Linear and nonlinear Fokker–Planck–Kolmogorov equations for measures on infinite-dimensional spaces

Vladimir Bogachev,
Moscow State University, Russia

Bogachev V.I., Da Prato G., Röckner M., Shaposhnikov S.V.
"An analytic approach to infinite-dimensional continuity and Fokker–Planck–Kolmogorov equations," Annali Scuola Norm. Super.
"Nonlinear evolution equations for measures on infinite dimensional spaces," in: Stochastic Partial Differential Equations and Applications, Quaderni di Matematica, V. 25, Napoli, 2010, pp. 51–64.

Bogachev V., Krylov N., Röckner M., Shaposhnikov S. V.
Fokker–Planck–Kolmogorov equations.
Moscow, 2013.

EQUATIONS ON $\mathbb{R}^d$:

$$L\varphi = a^{ij}\partial_{x_i}\partial_{x_j}\varphi + b^i\partial_{x_i}\varphi,\ \Delta\varphi + (b, \nabla\varphi)$$

STATIONARY EQUATION

$$L^*\mu = 0$$

$$\partial_{x_i}\partial_{x_j}(a^{ij}\mu) - \partial_{x_i}(b^i\mu) = 0.$$

$$\Delta\mu - \operatorname{div}(b\mu) = 0$$

PARABOLIC EQUATION

$$(\partial_t + L)^* \mu = 0$$

$$\partial_t \mu = \partial_{x_i} \partial_{x_j} (a^{ij} \mu) - \partial_{x_i} (b^i \mu) = 0.$$

$$\int (\partial_t \varphi + L\varphi) \, d\mu = 0.$$

$$\mu(dxdt) = \mu_t(dx) \; dt$$

INITIAL DATA: $\mu_0 = \nu$:

$$\int \psi d\mu_t \to \int \psi d\nu, \quad t \to 0.$$

$$\int \psi \, d\mu_t - \int \psi \, d\nu = \int L\psi \, d\mu_t \, dt$$

$b \in C^\infty$: UNIQUENESS OF PROBABILITY SOLUTION?

$d = 1$: **YES** for stationary; **OPEN** for parabolic

$d = 2$: **NO** for stationary; **OPEN** for parabolic

$d > 2$: **NO** for stationary; **NO** for parabolic

OPEN: $d \geq 1$: parabolic with Dirac's initial ν

Lyapunov functions help:

$V \in C^2$, $\lim\limits_{|x|\to\infty} V(x) = +\infty$,

$LV \le C \cdot V$: uniqueness

$LV \le -1$ outside a ball: existence

NONLINEARITY ENTERS:

A and b become dependent on the solution μ

E.g. $\Delta\mu + \operatorname{div}(b(\mu)\mu) = 0$

E.g. $b(\mu)(x) = \displaystyle\int \beta(x, y)\, \mu(dy)$

FIX POINT: given σ, solve $L^*_\sigma \mu = 0$, obtain

$$\sigma \mapsto \mu_\sigma$$

and find σ with $\mu_\sigma = \sigma$

RECENT PAPERS:

O. Manita, S. Shaposhnikov

Continuity equation for probability measures on a separable Hilbert space X with an orthonormal basis $\{e_i\}$:

$$\partial_t\mu_t + \sum_{i=1}^{\infty} \partial_{e_i}(b^i(\mu,\ \cdot\ ,\ \cdot\ )\mu_t) = 0,$$

$$\mu_0 = \nu, \qquad (1)$$

$b^i\colon\ \mathcal{P}(X\times[0,1])\times X\times[0,1] \to \mathbb{R}^1,\ i \in \mathbb{N}$, are Borel functions, $\mathcal{P}(X \times [0,1])$ is the space of probability measures on $X \times [0,1]$ with the weak topology

A family $\mu := (\mu_t)_{t\in[0,1]}$ of Borel probability measures on X (regarded also as a measure on $X \times [0,1]$) satisfies (1) if

$$b^i(\mu, \,\cdot\,, \,\cdot\,) \in L^1(\mu)$$

and, for all $t \in [0,1]$, one has

$$\int_X \varphi \, d\mu_t - \int_X \varphi \, d\nu =$$

$$= \int_0^t \int_X \sum_{i=1}^m b^i(\mu, x, s) \partial_{e_i} \varphi(x) \, \mu_s(dx) \, ds$$

for every function φ of the form

$$\varphi(x) = \varphi_0(x_1, \ldots, x_m), \ x_i = (x, e_i),$$

$$\varphi_0 \in C_b^\infty(\mathbb{R}^m), \ m \in \mathbb{N}.$$

In $\mathbb{R}^d$:

