Mean field games

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Outline

1. Introduction
2. A few examples
3. Derivation of mean-field games
4. Stationary problems
   - Variational mean-field games
   - Extended mean-field games
   - Congestion models
5. Time dependent mean-field games
6. Time dependent Hamiltonians - the proof
   - Subquadratic case
   - Superquadratic case

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Mean-field games

- Mean field games is a class of problems which attempts to model and understand the behaviour of large groups of rational agents.
- These models were developed in the mathematical community by P.L. Lions and J.M. Lasry, and in the engineering community by P. Caines, M. Huang, and R. Malhamé.
- This research area has a wide range of applications as well as a large number of non-trivial mathematical challenges.

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Applications

- Non-renewable resources (Lions, Lasry, and Guéant)
- Planning problems (Porretta; Achdou, Camilli, and Dolcetta)
- Growth theory (Lions, Lasry, and Guéant; Moll, and Lucas; Lachapelle, and Turinici)
- Price formation models (Lions,and Lasry; Markowich, Caffarelli, Wolfram, and Pietschmann)
- Social Network Dynamics (G., Mohr, and Souza; Guéant)

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Related Models

- Crowd and pedestrian motion (Kamareddine, and Hughes; Burger, Di Francesco, Markowich, Pietschmann, and Wolfram)
- Chemotaxis (Perthame et al; Carrillo et al)
- Flocking and swarming (Slepcev et al, Carrillo et al; Bertozzi et al)
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Common structure

- A probability density m enconding a population distribution;
- A "potential" or "value function" u which measures the effects of the population in the environment;
- A partial differential equation for u which depends on m (tipically a nonlinear elliptic or parabolic equation)
- A partial differential equation for m which is driven by the potential u.

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Hughes-like models

- Consider a region Ω, a population distribution $m(x,t)$. Each agent wants to leave Ω as fast as possible.
- Taking into account the congestion effects the time to the exit of a single agent if the whole population is frozen satisfies the equation

$$\frac{1}{2}|Du|^2 = \frac{1}{(6-m)^2}.$$

- Each agent follows the "instantaneous" shortest exit path and so the population evolves according to

$$m_t - \operatorname{div}(mDu) = 0$$

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Keller-Segel Chemotaxis model

- Consider a biological population described by a density m.
- This population produces chemicals which attract or repel other members of the population and which difuse very fast. The spatial distribution u of these chemicals solves

$$-\Delta u = f(m).$$

- The population reacts by moving in the direction of the gradient of this distribution:

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Benamou-Brenier optimal transport method

- Given two probability measures ρ_0, ρ_1;
- The quadratic cost optimal transport problem can be solved in terms of the system

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- a Hamilton-Jacobi type equation
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Standard MFG

- Time dependent MFG

$$\begin{cases} -u_t + H(Du, x) = \Delta u + F(m) \\ m_t - \operatorname{div}(D_p H m) = \Delta m \end{cases}$$

with $m(x, 0)$ and $u(x, T)$ given.

- Stationary version

$$\begin{cases} H(Du, x) = \Delta u + F(m) + \overline{H} \\ -\operatorname{div}(D_p H m) = \Delta m \end{cases}$$

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Typical non-linearity F:

- Non-local: $F(m) = G(\eta * m)$.
- Power-like: $F(m) = m^\alpha$.
- Logarithm: $F(m) = \ln m$.

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Typical Hamiltonian: $H(x,p) = a(x)(1+|p|^2)^{\gamma/2} + V(x)$

- subquadratic $1 \leq \gamma < 2$
- quadratic $\gamma = 2$ and $a = 1$
- superquadratic $2 \leq \gamma < 3$.

