \frac{1}{q}\sup\Big\{\mathbb{E}\left[\int_0^T dt \sum_{j=1}^m \left(\sum_{k=1}^\ell \frac{D\tilde\Theta_t(\varphi_j, \psi_k)^2}{\left(\int_M \varphi_j\right)^{\frac{2(q-1)}{q}}}\right)^{q/2}\right],$$

$$m, \ell \geq 1,\ \sum_j \varphi_j = 1,\ \varphi_j \geq 0,\ \forall v \in TM,\ \sum_k \langle \nabla\psi_k, v\rangle^2 \leq \|v\|^2\Big\}.$$

Proposition

$$\mathcal{E}_q'(\Theta^g) = \mathcal{E}_q(g), \quad q > 1$$

Properties of Θ

- $\mathbb{E}\left[\Theta_T(\varphi,\psi)\right] = \int_{M\times M} \varphi(x)\psi(y)\,\eta(dx,dy);$
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Definition

A generalized flow with diffusion coefficient σ and final configuration η is a bilinear map Θ which to $\varphi,\psi \in L^2(M)$ associates a continuous seminmartingale $\Theta_t(\varphi,\psi)$ satisfying all the properties above. Its kinetic q-energy is $\mathcal{E}'_q(\Theta)$. The set of laws of generalized flows with finite kinetic energy is denoted by $\mathcal{H}'_q = \mathcal{H}'_q(\sigma,\eta,T)$.

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- Θ^g with $g(t)(x)$ an incompressible Brownian flow:

$$dg(t)(x) = \sigma(g(t)(x))dW_t + u(t, g(t)(x))\, dt;$$

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(Fang-Luo, Stoch. Analysis Appl., 2007)

- If $\Theta_t^1(\omega_1)$ and $\Theta_t^2(\omega_2)$ are two generalized flows defined on Ω_1 and Ω_2, then on $\Omega_1 \times \Omega_2 \times \{1, 2\}$ with product probability and any law on $\{1, 2\}$,

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with convention $\mathcal{E}'_q(\sigma,\eta,T)=\infty$ if the set is empty

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If $\mathcal{E}'_q(\sigma,\eta,T)<\infty$ then there exists a generalized flow Θ with law belonging to $\mathcal{H}'_q(\sigma,\eta,T)$, and such that

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Sketch of proof: use construction $\Theta_t(\omega_1,\omega_2,i) = \Theta^i_t(\omega_i)$.

Define

$$\mathcal{E}'_q(\sigma,\eta,T) = \inf\left\{\mathcal{E}'_q(\Theta),\ \mathrm{Law}(\Theta) \in \mathcal{H}'_q(\sigma,\eta,T)\right\}$$

with convention $\mathcal{E}'_q(\sigma,\eta,T) = \infty$ if the set is empty

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If $\mathcal{E}'_q(\sigma,\eta,T) < \infty$ then there exists a generalized flow Θ with law belonging to $\mathcal{H}'_q(\sigma,\eta,T)$, and such that

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(2) $\dfrac{\partial\theta^{\beta,\varphi}}{\partial t} = -(\tilde{u} \cdot \nabla)\theta^{\beta,\varphi}$, $\theta_0^{\beta,\varphi} = \varphi$.
Conversely, to solve (1), first solve (2) (DiPerna Lions 89)) then let
$\Theta_t^{\sigma,u}(\varphi, \psi) = \Theta_t^{\tilde{g}}\left(\theta^{\beta,\varphi}, \psi\right)$.

Proposition

We have

$$D\tilde{\Theta}_t^{\sigma,u}(\varphi, \psi) = \Theta_t^{\sigma,u}(\varphi, \langle d\psi, u(t,\cdot)\rangle) \qquad \text{and} \qquad \mathcal{E}'(\Theta^{\sigma,u}) \leq \frac{1}{q}\|u\|_q^q.$$