$\partial_t \mu_t + \mathrm{div}_x(b(\mu,\,\cdot\,,\,\cdot\,)\mu_t) = 0, \quad \mu_0 = \nu,$

where

$b = (b^i)_{i=1}^d \colon\ \mathcal{P}(\mathbb{R}^d \times [0,1]) \times \mathbb{R}^d \times [0,1] \to \mathbb{R}^d$

is a Borel mapping. A family $\mu := (\mu_t)_{t\in[0,1]}$ of Borel probability measures on $\mathbb{R}^d$ satisfies our equation if

$b^i(\mu,\,\cdot\,,\,\cdot\,) \in L^1(S \times [0,1], \mu_t(dx)\,dt)$

for every compact set $S \subset \mathbb{R}^d$, that is, the function $(x,t) \mapsto |b(\mu,x,t)|$ is integrable with respect to $|\mu|$ on every compact set in $\mathbb{R}^d \times [0,1]$, and for all $t \in [0,1]$

$$\int_{\mathbb{R}^d} \varphi \, d\mu_t - \int_{\mathbb{R}^d} \varphi \, d\nu =$$

$$= \int_0^t \int_{\mathbb{R}^d} (b(\mu, \,\cdot\,, \,\cdot\,), \nabla\varphi) \, d\mu_s \, ds$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d)$.

Theorem 1. *Let ν be a probability measure on $\mathbb{R}^d$. Suppose that*

(A1) *for every fixed $\mu \in \mathcal{P}(\mathbb{R}^d \times [0,1])$, the mapping $x \mapsto b(\mu, x, t)$ is continuous for a.e. t and one has uniform convergence $b(\mu_j, \cdot, \cdot) \to b(\mu, \cdot, \cdot)$ on compact sets whenever $\mu_j \to \mu$ weakly;*

(B1) *there are $c > 0$ and $\kappa \geq 2$ such that for all $(x,t) \in \mathbb{R}^d \times [0,1]$ and all $\mu \in \mathcal{P}(\mathbb{R}^d \times [0,1])$ one has*

$$(b(\mu, x, t), x) \leq c(1 + |x|^2),$$

$$|b(\mu, x, t)| \leq c(1+|x|^\kappa), \quad \int_{\mathbb{R}^d} |x|^\kappa \, \nu(dx) < \infty.$$

Then there is a family $\mu = (\mu_t)_{t\in[0,1]}$ *of probability measures satisfying* (1) *and*

$$\sup_{t\in[0,1]} \int_{\mathbb{R}^d} |x|^\kappa \, \mu_t(dx) \leq M < \infty,$$

where M *depends only on* c *and the moment of* ν *of order* κ*.*

Under (B1), condition (A1) can be reformulated as follows: for every fixed measure μ, the mapping $x \mapsto b(\mu, x, t)$ is continuous for a.e. t and for each compact set $S \subset \mathbb{R}^d$, the mapping b generates a continuous mapping F from the space $\mathcal{P}(\mathbb{R}^d \times [0,1])$ to $L^\infty([0,1], C(S))$ defined by $F(\mu)(t)(x) := b(\mu, x, t)$.

The method of "vanishing viscosity": the parabolic equation

$$\partial_t \mu_t - \varepsilon \Delta \mu_t + \mathrm{div}_x (b(\mu, \,\cdot\,, \,\cdot\,)\mu_t) = 0, \qquad \mu_0 = \nu. \tag{2}$$

Under the stated assumptions this equation has a unique solution for any given mapping b independent of μ. Hence we obtain the corresponding drift $b(\mu, \cdot, \cdot)$ generating a solution to the linear equation with that $b(\mu, \cdot, \cdot)$. By using this and Schauder's fixed point theorem we prove that (2) is solvable. Finally, letting $\varepsilon \to 0$, we obtain a solution with $\varepsilon = 0$.

The infinite-dimensional case. Let $\{e_i\}$ be an orthonormal basis in X. Set

$$P_n x := \sum_{i=1}^{n} x_i e_i, \quad X_n := P_n(X).$$

We identify X_n with $\mathbb{R}^n$.

Two conditions on b:

(A2) for each fixed measure $\mu \in \mathcal{P}(X \times [0,1])$ and each fixed i, the functions

$$x \mapsto b^i(\mu, x, t)$$

are weakly continuous on balls for a.e. t and one has uniform convergence

$$b^i(\mu_j, x, t) \to b^i(\mu, x, t)$$

on bounded sets in $\mathbb{R}^n \times [0,1]$ whenever $\mu_j \to \mu$ weakly with respect to the weak topology on X;

(B2) there are $\alpha > 0$, $c_i > 0$, $\kappa \geq 2$ such that for all $(x, t) \in X \times [0, 1]$ and $\mu \in \mathcal{P}(X \times [0, 1])$ one has

$$\sum_{i=1}^{n} b^i(\mu, x, t)x_i \leq \alpha(1+|x|^2) \quad \forall\, x \in X_n,$$

$$|b^i(\mu, x, t)| \leq c_i(1 + |x|^\kappa).$$

The weak continuity on balls means that $b^i(\mu, x^j, t) \to b^i(\mu, x, t)$ if $x^j \to x$ weakly.