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Extensions

- These standard models have been studied extensively (local and non-local F, first order cases, planning problems...).
- However, in real world applications a few additional complications arise since many interesting models have different structure
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Non-standard structures I

- Congestion effects. An example would be

$$\begin{cases} V(x) + \frac{H(Du)}{m^\alpha} = \Delta u + F(m) + \overline{H} \\ -\operatorname{div}(D_p H m^{1-\alpha}) = \Delta m \end{cases}$$

- Extended models which arise when costs depend on the actions of the players

$$\begin{cases} H(Du, x, B) = \Delta u + F(m) + \overline{H} \\ -\operatorname{div}(Bm) = \Delta m \\ B = -D_p H(Du, xB). \end{cases}$$

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Non-standard structures II

- Lack of differentiability of the Hamilton-Jacobi equation which arises in optimal stopping and gives rise to study

$$\begin{cases} \beta_\epsilon(u) + H(Du, x) = \Delta u + F(m) + \overline{H} \\ \beta_\epsilon'(u)m - \operatorname{div}(D_p H m) = \Delta m \end{cases}$$

  where $\beta_\epsilon(z) = 0$ if $z < 0$ and $\lim_{\epsilon\to 0} \beta_\epsilon(z) = +\infty$.

- Lack of adjoint structure (eg Hughes-type models in crowd dynamics)

$$\begin{cases} H(Du, x) = \Delta u + F(m) \\ m_t - \operatorname{div}(D_p H m) = \Delta m \end{cases}$$

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Stochastic optimal control

Define

$$u(x,t) = \inf_{\mathbf{v}} E \int_t^T L(\mathbf{x}, \mathbf{v}) ds + \psi(\mathbf{x}(T)),$$

where the infimum is taken, for instance, over all progressively measurable controls $\mathbf{v}$ with respect to the filtration generated by a Brownian motion W_t, and

$$d\mathbf{x} = \mathbf{v}dt + \sigma dW_t, \qquad \mathbf{x}(t) = x.$$

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Hamilton-Jacobi equation

The Hamiltonian is defined as

$$H(p, x) = \sup_{v \in \mathbb{R}^d} -v \cdot p - L(x, v).$$

If the value function u is smooth then it solves the Hamilton-Jacobi equation

$$-u_t + H(x, D_x u) = \frac{\sigma^2}{2} \Delta u,$$

together with the terminal condition

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Verification theorem

Theorem

If u is a smooth solution to the Hamilton-Jacobi equation then

$$\mathbf{v} = -D_pH(\mathbf{x}, D_xu(\mathbf{x}, t))$$

is an optimal control.

Consider a diffusion

$$d\mathbf{x} = b(\mathbf{x}, t)dt + \sigma dW_t$$

with initial distribution of $\mathbf{x}(0)$ given by a probability measure $m(x, 0)$. That is

$$P(\mathbf{x}(0) \in A) = \int_A m(x, 0)dx.$$

Define $m(x, t)$ by

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The probability measure m solves the Focker Plank equation

$$m_t + \operatorname{div}(b(x,t)m) = \frac{\sigma^2}{2}\Delta m.$$

In particular if $b = -D_pH(D_xu(x,t),x)$ is the optimal feedback for the control problem above

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The mean field game problem arises when a optimal (deterministic or stochastic) control problem has a Lagrangian L or terminal cost ψ depending on a population distribution m.

$$u(x,t) = \inf_{\mathbf{v}} E \int_t^T L(\mathbf{x}, \dot{\mathbf{x}}, m(\cdot, s))ds + \psi(\mathbf{x}(T), m(\cdot, T)).$$

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Then the value function u solves the Hamilton-Jacobi equation

$$-u_t + H(D_x u, x, m) = \frac{\sigma^2}{2}\Delta u$$

In this setting one assumes the following rationality hypothesis, that is, that each agent in the population follows the optimal dynamics and then m is a solution of

$$m_t - \operatorname{div}(D_p H m) = \frac{\sigma^2}{2}\Delta m.$$

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In addition to the PDE, the value function u and the probability measure m must satisfy certain boundary conditions:

