On Two-Phase Forchheimer Flows of Incompressible Fluids

Luan Hoang, Akif Ibragimov, Thinh Kieu

Department of Mathematics and Statistics, Texas Tech University
http://www.math.ttu.edu/∼lhoang/
`luan.hoang@ttu.edu`

July 1, 2014
Advances in Mathematical Fluid Mechanics, Stochastic and Deterministic Methods
Lisbon, Portugal, June 30 - July 5, 2014

Outline

Introduction: Darcy's and Forchheimer's flows

Fluid flows in porous media with velocity u and pressure p:

- Darcy's Law:
$$\alpha u = -\nabla p,$$
- Forchheimer's "two term" law
$$\alpha u + \beta |u|\, u = -\nabla p,$$
- Forchheimer's "three term" law
$$\mathcal{A} u + \mathcal{B}\, |u|\, u + \mathcal{C} |u|^2\, u = -\nabla p.$$
- Forchheimer's "power" law
$$a u + c^n |u|^{n-1}\, u = -\nabla p,$$

Here $\alpha, \beta, a, c, n, \mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ are empirical positive constants.

Generalized Forchheimer equations

[Aulisa-Bloshanskaya-**H.**-Ibragimov 2009]
Generalizing the above equations as follows

$$g(|u|)u = -\nabla p.$$

Let $G(s) = sg(s)$. Then $G(|u|) = |\nabla p| \Rightarrow |u| = G^{-1}(|\nabla p|)$. Hence

$$u = -\frac{\nabla p}{g(G^{-1}(|\nabla p|))} \Rightarrow u = -K(|\nabla p|)\nabla p,$$

$$K(\xi) = K_g(\xi) = \frac{1}{g(s)} = \frac{1}{g(G^{-1}(\xi))}, \quad sg(s) = \xi.$$

Class $FP(N, \vec{\alpha})$. Let $N > 0$, $0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_N$,

$$FP(N, \vec{\alpha}) = \Big\{ g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \ldots + a_N s^{\alpha_N} \Big\},$$

where $a_0, a_N > 0$, $a_1, \ldots, a_{N-1} \geq 0$. Notation: $\alpha_N = \deg(g)$,
$\vec{a} = (a_0, a_1, \ldots, a_N)$, $a = \frac{\alpha_N}{\alpha_N + 1} \in (0,1)$, $b = \frac{\alpha_N}{\alpha_N + 2} \in (0,1)$.

Historical remarks

- Darcy-Dupuit: 1865
- Forchheimer: 1901
- Other nonlinear models: 1940s–1960s
- Incompressible fluids: Payne, Straughan and collaborators since 1990's, Celebi-Kalantarov-Ugurlu since 2005 (Brinkman-Forchheimer)
- Derivation of non-Darcy, non-Forchheimer flows: Marusic-Paloka and Mikelic 2009 (homogenization for Navier–Stokes equations), Balhoff et. al. 2009 (computational)

Works on generalized Forchheimer flows

A. Single-phase flows.

- 1990's Numerical study
- L^2-theory (for slightly compressible flows): Aulisa-Bloshanskaya-**H.**-Ibragimov (2009), **H.**-Ibragimov: Dirichlet B.C. (2011), **H.**-Ibragimov Flux B.C. (2012), Aulisa-Bloshanskaya-Ibragimov total flux, productivity index (2011, 2012), Inhomogeneous media Celik-**H.**(in preparation).
- L^α-theory: **H.**-Ibragimov-Kieu-Sobol (2012-preprint)
- L^∞, $W^{1,p}$-theory: **H.**-Kieu-Phan (2014-to appear), Celik-**H.**(in preparation).
- $W^{1,\infty}$-theory: interior **H.**-Kieu (2014-preprint), global Celik-**H.**-Kieu (in preparation).

B. Multi-phase flows.

- One-dimensional case: **H.**-Ibragimov-Kieu (2013).
- Multi-dimensional case: **H.**-Ibragimov-Kieu (this talk-preprint).

