Euler Poincaré equations for stochastic processes defined on semi-direct product Lie algebras

Madrid, July 2014

Ana Bela Cruzeiro
Dep. Mathematics IST and
Grupo de Física-Matemática Univ. Lisboa

Joint work (**in progress**) with X. Chen (U. Lisbon) and T. Ratiu (EPFL, Lausanne)

General idea: To derive certain deterministic equations of motion corresponding to dissipative systems (that cannot be obtained in a classical setting) by deforming stochastically the underlying Lagrangian paths and interpreting the velocities in a generalized sense.

Particular case: deterministic Euler-Poincaré equations.

Lagrangian is the classical one, but computed over stochastic processes (inspired by Feynman path integral approach to QM - Yasue, Zambrini)

Different stochastic geometric mechanics: the Lagrangian is randomly perturbed, velocities are random - Bismut, Ortega)

Semi-martingales on a Lie group

Some stochastic analysis notions:

Fix a probability space $(\Omega, \mathcal{P}, P)$ and an increasing filtration $(\mathcal{P}_t)$, $t \geq 0$. A (real-valued) stochastic process $X : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ is **adapted** if $X(t)$ is $\mathcal{P}_t$-measurable for every t.

A (real valued) adapted process $M(t)$ is a **martingale** if

(i) $E|M_\omega(t)| < \infty$ foar all t

(ii) $E_s(M_\omega(t)) = M_s(\omega)$ a.s. for all $0 \leq s < t$ where E denotes expectation and E_s conditional expectation with respect to $\mathcal{P}_s$.

A real-valued process $X(t)$ is a **semimartingale** if it is of the form

$$X(t) = X(0) + M(t) + A(t)$$

where M is a martingale and A an adapted process of bounded variation with $A(0) = 0$.

Itô stochastic integral

$$\int_0^t X(s)dY(s) = \lim_n \sum X(t_i)[Y(t_{i+1}) - Y(t_i)]$$

Statonovich stochastic integral

$$\int_0^t X(s)\delta Y(s) = \lim_n \sum \frac{1}{2}[X(t_i) + X(t_{i+1})][Y(t_{i+1}) - Y(t_i)]$$

X, Y semimartingales, limits taken in probability.

$$X(t)\delta Y(t) = X(t)dY(t) + \frac{1}{2}d[X, Y]_t$$

$[X, Y]_t$ covariation of X and Y.

Itô's formula:
For $f \in C^2(\mathbb{R})$,

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))dX(s) + \frac{1}{2}\int_0^t f''(X(s))d[X,X]_s$$

or

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))dX(s)$$

Recall that **Brownian motion** is a continuous martingale $W(t)$ s.t. $[W,W]_t = t$.

Consider G (finite dimensional) Lie group with
$< >$ left (right) invariant metric
∇ left (right) invariant connection, torsion free
e identity element
Semimartingale on G: process $g : \Omega \times \mathbb{R}^+ \to G$ s.t.

$$f(g(t)) - f(g(0)) - \int_0^t \text{Hess} f(g(s)) d[g, g]_s$$

for all $f \in C^2(G)$ and where

$$\text{Hess} f(g)(v_1, v_2) = \tilde{v}_1 \tilde{v}_2 f(g) - \nabla_{\tilde{v}_1} \tilde{v}_2 f(g),$$

is a martingale; $v_1 v_2 \in T_e G$, $\tilde{v}_i$ smooth vector fields on G s.t. $\tilde{v}_i(g) = v_i$ and

$$d[g, g]_t = d[\int_0^t P_s^{-1} \delta g(s), \int_0^t P_s^{-1} \delta g(s)]_t$$

$P_t : T_{g(0)} G \to T_{g(t)} G$ parallel transport over $t \to g(t)$ associated with ∇.