1. Initial-terminal problem

$$u(x,T)=\psi(x) \qquad m(x,0)=m_0(x).$$

2. Planning problem

$$m(x,0)=m_0 \qquad m(x,T)=m_T$$

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Consider the variational problem

$$\min \int_{\mathbb{T}^d} e^{H(Du,x)} dx.$$

The Euler-Lagrange equation is the Mean Field Game:

$$\begin{cases} H(Du,x) = \ln m + \overline{H} \\ -\operatorname{div}(D_pHm) = 0, \end{cases}$$

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One can consider also various other extensions:

- Time-dependent problems

$$\int_0^T \int_{\mathbb{T}^d} e^{-u_t+H(D_x u,x)}dxdt$$

- Second order problems

$$\int_{\mathbb{T}^d} e^{-\Delta u+H(Du,x)}dx$$

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The stochastic Evans-Aronsson problem concerns the variational problem

$$\min_u \int_{\mathbb{T}^d} e^{-\Delta u + H(Du,x)}$$

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Theorem (G. , Sanchez-Morgado)

Suppose $H(p,x)=\frac{1}{2}|P+p|^2+V(x)$. *Let u and v be periodic solutions to*

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The general case, where $F(m) = \ln m$ or $F(m) = m^\alpha$ in higher dimensions:

$$\begin{cases} -\Delta u + H(Du, x) = F(m) \\ -\Delta m - \operatorname{div}(D_p H m) = 0, \end{cases}$$

was addressed by G. , Patrizi, Voskanyan.

Extended mean-field games

We consider periodic solutions to

$$\begin{cases} H(x, D_x u, m, V) = \Delta u + \bar{H} \\ \operatorname{div}(Vm) = \Delta m \\ V = -D_p H(x, D_x u, m, V). \end{cases}$$

- H is quasivariational:

$$|H(x,p,m,V) - H_0(x,p,m,V) + g(m(x))| \leq C, \quad (A1)$$

H_0 non-local in m, and

$$g(m) = \ln m, \text{ or } g(m) = m^\gamma, \text{ with } 0 < \gamma < \frac{1}{d-4}$$

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Example

$$H(x, p, m, V) = \alpha(x)\frac{|p|^2}{2} + \beta \int_{\mathbb{T}^d} p \cdot V dm - g(m),$$

with β small.

Theorem (G., Patrizi, Voskanyan)

Under the previous hypothesis, there exists a classical solution $(u, m, V, \overline{H})$.

Running costs such as

$$L(x, v, m) = m^{\alpha}(x)\frac{|v|^2}{2} - V(x)$$

correspond to the congestion MFG:

$$\begin{cases} u + V(x) + \frac{|Du|^2}{2m^{\alpha}} = \Delta u + \overline{H} \\ m - \operatorname{div}(D_p H m^{1-\alpha}) = \Delta m + 1 \end{cases}$$

Theorem (G., H. Mitake)

Under the previous hypothesis, there exists a classical solution (u, m) with m bounded by below if $0 < \alpha < 1$.

- The proof relies on an a-priori bound for $\frac{1}{m}$ in L^∞.
- This bound depends on an explicit cancellation.
- It is not known if similar results hold for general models or time-dependent problems.

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2. A few examples
3. Derivation of mean-field games
4. Stationary problems
   - Variational mean-field games
   - Extended mean-field games
   - Congestion models
5. Time dependent mean-field games
6. Time dependent Hamiltonians - the proof
   - Subquadratic case
   - Superquadratic case

Initial-terminal value problem

$$-u_t + H(D_x u, x) = \frac{\sigma^2}{2}\Delta u + m^\alpha$$

$$m_t - \operatorname{div}(D_p H m) = \frac{\sigma^2}{2}\Delta m.$$

Together with initial conditions for m and terminal conditions for u.