Note: there are more works on Forchheimer flows (2-terms or 3 terms).

A. Single-phase Forchheimer flows

Let ρ be the density. Continuity equation

$$\frac{d\rho}{dt} + \nabla \cdot (\rho u) = 0.$$

For slightly compressible fluid:

$$\frac{d\rho}{dp} = \frac{1}{\kappa}\rho,$$

where $\kappa \gg 1$. Then

$$\frac{dp}{dt} = \kappa \nabla \cdot \Big(K(|\nabla p|)\nabla p \Big) + K(|\nabla p|)|\nabla p|^2.$$

Since $\kappa \gg 1$, we neglect the last terms, after scaling the time variable:

$$\frac{dp}{dt} = \nabla \cdot \Big(K(|\nabla p|)\nabla p \Big).$$

Degeneracy

$$K(\xi) \equiv (1+\xi)^{-a}, \quad a = \frac{\alpha_N}{\alpha_N + 1}.$$

B. Two-phase incompressible Forchheimer flows

For each ith-phase ($i = 1, 2$), saturation $S_i \in [0, 1]$, density $\rho_i \geq 0$, velocity $\mathbf{u_i} \in \mathbb{R}^\mathbf{n}$, and , and pressure $p_i \in \mathbb{R}$. The saturations satisfy

$$S_1 + S_2 = 1.$$

Each phase's velocity obeys the generalized Forchheimer equation. Conservation of mass holds for each of the phases:

$$\partial_t(\phi\rho_i S_i) + \mathrm{div}(\rho_i \mathbf{u}_i) = 0, \quad i = 1, 2.$$

Due to incompressibility of the phases, i.e. $\rho_i = const. > 0$, it is reduced to

$$\phi\partial_t S_i + \mathrm{div}\ \mathbf{u}_i = 0, \quad i = 1, 2.$$

Let p_c be the capillary pressure between two phases, more specifically,

$$p_1 - p_2 = p_c.$$

Darcy's flows. Kruzkov, Sukorjanski, Alt, DiBenedetto, Cances, Mikelic, Galusinski, Saad, Chemetov, Neves ...

Denote $S = S_1$ and $p_c = p_c(S)$. Then

$$g_i(|\mathbf{u}_i|)\mathbf{u}_i = -f_i(S)\nabla p_i, \quad i = 1, 2,$$

$$\nabla p_1 - \nabla p_2 = p_c'(S)\nabla S.$$

Hence

$$F_2(S)g_2(|\mathbf{u}_2|)\mathbf{u}_2 - F_1(S)g_1(|\mathbf{u}_1|)\mathbf{u}_1 = \nabla S,$$

where

$$F_i(S) = \frac{1}{p_c'(S)f_i(S)}, \quad i = 1, 2.$$

In summary,

$$\begin{aligned} &0 \le S = S(\mathbf{x}, t) \le 1, \\ &S_t = -\mathrm{div}\ \mathbf{u}_1, \\ &S_t = \mathrm{div}\ \mathbf{u}_2, \\ &\nabla S = F_2(S)\mathbf{G}_2(\mathbf{u}_2) - F_1(S)\mathbf{G}_1(\mathbf{u}_1). \end{aligned}$$

One-dimensional problem

Assumption A.

$$f_1, f_2 \in C([0,1]) \cap C^1((0,1)),$$

$$f_1(0) = 0, \quad f_2(1) = 0,$$

$$f_1'(S) > 0, \quad f_2'(S) < 0 \text{ on } (0,1).$$

Assumption B.

$$p_c' \in C^1((0,1)), \quad p_c'(S) > 0 \text{ on } (0,1).$$

Theorem (H.-Kieu-Ibragimov 2013)

- *There are 16 types of non-constant steady states (based on their monotonicity and asymptotic behavior as $x \to \pm\infty$).*
- *The steady states which are never zero nor one are linearly stable.*