Consider $H_k \in T_eG$, $u(\cdot) \in C^1([0,T]; T_eG)$ and the following process, solution of the stochastic differential equation on G, for $t \in [0,T]$:

$$dg(t) = T_eL_{g(t)}\Big(\sum_k H_k \delta W_t^k - \frac{1}{2}\sum_i \nabla_{H_k} H_k dt + u(t)dt\Big), ..., g(0) = e$$

where $T_hL_{g(t)} : T_hG \to T_{g(t)h}G$ is the differential of the left translation $L_{g(t)}(y) := g(t)y, \forall y \in G$ at the point $y = h \in G$, $W^k(t)$ iid $\mathbb{R}^k$ valued Brownian motions.

$$dg(t) = T_eL_{g(t)}\Big(\sum_k H_k dW^k(t) + u(t)dt\Big), ..., g(0) = e$$

Remark: If H_k is an o.n. basis of $\mathcal{G}$, ∇ the Levi-Civita connection and $\nabla_{H_k} H_k = 0$, $u = 0$, then g is the Brownian motion on G, generated by Laplace-Beltrami operator.

Derivative in time for ξ G-valued semi-martingale $\xi(0) = x$ replaced by **generalized derivative** :

Take $\eta(t) := \int_0^t P_s^{-1}\delta\xi(s)$ is a $T_x(G)$ valued semi-martingale.
Take the derivative of bounded variation part,

$$D_t\eta := \lim_{\epsilon\to 0} E\left[\frac{\eta(t+\epsilon)-\eta(t)}{\epsilon}|\mathcal{P}_t\right]$$

and define

$$D_t^{\nabla}\xi(t) := t_{t\leftarrow 0}^{\nabla} D_t\eta$$

Then

$$D_t^{\nabla} g(t) = T_e L_{g(t)} u(t)$$

Semi-direct products:

U (finite dimensional) vector space, U^* dual
$\diamond : U \times U^* \to T_e^* G$
Suppose G has a left representation on U.

$$< a \diamond \alpha, v >_{T_eG} = - < \alpha v, a >_U = < \alpha, va >_U$$

$a \in U, \ \alpha \in U^*, \ v \in T_eG$
On the set $\mathcal{S}(G)$ of all G- valued semimartingales defined for $t \in [0, T]$, define the **action functional**

$J^{\nabla, \alpha_0, l} : \mathcal{S}(G) \times \mathcal{S}(G) \to \mathbb{R}_+$

$$J^{\nabla, \alpha_0, l}(g^1_\omega(\cdot), g^2_\omega(\cdot)) := E \int_0^T l \left(T_{g^1_\omega(t)} L_{g^1_\omega(t)^{-1}} D_t^\nabla g^1_\omega(t), \alpha(t) \right) dt,$$

where

$$\alpha(t) = E[\tilde{\alpha}_\omega(t)], \quad \alpha_\omega(t) = (g^2_\omega(t))^{-1} \alpha_0,$$

$\alpha_0 \in U^*$.

Variations

For a deterministic curve $v(\cdot) \in C^1([0,T]; T_eG)$, $v(0) = v(T) = 0$ and $\epsilon \in (0,1)$, $e_{\epsilon,v}(\cdot) \in C^1([0,T]; G$ solution of the deterministic equation on G:

$$\frac{d}{dt} e_{\epsilon,v}(t) = \epsilon T_e L_{e_{\epsilon,v}(t)} \dot{v}(t), \quad e_{\epsilon,v}(0) = e$$

Then

$(g^1_\omega(\cdot), g^2_\omega(\cdot)) \in \mathcal{S}(G) \times \mathcal{S}(G)$ is **critical** for the action functional $J^{\nabla,\alpha_0,l}$ if

$$\frac{d}{d\epsilon}|_{\epsilon=0} J^{\nabla,\alpha_0,l}(g^1_\omega(\cdot)e_{\epsilon,v}(\cdot), g^2_\omega(\cdot)e_{\epsilon,v}(\cdot)) = 0$$

We shall consider the semimartingales g^1, g^2 of the form

$$dg^1_\omega(t) = T_e L_{g^1_\omega(t)} \Big(\sum_k H^1_k dW^{k,1}_\omega(t) + u(t)dt \Big), ..., g^1_\omega(0) = e$$

$$dg^2_\omega(t) = T_e L_{g^2_\omega(t)} \Big(\sum_k H^2_k dW^{k,2}_\omega(t) + u(t)dt \Big), ..., g^2_\omega(0) = e$$

for fixed H^i_k, fixed Brownian motions, and some $u \in C^1([0, T]; T_e G)$.

