The motion of the rigid body in viscous fluid including collisions

Nikolai Chemetov
Joint work with S. Necasova

CMAF-Universidade de Lisboa

Equations of Motion

$\mathbf{q} = \mathbf{q}(t) \in \mathbb{R}^N$-the body mass center at a time $t \in [0, T]$;

$\mathbb{Q} = \mathbb{Q}(t)$ - a rotation matrix: $\mathbb{Q}(t)\mathbb{Q}(t)^T = \mathbb{I}$, $\mathbb{Q}(0) = \mathbb{I}$ with $\mathbb{I}$ being the identity matrix.

The trajectories of all points of the body are described by a isometry

$$A(t, \mathbf{y}) = \mathbf{q}(t) + \mathbb{Q}(t)(\mathbf{y} - \mathbf{q}(0)) \qquad \text{for any} \quad \mathbf{y} \in S_0 \tag{1}$$

and the body occupies the set

$$S(t) = \{\mathbf{x} \in \mathbb{R}^N : \mathbf{x} = A(t, \mathbf{y}), \quad \mathbf{y} \in S_0\} = A(t, S_0). \tag{2}$$

The velocity of the body is

$$\mathbf{u}_s = \mathbf{a}(t) + \mathbb{P}(t)(\mathbf{x} - \mathbf{q}(t)) \qquad \text{for all} \quad \mathbf{x} \in S(t), \qquad (3)$$

$\mathbf{a} = \mathbf{a}(t) \in \mathbb{R}^N$-the translation velocity;
$\mathbb{P} = \mathbb{P}(t)$ -the angular velocity:

$$\frac{d\mathbf{q}}{dt} = \mathbf{a}, \qquad \frac{d\mathbb{Q}}{dt}\mathbb{Q}^T = \mathbb{P} \quad \text{in } [0, T].$$

$\mathbb{P}$ is a skew-symmetric matrix, i.e. there exists $\boldsymbol{\omega} = \boldsymbol{\omega}(t) \in \mathbb{R}^N$:

$$\mathbb{P}(t)\mathbf{x} = \boldsymbol{\omega}(t) \times \mathbf{x}, \qquad \forall \mathbf{x} \in \mathbb{R}^N.$$

The motion of the body for $\mathbf{x} \in S(t)$:

$$m\frac{d\mathbf{a}}{dt} = -\int_{\partial S(t)} P_f \mathbf{n}\, d\mathbf{x} + \int_{S(t)} \rho_s \mathbf{g}\, d\mathbf{x}$$

$$\rho_s \frac{d(\mathbb{J}\boldsymbol{\omega})}{dt} = -\int_{\partial S(t)} (\mathbf{x} - \mathbf{q}(t)) \times P_f \mathbf{n}\, d\mathbf{x} + \int_{S(t)} \rho_s (\mathbf{x} - \mathbf{q}(t)) \times \mathbf{g}\, d\mathbf{x},$$

and the motion of the fluid for $\mathbf{x} \in F(t) = \Omega \backslash \overline{S(t)}$:

$$\partial_t \rho_f + (\mathbf{u}_f \cdot \nabla)\rho_f = 0, \qquad \mathrm{div}\mathbf{u}_f = 0$$

$$\rho_f(\partial_t \mathbf{u}_f + (\mathbf{u}_f \cdot \nabla)\mathbf{u}_f) = \mathrm{div} P + \rho_f \mathbf{g}.$$

ρ_f and ρ_s-densities of the fluid and the body;
$m = \int_{S(t)} \rho_s \, d\mathbf{x}$-the mass of the body;
P_f-the value of the stress tensor P of the fluid on $\partial S(t)$;
$\mathbf{n}(\mathbf{x})$-the unit *interior* normal at $\mathbf{x} \in \partial S(t)$;
$\mathbf{g}$-the external force;

$$\mathbb{J} = \int_{S(t)} (|\mathbf{x} - \mathbf{q}(t)|^2 \mathbb{I} - (\mathbf{x} - \mathbf{q}(t)) \otimes (\mathbf{x} - \mathbf{q}(t))) \, d\mathbf{x}$$