Multi-dimensional problem

In $\mathbb{R}^n$, steady states:

$$\mathrm{div}\ \mathbf{u}_1 = \mathrm{div}\ \mathbf{u}_2 = 0, \quad \nabla S = F_2(S)\mathbf{G}_2(\mathbf{u}_2) - F_1(S)\mathbf{G}_1(\mathbf{u}_1).$$

Steady states with geometric constraints:

$$\mathbf{u}_1^*(\mathbf{x}) = c_1|\mathbf{x}|^{-n}\mathbf{x}, \quad \mathbf{u}_2^*(\mathbf{x}) = c_2|\mathbf{x}|^{-n}\mathbf{x}, \quad S_*(\mathbf{x}) = S(|\mathbf{x}|),$$

where c_1, c_2 are constants and $S(r)$ is a solution of the following ODE:

$$S' = F(r, S(r)) \quad \text{for } r > r_0, \quad S(r_0) = s_0, \quad 0 < S(r) < 1.$$

where s_0 is always a number in $(0,1)$ and

$$F(r, S(r)) = G_2(c_2 r^{1-n})F_2(S) - G_1(c_1 r^{1-n})F_1(S).$$

Theorem

There exists a maximal interval of existence $[r_0, R_{\max})$*, where* $R_{\max} \in (r_0, \infty]$*, and a unique solution* $S \in C^1([r_0, R_{\max}); (0,1))$*. Moreover, if* $R_{\max}$ *is finite then either*

$$\lim_{r \to R_{\max}^-} S(r) = 0 \quad \text{or} \lim_{r \to R_{\max}^-} S(r) = 1.$$

Theorem

If solution $S(r)$ *exists in* $[r_0, \infty)$*, then it eventually becomes monotone and, consequently,* $s_\infty = \lim_{r\to\infty} S(r)$ *exists.*
In case $n = 2$ *and* $c_1^2 + c_2^2 > 0$*, let* $s^* = (f_1/f_2)^{-1}\left(\frac{c_1 a_1^0}{c_2 a_2^0}\right)$*.*
(i) *If* $c_1 \le 0$ *and* $c_2 \ge 0$ *then* $s_\infty = 1$*.*
(ii) *If* $c_1 \ge 0$ *and* $c_2 \le 0$ *then* $s_\infty = 0$*.*
(iii) *If* $c_1, c_2 < 0$ *then* $s_\infty = s^*$*.*
(iv) *If* $c_1, c_2 > 0$ *then* $s_\infty \in \{0, 1, s^*\}$*.*

Linearized problem

The formal linearized system at the steady state $(\mathbf{u}_1^*(\mathbf{x}), \mathbf{u}_2^*(\mathbf{x}), S_*(\mathbf{x}))$ is

$$\begin{aligned} \sigma_t &= -\mathrm{div}\ \mathbf{v}_1, \quad \sigma_t = \mathrm{div}\ \mathbf{v}_2, \\ \nabla\sigma &= F_2(S_*)\mathbf{G}_2'(\mathbf{u}_2^*)\mathbf{v}_2 + F_2'(S_*)\sigma\mathbf{G}_2(\mathbf{u}_2^*) \\ &\quad - \Big(F_1(S_*)\mathbf{G}_1'(\mathbf{u}_1^*)\mathbf{v}_1 + F_1'(S_*)\sigma\mathbf{G}_1(\mathbf{u}_1^*)\Big). \end{aligned}$$

Let $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. Then $\mathrm{div}\ \mathbf{v} = 0$. Assume $\mathbf{v} = \mathbf{V}(\mathbf{x}, t)$ is given . Let

$$\begin{aligned} \underline{\mathbf{B}} &= \underline{\mathbf{B}}(\mathbf{x}) = F_2(S_*)\mathbf{G}_2'(\mathbf{u}_2^*) + F_1(S_*)\mathbf{G}_1'(\mathbf{u}_1^*), \\ \underline{\mathbf{A}} &= \underline{\mathbf{A}}(\mathbf{x}) = \underline{\mathbf{B}}(\mathbf{x})^{-1} \\ \mathbf{b} &= \mathbf{b}(\mathbf{x}) = F_2'(S_*)\mathbf{G}_2(\mathbf{u}_2^*) - F_1'(S_*)\mathbf{G}_1(\mathbf{u}_1^*), \\ \mathbf{c} &= \mathbf{c}(\mathbf{x}, t) = F_1(S_*)\mathbf{G}_1'(\mathbf{u}_1^*)\mathbf{V}(\mathbf{x}, t). \end{aligned}$$