We DO NOT assume that $b = (b^i)$ corresponds to a vector field in X: it is merely a collection of scalar functions b^i.

Theorem 2. *Let ν be a Borel probability measure on X such that for some $p > \kappa$ one has*

$$\int_X |x|^p \, \nu(dx) < \infty.$$

Let the collection $b = (b^i)$ have properties (A2) *and* (B2) *above. Then there exists a family $\mu = (\mu_t)_{t\in[0,1]}$ of probability measures satisfying* (1) *and*

$$\sup_{t\in[0,1]} \int_X |x|^p \, \mu_t(dx) < \infty.$$

A possible disadvantage of our hypotheses can be the requirement of weak continuity of the functions b^i on balls (excludes functions depending on the norm). Is this hypothesis really needed?

This hypothesis is naturally connected with the assumption that b^i is weakly continuous in μ with respect to the weak topology, which again is stronger than the continuity associated with the norm topology.

We now relax these two assumptions to probably more natural continuities associated with the norm topology at the expense of certain stronger dissipativity of the drift.

Let us consider the Borel function

$$V(x) = \sum_{n=1}^{\infty} \lambda_i x_i^2, \quad \text{where } \lambda_i > 0 \text{ and } \lambda_i \to +\infty,$$

defined on the compactly embedded weighted Hilbert space X_V of sequences $x = (x_i)$ with finite norm $V^{1/2}$.

Modify our previous assumptions (A2) and (B2) as follows.

(A3) for every fixed $\mu \in \mathcal{P}(X \times [0,1])$ and every fixed i, the functions

$$x \mapsto b^i(\mu, x, t)$$

are defined and continuous on the compacts sets $\{V \leq R\}$ with respect to the norm on X for a.e. t and one has uniform convergence

$$b^i(\mu_j, x, t) \to b(\mu, x, t)$$

on the sets $\{V \leq R\} \times [0,1]$ whenever $\mu_j \to \mu$ weakly with respect to the norm topology on X;

(B3) there are $\alpha > 0$, $c_i > 0$, $\kappa \geq 1$ such that for all $(x,t) \in X_V \times [0,1]$ and $\mu \in \mathcal{P}(X \times [0,1])$ one has

$$\sum_{i=1}^{n} \lambda_i b^i(\mu, x, t) x_i \leq \alpha(1+V(x)) \quad \forall\, x \in X_n,$$

$$|b^i(\mu, x, t)| \leq c_i(1 + V(x)^\kappa).$$

Theorem 3. *Let ν be a Borel probability measure on X such that for some $p > \kappa$ one has*

$$\int_X V(x)^p \, \nu(dx) < \infty.$$

Then, under assumptions (A3) *and* (B3), *there exists a family $\mu = (\mu_t)_{t\in[0,1]}$ of probability measures satisfying* (1) *such that $\mu_t(X_V) = 1$ for all t and*

$$\sup_{t\in[0,1]} \int_X V(x)^p \, \mu_t(dx) < \infty.$$

Modification of (B3):

(B4) There are $\alpha > 0$, $c_i > 0$, $\kappa \geq 1$ such that for all $(x,t) \in X_V \times [0,1]$ and $\mu \in \mathcal{P}(X \times [0,1])$ one has

$$\sum_{i=1}^{n} b^i(\mu, x, t) x_i \leq \alpha - \alpha V(x) \quad \forall\, x \in X_n,$$

$$|b^i(\mu, x, t)| \leq c_i(1 + V(x)\|x\|^\kappa).$$

Theorem 4. *Let ν be a Borel probability measure on X such that for some $p > \kappa$ one has*

$$\int_X V(x)\|x\|^{2p}\,\nu(dx) < \infty.$$

Then, under assumptions (A3) *and* (B4)*, there exists a family $\mu = (\mu_t)_{t\in[0,1]}$ of probability measures satisfying* (1)*. Moreover, $\mu_t(X_V) = 1$ for all t,*

$$\sup_{t\in[0,1]} \int_X V(x)\|x\|^{2p}\,\mu_t(dx) < \infty.$$

Example 1. Let U be a bounded domain in $\mathbb{R}^2$ with regular boundary, let Δ be the Laplacian with zero boundary condition having an eigenbasis $\{e_i\}$ with the corresponding eigenvalues $\{\lambda_i\}$. Let $X = L^2(U)$ and let

$$V(x) = \int_U |\nabla x(u)|^2 \, du.$$

Then X_V is the Sobolev class $W_0^{2,1}(U)$.