-the matrix of the inertia moments of the body $S(t)$.
$\mathbf{u}_f$-the fluid velocity;
$P = -pI + 2\mu_f \, \mathbb{D}\mathbf{u}_f$-the stress tensor;
$\mathbf{u}_f = \frac{1}{2} \left(\nabla \mathbf{u}_f + (\nabla \mathbf{u}_f)^T \right)$-the deformation-rate tensor;
p-the fluid pressure;
$\mu_f > 0$-the viscosity of the fluid.

We define common velocity and density for the body and the fluid in the whole domain $\Omega_T = (0, T) \times \Omega$ as

$$(\mathbf{u}, \rho) = \begin{cases} (\mathbf{u}_s, \rho_s), & \mathbf{x} \in S(t); \\ (\mathbf{u}_f, \rho_f), & \mathbf{x} \in F(t). \end{cases}$$

In addition the initial-boundary conditions are prescribed

$$S = S_0, \quad \rho = \rho_0(\mathbf{x}), \quad \mathbf{u} = \mathbf{u}_0 \quad \text{at } t = 0$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega$$

Dirichlet condition

Dirichlet boundary condition on $\partial S(t)$

$$\mathbf{u}_f \cdot \mathbf{n} = \mathbf{u}_s \cdot \mathbf{n}, \qquad \mathbf{u}_f \cdot \boldsymbol{\tau} = \mathbf{u}_s \cdot \boldsymbol{\tau},$$

$\mathbf{u}_f$ and $\mathbf{u}_s$-the velocity values of the fluid and the body on $\partial S(t)$;

$\boldsymbol{\tau}(\mathbf{x})$-any tangent vector to $S(t)$ at $\mathbf{x} \in \partial S(t)$.

Solvability

- Hoffmann, Starovoitov (1999)
- Desjardins, Esteban (1999, 2000)
- San Martin, Starovoitov, Tucsnak (2002)

No collisions - Influence of regularity of boundaries

- Starovoitov (2003)
- Hillairet (2003), Hillairet, Takahashi (2009)
- Hesla (2005)
- Gerard-Varet, Hillairet (2010)

Navier's boundary conditions on $\partial S(t)$

- Neustupa, Penel (2009, 2010)

$$\mathbf{u}_f \cdot \mathbf{n} = \mathbf{u}_s \cdot \mathbf{n}, \qquad (P_f \mathbf{n} + \gamma(\mathbf{u}_f - \mathbf{u}_s)) \cdot \boldsymbol{\tau} = 0,$$

$\gamma > 0$-the friction coefficient;

and also Navier's boundary condition

$$(P\mathbf{n} + \lambda \mathbf{u}) \cdot \boldsymbol{\tau} = 0 \quad \text{on } \partial\Omega.$$

Global existence, but in the whole $\mathbb{R}^N$

- G. Planas, F. Suer (2014)

Local-in-time existence, until touching

- Gérard-Varet, Hillairet (2014)

Notations and Main results

We define the spaces

$$V^{0,2}(\Omega) = \{\mathbf{v} \in L^2(\Omega) : \operatorname{div}\mathbf{v} = 0 \quad \text{in } \mathcal{D}'(\Omega), \quad \mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega\},$$

$$V^{1,2}(\Omega) = \{\mathbf{v} \in W^{1,2}(\Omega) : \operatorname{div}\mathbf{v} = 0 \ \text{ a.e. in } \Omega\},$$

$$BD(\Omega) = \left\{\mathbf{v} \in L^1(\Omega) : \mathbb{D}\mathbf{v} \in \mathcal{M}(\Omega)\right\}.$$

$\mathcal{M}(\Omega)$ - the space of bounded Radon measures.