Decoupling the linearized system:

$$\sigma_t = \nabla \cdot \Big[\underline{\mathbf{A}}(\nabla\sigma - \sigma\mathbf{b})\Big] + \nabla \cdot (\underline{\mathbf{A}}\mathbf{c}),$$

$$\mathbf{v}_2 = \underline{\mathbf{A}}(\nabla\sigma - \sigma\mathbf{b}) + \underline{\mathbf{A}}\mathbf{c}, \quad \mathbf{v}_1 = \mathbf{V} - \mathbf{v}_2.$$

Lemma

For any $c_1^2 + c_2^2 > 0$ and $\mathbf{x} \neq 0$, matrices $\underline{\mathbf{B}}(\mathbf{x})$ and $\underline{\mathbf{A}}(\mathbf{x})$ are symmetric, invertible and positive definite.

Also, matrix $\underline{\mathbf{B}}$ has the following special property:

$$\underline{\mathbf{B}}(\mathbf{x})\mathbf{x} = \sum_{i=1}^{2} \Big\{ F_i(\hat{S}(|\mathbf{x}|)) \left[g_i(|c_i||\mathbf{x}|^{1-n}) + g_i'(|c_i||\mathbf{x}|^{1-n})|c_i||\mathbf{x}|^{1-n} \right] \Big\} \mathbf{x} = \phi(|\mathbf{x}|)\mathbf{x},$$

where

$$\phi(r) = \sum_{i=1}^{2} F_i(\hat{S}(r)) \left[g_i(|c_i|r^{1-n}) + g_i'(|c_i|r^{1-n})|c_i|r^{1-n} \right].$$

Now consider "good" steady states.

In Bounded domains

Let $R > r_0 > 0$, $U \subset \mathcal{U} \stackrel{\text{def}}{=\!=} B_R \setminus \bar{B}_{r_0}$. Denote $\Gamma = \partial U$, $D = U \times (0, \infty)$ and $\mathcal{D} = \mathcal{U} \times (0, \infty)$.

Initial-boundary value problem (IBVP):

$$\begin{cases} \sigma_t = \nabla \cdot \Big[\underline{\mathbf{A}}(\nabla \sigma - \sigma \mathbf{b}) \Big] + \nabla \cdot (\underline{\mathbf{A}}\mathbf{c}) & \text{on } U \times (0, \infty), \\ \sigma = g(\mathbf{x}, t) & \text{on } \Gamma \times (0, \infty), \\ \sigma = \sigma_0(\mathbf{x}) & \text{on } U \times \{t = 0\}. \end{cases}$$

Condition (E1). $F_1, F_2 \in C^7((0,1))$ and $V \in C^6_{\mathbf{x}}(\bar{D})$; $V_t \in C^3_{\mathbf{x}}(\bar{D})$.

Theorem

Assume **(E1)** *and* $\Delta_4 \stackrel{\text{def}}{=\!=} \sup_D(|\mathbf{V}(\mathbf{x},t)| + |\nabla \mathbf{V}(\mathbf{x},t)|) + \sup_{\Gamma\times[0,\infty)} |g(\mathbf{x},t)|$ *is finite. Then the solution* $\sigma(\mathbf{x},t)$ *of the linearized equation satisfies*

$$\sup_{\mathbf{x}\in U} |\sigma(\mathbf{x},t)| \le C\left[e^{-\eta_1 t}\sup_U |\sigma_0(\mathbf{x})| + \Delta_4\right] \quad \textit{for all } t>0.$$