Finally, let b be given by a heuristic expression

$$b(\mu, x, t) = \Delta x + \alpha_3(\mu, x, t)x^3 + \alpha_2(\mu, x, t)x^2 + \\ + \alpha_1(\mu, x, t)x + \alpha_0(\mu, x, t),$$

where the functions $\alpha_3, \ldots, \alpha_0$ are Borel measurable, uniformly bounded, continuous in x on balls in $W_0^{2,1}(U)$ with respect to the L^2-norm, and satisfy the estimate

$$\alpha_3(\mu, x, t) \leq -M, \quad M > 0 \text{ is a constant.}$$

Suppose also that if $\mu_j \to \mu$ weakly, then $\alpha_k(\mu_j, x, t) \to \alpha_k(\mu, x, t)$ uniformly in $t \in [0, 1]$ and x from every fixed ball in $W_0^{2,1}(U)$, $0 \leq k \leq 3$.

The corresponding functions b^i are

$$b^i(\mu, x, t) := \lambda_i \int_U x(u) e_i(u)\, du +$$

$$+ \int_U [\alpha_3(\mu, x, t) x^3(u) + \cdots + \alpha_0(\mu, x, t)] e_i(u)\, du.$$

Then there exists a family $\mu = (\mu_t)_{t \in [0,1]}$ of probability measures satisfying (1).

LINEAR EQUATIONS

$$L\varphi = \sum_{i,j} a^{ij}\partial_{x_i}\partial_{x_j}\varphi + \sum_i b^i \partial_{x_i}\varphi,$$

$$\partial_t \mu = L^* \mu, \quad \mu|_{t=0} = \nu.$$

The stochastic equation of Navier–Stokes type is considered in the space V_2 of $\mathbb{R}^d$-valued mappings $u = (u^1, \ldots, u^d)$ such that $u^j \in W_0^{2,1}(D)$ and $\operatorname{div} u = 0$, where $D \subset \mathbb{R}^d$ is a bounded domain with smooth boundary. Norm on V_2:

$$\|u\|_{V_2}^2 := \sum_{j=1}^{d} \|\nabla_z u^j\|_2^2.$$

Let H be the closure of V_2 in $L^2(D, \mathbb{R}^d)$ and let P_H denote the orthogonal projection on H in $L^2(D, \mathbb{R}^d)$.

The stochastic Navier–Stokes equation

$$du(z,t) = \sqrt{2}dW(z,t) + \\ + P_H\Big[\Delta_z u(z,t) - \\ - \sum_{j=1}^{d} u^j(z,t)\partial_{z_j} u(z,t) + F(z,u(z,t),t)\Big]dt,$$

where W is a Wiener process of the form $W(z,t) = \sum_{n=1}^{\infty} \sqrt{\alpha_n} w_n(t)\eta_n(z)$, where

$$\alpha_n \geq 0, \quad \sum_{n=1}^{\infty} \alpha_n < \infty,$$

w_n are independent Wiener processes, and $\{\eta_n\}$ is an orthonormal basis in H, and

$$F\colon\ D \times \mathbb{R}^d \times (0,T_0) \to \mathbb{R}^d$$

is a bounded continuous mapping.

The case $F = 0$ is the classical stochastic Navier–Stokes equation. Since Δ is not defined on all of V_2, this equation requires some interpretation.

There is an orthonormal basis $\{\eta_n\}$ in H formed by eigenfunctions of Δ with eigenvalues $-\lambda_n < 0$ such that $\eta_n \in V_2$. Employing the fact that $\langle P_H w, \eta_n \rangle_2 = \langle w, \eta_n \rangle_2$ for any $w \in L^2(D, \mathbb{R}^d)$, we introduce the "coordinate" functions

$$b^n(u,t) = \langle u, \Delta\eta_n \rangle_2 - \sum_{j=1}^{d} \langle P_H(u^j \partial_{z_j} u), \eta_n \rangle_2 + $$

$$+ \langle P_H F(\,\cdot\,, u(\,\cdot\,, t), t), \eta_n \rangle_2 =$$

$$= \langle u, \Delta\eta_n \rangle_2 - \sum_{j=1}^{d} \langle \partial_{z_j} u, u^j \eta_n \rangle_2 +$$

$$+ \langle F(\,\cdot\,, u(\,\cdot\,, t), t), \eta_n \rangle_2.$$

These functions are defined by the last line on all of V_2. They are continuous on balls in V_2 with respect to the topology of $L^2(D, \mathbb{R}^d)$ by the compactness of the embedding of $W^{2,1}(D) \to L^2(D)$. We arrive at the operator

$$L\varphi(u,t) = \sum_{n=1}^{\infty} \alpha_n \partial_{\eta_n}^2 \varphi(u,t) +$$

$$+ \sum_{n=1}^{\infty} b^n(u,t) \partial_{\eta_n} \varphi(u,t).$$

$d = 2$ existence and uniqueness

$d \geq 3$ existence