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Weak solutions

- Lions (unpublished), Cardaliaguet (variational methods, first order problems), Porretta (upcoming paper)
- Planning problem - Porretta.

Regularity for time-dependent problems

Existence of smooth solutions holds for:

- Subquadratic Hamiltonians and $\alpha < \alpha_\sigma$ where $\alpha_\sigma > \frac{2}{d-2}$ (G., Morgado and Pimentel); the case $\alpha < \frac{2}{d-2}$ was previous addressed by Lions.
- Exactly quadratic, no conditions of α (Cardaliaguet, Lasry, Lions, and Porretta).
- Superquadratic Hamiltonians $\alpha < \alpha_\Sigma$, where $\alpha_\Sigma > \frac{1}{d-2}$ (G., Morgado and Pimentel).

Logarithmic nonlinearity

The logarithmic nonlinearity:

$$-u_t + H(D_x u, x) = \frac{\sigma^2}{2}\Delta u + \ln m$$

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All the proofs in three lines

- $\|u\| \leq C + C\|m\|^{\beta_1}$
- $\|m\| \leq C + C\|u\|^{\beta_2}$
- "result" ($\|u\|$ bounded) follows if $\beta_1\beta_2 < 1$.

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Subquadratic case

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Polynomial estimates for the Fokker-Planck equation

Theorem

Let (u, m) *be a solution and* $\|m\|_{L^\infty([0,T],L^{\beta_0}(\mathbb{T}^d))} \leq C$*, for some* $\beta_0 \geq 1$*. Suppose further that* $p > \frac{d}{2}$ *and* $r = \frac{p(d(\theta-1)+2)}{2p-d}$*. Then,*

$$\int_{\mathbb{T}^d} (m)^{\beta_n}(\tau, x)\,dx \leq C + C \left\| |D_pH|^2 \right\|^{r_n}_{L^r(0,T;L^p(\mathbb{T}^d))},$$

where

$$r_n = r\frac{\theta^n - 1}{\theta - 1}, \ \theta > 1 \text{ and } \beta_n = \theta^n \beta_0.$$

Gagliardo-Nirenberg inequality

Theorem

Let (u, m) be a solution and assume that H is subquadratic. For $1 < p, r < \infty$ there are positive constants c and C such that

$$\|D^2 u\|_{L^r(0,T;L^p(\mathbb{T}^d))} \leq c\|F(m)\|_{L^r(0,T;L^p(\mathbb{T}^d))} + c\|u\|^{\frac{\gamma}{2-\gamma}}_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} + C.$$

Superquadratic case

1. Polynomial estimates for the Fokker-Planck equation
2. Estimates by the non-linear adjoint method

Polynomial estimates for the Fokker-Planck equation

Theorem

Let (u, m) be a solution. Assume that H is superquadratic. Assume further that $0 < \mu < 1 < \beta_0$, θ, p, r, and $0 \leq \upsilon \leq 1$ satisfy

$$\alpha p = \frac{\theta^n \beta_0}{\theta^n + \upsilon - \theta^n \upsilon},$$

and $r = \frac{d(\theta-1)+2}{2}$. Then

$$\|F\|_{L^\infty(0,T;L^p(\mathbb{T}^d))} \leq C + C\,\|Du\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}^{\frac{(2+2\mu)(\theta^n-1)r\upsilon\alpha}{\theta^n\beta_0(\theta-1)}}.$$

Estimates by the non-linear adjoint method

Theorem

Suppose that H is superquadratic. Let (u, m) be a solution and assume that $p > d$. Then

$$\begin{aligned}\|Du\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} \leq & C + C\|F(m)\|_{L^\infty(0,T;L^p(\mathbb{T}^d))}^{\frac{1}{1-\mu}} \\ & + C\|F(m)\|_{L^\infty(0,T;L^p(\mathbb{T}^d))}^{\frac{1}{1-\mu}}\|u\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}^{\frac{1}{1-\mu}}.\end{aligned}$$