Let S be an open simply-connected subset of Ω with $\partial S \in C^2$.

$$KB(S) = \left\{\mathbf{v} \in BD(\Omega) : \mathbb{D}\mathbf{v} \in L^2(\Omega\backslash\overline{S}), \quad \mathbb{D}\mathbf{v} = 0 \text{ a.e. on } S, \right.$$

$$\left. \operatorname{div}\mathbf{v} = 0 \quad \text{in} \quad \mathcal{D}'(\Omega)\right\}.$$

Definition

The triple $\{A, \rho, \mathbf{u}\}$ is a weak solution, if :
1) $A(t, \cdot) : \Omega \to \Omega$ is isometry (1), which defines the set $S(t)$ by (2).
The isometry A is compatible with $\mathbf{u} = \mathbf{u}_s$ on $S(t)$ by (1)-(3);
2) $\rho \in L^\infty(\Omega_T)$ satisfies

$$\int_{\Omega_T} \rho(\xi_t + (\mathbf{u} \cdot \nabla)\xi)\, dtd\mathbf{x} = -\int_{\Omega} \rho_0 \xi(0, \cdot)\, d\mathbf{x}$$

for any $\xi \in C^1(\Omega_T)$, $\xi(T, \cdot) = 0$;

3) $\mathbf{u} \in L^2(0,T;KB(S(t))) \cap L^\infty(0,T;V^{0,2}(\Omega))$ satisfies

$$\int_0^T \left\{ \int_{\Omega\setminus\partial S(t)} \rho\mathbf{u}\{\psi_t + (\mathbf{u}\cdot\nabla)\psi\} - 2\mu_f\,\mathbb{D}\mathbf{u} : \mathbb{D}\psi + \rho\mathbf{g}\psi d\mathbf{x} \right\} dt$$

$$= -\int_\Omega \rho_0\mathbf{u}_0\psi(0,\cdot)\,d\mathbf{x} + \int_0^T \left\{ \int_{\partial S(t)} \gamma(\mathbf{u}_s - \mathbf{u}_f)(\psi_s - \psi_f)\,d\mathbf{x} + \int_{\partial\Omega} \lambda\mathbf{u}\psi\,d\mathbf{x} \right\} dt$$

for any

$$\psi \in L^{2(N-1)}(0,T;KB(S(t))),$$

$$\psi_t \in L^2(0,T;L^2(\Omega\setminus\partial S(t))), \quad \psi(T,\cdot) = 0.$$

Main result

Let $\mathbf{u}_0 \in V^{0,2}(\Omega)$, $\mathbf{g} \in L^2(\Omega_T)$,

$$\rho_0(\mathbf{x}) = \begin{cases} \rho_s(\mathbf{x}) \geqslant const > 0, & \mathbf{x} \in S_0; \\ \rho_f = const > 0, & \mathbf{x} \in F_0, \end{cases} \quad \text{and} \quad \rho_s \in L^\infty(S_0),$$

There exists a weak solution $\{A, \rho, \mathbf{u}\}$:
$A(t, \cdot)$ - Lipschitz continuous on $t \in [0, T]$,

$$\rho(t, \mathbf{x}) = \begin{cases} \rho_s(A^{-1}(t, \mathbf{x})), & \mathbf{x} \in S(t); \\ \rho_f = const, & \mathbf{x} \in F(t), \end{cases} \quad t \in (0, T),$$

$$\frac{1}{2}\int_\Omega \rho|\mathbf{u}|^2(r)\, d\mathbf{x} + \int_0^r \left\{ \int_{\Omega \setminus \partial S(t)} 2\mu_f \left|\mathbb{D}\,\mathbf{u}\right|^2 \, d\mathbf{x} + \int_{\partial S(t)} \gamma |\mathbf{u}_f - \mathbf{u}_s|^2 \, d\mathbf{x} \right.$$

$$\left. + \int_{\partial\Omega} \lambda |\mathbf{u}|^2 \, d\mathbf{x} \right\} dt \leqslant \frac{1}{2}\int_\Omega \rho_0 |\mathbf{u}_0|^2 \, d\mathbf{x} + \int_0^r \int_\Omega \rho \mathbf{g} \mathbf{u} \, dt d\mathbf{x}.$$

Approximate problem

$$\overline{f}^{\tau}(\mathbf{x}) = \int_{\mathbb{R}^N} f(\mathbf{y})\, \sigma^{\tau}(\mathbf{x}-\mathbf{y})\, d\mathbf{y}, \quad \forall \mathbf{x} \in \mathbb{R}^N$$

- a standard regularization of a function $f \in L^1(\Omega)$.