Moreover,

$$\limsup_{t\to\infty}\left[\sup_{\mathbf{x}\in U}|\sigma(\mathbf{x},t)|\right] \le C\Delta_5,$$

where

$\Delta_5 = \limsup_{t\to\infty}\left[\sup_{\mathbf{x}\in U}(|\mathbf{V}(\mathbf{x},t)| + |\nabla\mathbf{V}(\mathbf{x},t)|) + \sup_{\mathbf{x}\in\Gamma}|g(\mathbf{x},t)|\right].$

Theorem

Assume **(E1)**, *and* $\Delta_6 \stackrel{\text{def}}{=\!=} \sup_D(|\mathbf{V}(\mathbf{x},t)| + |\nabla \mathbf{V}(\mathbf{x},t)| + |\nabla^2 \mathbf{V}(\mathbf{x},t)|)$ *and* $\Delta_7 \stackrel{\text{def}}{=\!=} \sup_{\Gamma\times[0,\infty)} |g(\mathbf{x},t)|$ *are finite. Then for any* $U' \Subset U$, *there is* $\tilde{M} > 0$ *such that for* $i = 1, 2$, $\mathbf{x} \in U'$ *and* $t > 0$,

$$\sup_{\mathbf{x}\in U'} |\mathbf{v}_i(\mathbf{x},t)| \le \tilde{M}\Big(1 + \frac{1}{\sqrt{t}}\Big)\Big[e^{-\eta_1 t} \sup_U |\sigma_0(\mathbf{x})| + \Delta_6 + \sqrt{\Delta_6} + \Delta_7\Big].$$

Consequently, if

$$\lim_{t\to\infty} \Big\{ \sup_{\mathbf{x}\in U}(|\mathbf{V}(\mathbf{x},t)| + |\nabla \mathbf{V}(\mathbf{x},t)| + |\nabla^2 \mathbf{V}(\mathbf{x},t)|) + \sup_{\mathbf{x}\in\Gamma} |g(\mathbf{x},t)| \Big\} = 0,$$

then for any $\mathbf{x} \in U$,

$$\lim_{t\to\infty} \mathbf{v}_1(\mathbf{x},t) = \lim_{t\to\infty} \mathbf{v}_2(\mathbf{x},t) = 0.$$

Structure and Transformation

Rewrite vector function $\mathbf{b}(\mathbf{x})$ explicitly as

$$\mathbf{b}(\mathbf{x}) = \Big(F_2'(S_*(\mathbf{x}))g_2(\frac{|c_2|}{|\mathbf{x}|^{n-1}})\frac{c_2}{|\mathbf{x}|^n} - F_1'(S_*(\mathbf{x}))g_1(\frac{|c_1|}{|\mathbf{x}|^{n-1}})\frac{c_1}{|\mathbf{x}|^n}\Big)\mathbf{x} = \lambda(|\mathbf{x}|)\mathbf{x},$$

where

$$\lambda(r) = F_2'(\hat{S}(r))g_2(\frac{|c_2|}{r^{n-1}})\frac{c_2}{r^n} - F_1'(\hat{S}(r))g_1(\frac{|c_1|}{r^{n-1}})\frac{c_1}{r^n}.$$

By defining

$$\Lambda(\mathbf{x}) = \frac{1}{2}\int_{r_0^2}^{|\mathbf{x}|^2} \lambda(\sqrt{\xi})d\xi = \int_{r_0}^{|\mathbf{x}|} r\lambda(r)dr,$$

we have for $\mathbf{x} \neq 0$ that

$$\mathbf{b}(\mathbf{x}) = \nabla\Lambda(\mathbf{x}).$$

Let

$$w(\mathbf{x}, t) = e^{-\Lambda(\mathbf{x})}\sigma(\mathbf{x}, t).$$

Then w satisfies

$$w_t - \nabla\cdot\Big(\underline{\mathbf{A}}\nabla w\Big) - \nabla\Lambda\cdot\underline{\mathbf{A}}\nabla w = e^{-\Lambda}\nabla\cdot(\underline{\mathbf{A}}\mathbf{c}).$$