Theorem.
$(g^1, g^2) \in\in \mathcal{S}(G) \times \mathcal{S}(G)$ is critical for $J^{\nabla,\alpha_0,l}$ iff $u(\cdot) \in C^1([0,T]; T_eG)$ satisfies the semidirect product Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\delta l}{\delta u} = \mathrm{ad}^*_{\tilde{u}(t)}\frac{\delta l}{\delta u} + \frac{\delta l}{\delta \alpha} \diamond \alpha(t) + K\left(\frac{\delta l}{\delta u}\right)$$

where $\frac{\delta l}{\delta u} \in T_e^*G$, $\frac{\delta l}{\delta \alpha} \in U$ are functional derivatives of l, $K : T_e^*G \to T_e^*G$ defined by

$$\langle K(\mu), v\rangle = -\left\langle \mu, \frac{1}{2}\sum_{j=1}^{k_1}\left(\nabla_{\mathrm{ad}_v H_j^1} H_j^1 + \nabla_{H_j^1}(\mathrm{ad}_v H_j^1)\right)\right\rangle$$

$\forall \mu \in T_e^*G, \quad \forall v \in T_eG$ and

$$\frac{d}{dt}\alpha(t) = -\frac{1}{2}\sum_{j=1}^{k_2} H_j^2\left(H_j^2\alpha(t)\right) - \tilde{u}^2(t)\alpha(t)$$

Above

$$\tilde{u}^i(t) = u(t) - \frac{1}{2}\sum_{j=1}^{k_1} \nabla_{H_j^i} H_j^i, \quad i = 1, 2$$

Remark In the case of right-invariant metric, the signs in the r.h.s. of the first and third terms of the E-P equations change. are changed.

Important remark

In some cases the operator $K(u)$) coincides with the de Rham-Hodge operator.

If G is a Lie group with right invariant metric, ∇ is the (right invariant) Levi-Civita connection with respect to $< >$ and $\nabla_{H_k} H_k = 0$ for each k, we have,

$$K(u) = -\frac{1}{2}\sum_k \left(\nabla_{H_k}\nabla_{H_k} u + R(u, H_k)H_k\right) \quad \forall u \in \mathcal{G}$$

Applications

Diffeomorphisms on the torus

$G = G^s = \{g := \mathbb{T}^3 \to \mathbb{T}^3$, bijection, $g, g^{-1} \in H^s\}$, where
H^s is the s-th order Sobolev space.
If $s > \frac{5}{2}$ G^s is an C^∞ is a topological group and an infinite dimensional Hilbert manifold (Ebin-Marsden).

$$T_e G^s = H^s(\mathbb{T}^3; \mathbb{T}^3)$$

Inner product

$$< u, v >^0 := \int_{\mathbb{T}^3} \langle u(x), v(x) \rangle_x dx, \quad u, v \in T_e G^s$$

Right invariant connection

$$\nabla^0_u v = \nabla^{LC}_u v$$

We consider

$$dg^{\nu}(t,x) = \sqrt{2\nu}dW(t) + u(t, g^{\nu}(t,x))dt, \quad g^{\nu}(0,x) = x$$

for a 3-dim Brownian motion $W(t)$, $\nu > 0$ constants.
Corresponds to taking $H_j = e_j, \;\; j = 1, 2, 3$ multiplied by constants $\sqrt{2\nu}$.

α_0 will be function, differential form;

$$\alpha(t) = E[\alpha g^{\nu}(t)^{-1}]$$

i.e., the action of G^s on U^* is the pull-back map.

For some choices of the constants ν and Lagrangian
we derive MHD equations, with dissipative viscosity (resistivity, diffusivity) terms.

- X. Chen, A. B. C., T. Ratiu, *work in progress*
- M. Arnaudon, X. Chen, A. B. C., *Stochastic Euler-Poincaré reduction*, preprint ArXiv
- M. Arnaudon, A. B. C., *Lagrangian Navier-Stokes diffusions on manifolds: variational principle and stability* , Bull. des Sc. Math. 138 (2012)
- F. Cipriano, A. B. C., *Navier-Stokes equation and diffusions on the group of homeomorphisms of the torus* , Comm. Math. Phys. 275 (2007)
- D.D. Holm, J.E. Marsden, T.S. Ratiu, *The Euler-Poincaré equations and semidirect products with applications to continuum theories*, Adv. in Math. 137 (1998)