We introduce the the characteristic functions:
$\varphi_{0,\delta}(\mathbf{x})$ of the set $[S_0]_\delta$

$\chi_{0,\delta}(\mathbf{x})$ of the set $U_\delta(\partial S_0) = S_0 \backslash \overline{[S_0]_\delta}$.

$$\partial_t \rho + \mathrm{div}(\rho \bar{\mathbf{u}}) = 0, \qquad \partial_t \varphi + \mathrm{div}(\varphi \bar{\mathbf{u}}) = 0, \qquad \partial_t \chi + \mathrm{div}(\chi \bar{\mathbf{u}}) = 0$$

$$\rho = \rho_{0,\varepsilon\delta} = (1 - \chi_{0,\delta})\rho_0 + \varepsilon \chi_{0,\delta}, \quad \varphi = \varphi_{0,\delta}, \quad \chi = \chi_{0,\delta} \quad \text{at } t = 0$$

and

$$\int_{\Omega_T} [\rho \mathbf{u} \partial_t \psi + \rho \mathbf{u} (\bar{\mathbf{u}} \cdot \nabla) \psi - \mu_\varepsilon \mathbb{D}\mathbf{u} : \mathbb{D}\psi + \rho \mathbf{g} \psi] \, dtd\mathbf{x},$$

$$= - \int_\Omega \rho_{0,\varepsilon\delta} \mathbf{u}_0 \psi(0, \cdot) \; d\mathbf{x},$$

for any $\psi \in L^2(0, T; V^{1,2}(\Omega)) \cap H^1(\Omega_T), \;\; \psi(T, \cdot) = 0.$

Here

$$\mu_\varepsilon = \frac{1}{\varepsilon}\varphi + 2\mu_f \theta + \gamma_0 \chi \int_\Omega \chi \; d\mathbf{x}, \;\; \theta = 1 - \varphi - \chi$$

$\gamma_0 = \frac{\gamma}{|\partial S_0|}, \; |\partial S_0| = \int_{\partial S_0} 1 d\mathbf{x}.$

In the cusp domain

$$V = \{\mathbf{x} = (x, y) \in \mathbb{R}^2 : 0 < x < 1, \quad 0 < y < x^2\}$$

there exists $\mathbf{u} \in LD^2(V)$, that is $\mathbf{u} \in L^2(V)$ and $\mathbb{D}\mathbf{u} \in L^2(V)$:

$$\mathbf{u} \notin L^r(V) \text{ for any } r > 2.$$

Using "Local" pressure we derive

$$\int_{\Omega_T} \rho_\delta \mathbf{u}_\delta (\overline{\mathbf{u}_\delta} \cdot \nabla) \psi_\delta \, dt d\mathbf{x} \to \int_0^T \left\{ \int_{\Omega \setminus \partial S(t)} \rho (\mathbf{u} \otimes \mathbf{u}) : \mathbb{D} \psi \, d\mathbf{x} \right\} dt,$$

$$\psi \in L^{2(N-1)}(0, T; KB(S(t))).$$

and there exists $\psi_F \in L^{2(N-1)}(0, T; C^1(\Omega) \cap V^{1,2}(\Omega))$, such that

$$\psi(t, \cdot) = \psi_F(t, \cdot) \quad \text{on} \quad F(t).$$

Conclusion

- Dirichlet boundary conditions - no collisions ???

Due to:

- **Navier boundary conditions - COLLISIONS !!!**