New system

Define the differential operator

$$\mathcal{L}w = \partial_t w - \nabla \cdot (\underline{\mathbf{A}}\nabla w) - \mathbf{b} \cdot \underline{\mathbf{A}}\nabla w.$$

Corresponding IBVP for $w(\mathbf{x}, t)$ is

$$\begin{cases} \mathcal{L}w = f_0 & \text{in } U \times (0, \infty), \\ w(\mathbf{x}, 0) = w_0(\mathbf{x}) & \text{in } U, \\ w(\mathbf{x}, t) = G(\mathbf{x}, t) & \text{on } \Gamma \times (0, \infty), \end{cases}$$

where $w_0(\mathbf{x})$ and $G(\mathbf{x}, t)$ are given initial data and boundary data, respectively, and $f_0(\mathbf{x}, t)$ is a known function.

- For the velocities, we have

$$\mathbf{v}_2 = \underline{\mathbf{A}}\big[\nabla(e^{\Lambda}w) - e^{\Lambda}w\mathbf{b}\big] + \underline{\mathbf{A}}\mathbf{c} = \underline{\mathbf{A}}\big[e^{\Lambda}\nabla w + we^{\Lambda}\nabla\Lambda - e^{\Lambda}w\mathbf{b}\big] + \underline{\mathbf{A}}\mathbf{c}.$$

Thus,

$$\mathbf{v}_2 = e^{\Lambda}\underline{\mathbf{A}}\nabla w + \underline{\mathbf{A}}\mathbf{c}, \quad \mathbf{v}_1 = \mathbf{V} - \mathbf{v}_2.$$

Lemma of growth in time of Landis type

Barrier function. Define

$$W(\mathbf{x},t) = \begin{cases} t^{-s} e^{-\frac{\varphi(\mathbf{x})}{t}} & \text{if } t>0, \\ 0 & \text{if } t\le 0, \end{cases}$$

where the number $s>0$ and the function $\varphi(\mathbf{x})>0$ will be decided later. Then

$$\mathcal{L}W = t^{-s-2} e^{-\frac{\varphi}{t}} \Big\{ t\big(-s + \nabla\cdot(\underline{\mathbf{A}}\nabla\varphi) + \mathbf{b}\cdot\underline{\mathbf{A}}\nabla\varphi\big) + \varphi - (\underline{\mathbf{A}}\nabla\varphi)\cdot\nabla\varphi \Big\}.$$

Thus, $\mathcal{L}W \le 0$ if

$$s \ge \nabla\cdot(\underline{\mathbf{A}}\nabla\varphi) + \mathbf{b}\cdot\underline{\mathbf{A}}\nabla\varphi \quad \text{and} \quad \varphi \le (\underline{\mathbf{A}}\nabla\varphi)\cdot\nabla\varphi.$$

We will choose φ to satisfy

$$\underline{\mathbf{A}}\nabla\varphi = \kappa_0 \mathbf{x},$$

where κ_0 is a positive constant selected later. Equivalently,

$$\nabla\varphi = \kappa_0 \underline{\mathbf{A}}^{-1}\mathbf{x} = \kappa_0 \underline{\mathbf{B}}\mathbf{x} = \kappa_0 \phi(|\mathbf{x}|)\mathbf{x}.$$

Define for $\mathbf{x} \in \bar{\mathcal{U}}$ the function

$$\varphi(\mathbf{x}) = \kappa_0\Big(\varphi_0 + \int_{r_0}^{|\mathbf{x}|} r\phi(r)dr\Big), \quad \text{where } \varphi_0 = \frac{C_0 r_0^2}{2} \text{ and } \kappa_0 = \frac{C_0}{2C_1}.$$

Select

$$s = s_R \stackrel{\text{def}}{=\!=} \kappa_0(n + C_2 R).$$

Lemma

The function $W(\mathbf{x},t)$ belongs to $C^{2,1}_{\mathbf{x},t}(\mathcal{D}) \cap C(\bar{\mathcal{D}})$ and satisfies $\mathcal{L}W \le 0$ in $\mathcal{D}$.

Lemma of growth in time

We fix $s = s_R$ and also the following two parameters

$$q = \frac{\kappa_0 C_0}{2s} \quad \text{and} \quad \eta_0 = \left(\frac{r_0}{R}\right)^{2s},$$

and denote $D_1 = U \times (0, qR^2]$.

Lemma (Lemma of growth in time)

Assume $w(\mathbf{x}, t) \in C^{2,1}_{\mathbf{x},t}(D_1) \cap C(\bar{D}_1)$. *If*

$$\mathcal{L}w \le 0 \text{ on } D_1 \quad \text{and} \quad w \le 0 \text{ on } \Gamma \times (0, qR^2),$$

then

$$\max\{0, \sup_U w(\mathbf{x}, qR^2)\} \le \frac{1}{1+\eta_0} \max\{0, \sup_U w(\mathbf{x}, 0)\}.$$

Proof.

Let $M = \max\{0, \sup_{\bar U} w(x,0)\}$, $\tilde W = M[1-\eta W]$, $\eta > 0$ selected later, $t_1 = qR^2$.
Applying maximum principle for $\tilde W$ gives

$$w(x,t_1) \le \tilde W(x,t_1) \le M(1-\eta C(s,R)) = M(1-\eta_0) \le M/(1+\eta_0).$$

Proposition (Homogeneous problem)

Assume $w(\mathbf{x},t) \in C^{2,1}_{\mathbf{x},t}(D) \cap C(\bar{D})$ *satisfies*

$$\mathcal{L}w = 0 \text{ in } D \quad \text{and} \quad w = 0 \text{ on } \Gamma \times (0,\infty).$$

Let $\eta_1 = \frac{\ln(1+\eta_0)}{qR^2}$. *Then*

$$-e^{-\eta_1 t} \inf_U |w(\mathbf{x},0)| \le w(\mathbf{x},t) \le (1+\eta_0)e^{-\eta_1 t} \sup_U |w(\mathbf{x},0)| \quad \forall (\mathbf{x},t) \in D.$$

Proposition (Non-homogeneous problem)

Assume $f_0 \in C(\bar{D})$ *and*
$\Delta_1 \stackrel{\text{def}}{=\!=} \sup_{U\times(0,\infty)} |f_0(\mathbf{x},t)| + \sup_{\Gamma\times(0,\infty)} |G(\mathbf{x},t)| < \infty$ *The solution* $w(\mathbf{x},t) \in C^{2,1}_{\mathbf{x},t}(D) \cap C(\bar{D})$ *satisfies*

$$|w(\mathbf{x},t)| \le C\left[e^{-\eta_1 t} \sup_U |w_0(\mathbf{x})| + \Delta_1\right] \quad \forall (\mathbf{x},t) \in D.$$

Bernstein's estimates

Proposition

Assume $f_0 \in C(\bar{D})$, $\nabla f_0 \in C(D)$, $\Delta_1 < \infty$ *and*

$$\Delta_3 \stackrel{\text{def}}{=\!=} \sup_D |\nabla f_0| < \infty.$$

For any $U' \Subset U$ *there is* $\tilde{M} > 0$ *such that if* $w(\mathbf{x}, t) \in C^{2,1}_{\mathbf{x},t}(D) \cap C(\bar{D})$ *is a solution of (19) that also satisfies* $w \in C^3_{\mathbf{x}}(D)$ *and* $w_t \in C^1_{\mathbf{x}}(D)$*, then*

$$|\nabla w(\mathbf{x}, t)| \le \tilde{M}\Big[1 + \frac{1}{\sqrt{t}}\Big]\Big[e^{-\eta_1 t} \sup_U |w(\mathbf{x}, 0)| + \Delta_1 + \sqrt{\Delta_3}\Big] \quad \forall (\mathbf{x}, t) \in U' \times (0, \infty$$

In unbounded domains

Outer domain $U = \mathbb{R}^n \setminus \bar{B}_{r_0}$.
Notation. For $R > r > 0$, denote $\mathcal{O}_r = \mathbb{R}^n \setminus \bar{B}_r$, $\mathcal{O}_{r,R} = B_R \setminus \bar{B}_r$.
Let $\Gamma = \partial U = \{\mathbf{x} : |\mathbf{x}| = r_0\}$ and $D = U \times (0, \infty)$.
Similar IBVP for σ and w.

Maximum principle for unbounded domain

Theorem

Let $T > 0$ and $w(\mathbf{x}, t)$ be a bounded function in $C^{2,1}_{\mathbf{x},t}(U_T) \cap C(\bar{U}_T)$ that solves $\mathcal{L}w = f_0$ in U_T, where $f_0 \in C(\bar{U}_T)$. Then

$$\sup_{\bar{U}_T} |w(\mathbf{x}, t)| \leq \sup_{\partial_p U_T} |w(\mathbf{x}, t)| + (T + 1) \sup_{\bar{U}_T} |f_0|.$$

Barrier function:

$$W(\mathbf{x}, t) \stackrel{\text{def}}{=\!=} (T - t)^{-s} e^{\frac{\varphi(\mathbf{x})}{T-t}} \quad \text{for } (\mathbf{x}, t) \in \mathcal{O}_{r_0,R} \times (0, T),$$

where constant $s > 0$ and function $\varphi(\mathbf{x}) > 0$ will be decided later.
Elementary calculations give

$$\mathcal{L}W = (T-t)^{-s-2} e^{\frac{\varphi}{T-t}} \Big\{ (T-t)\big(s - \nabla\cdot(\underline{\mathbf{A}}\nabla\varphi) - \mathbf{b}\cdot\underline{\mathbf{A}}\nabla\varphi\big) + \varphi - (\underline{\mathbf{A}}\nabla\varphi)\cdot\nabla\varphi \Big\}.$$

Then $\mathcal{L}W \geq 0$ if

$$s \geq \nabla \cdot (A\nabla\varphi) + \mathbf{b} \cdot \underline{\mathbf{A}}\nabla\varphi \quad \text{and} \quad \varphi \geq (\underline{\mathbf{A}}\nabla\varphi) \cdot \nabla\varphi.$$

Choose

$$\varphi(\mathbf{x}) = \kappa_1\Big(\varphi_1 + \int_{r_0}^{|\mathbf{x}|} r\phi(r)dr\Big),$$

where $\varphi_1 = \frac{C_1 r_0^2}{2} > 0$ and $\kappa_1 = \frac{C_1}{2C_0}$, and

$$s = s_R \overset{\text{def}}{=\!=} C_3(1+R).$$

Lemma of growth in spatial variables

Let $R > 0$ and $\ell \geq R + r_0$. Denote

$\mathcal{O}_R(\ell) = \mathcal{O}_{\ell-R,\ell+R} = \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x}| - \ell| < R\}$ and $\mathcal{S}_\ell = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = \ell\}$

Define the barrier function of Landis type

$$\mathcal{W}(\mathbf{x}, t) = \frac{1}{(t+1)^s} e^{-\frac{\psi(\mathbf{x})}{t+1}} \quad \text{for } |\mathbf{x}| \geq r_0, \ t \geq 0,$$

where parameter $s > 0$ and function $\psi > 0$. Then $\mathcal{LW} \leq 0$ if

$$s \geq \nabla \cdot (\underline{\mathbf{A}} \nabla \psi) + \mathbf{b} \cdot \underline{\mathbf{A}} \nabla \psi \quad \text{and} \quad \psi \leq (\underline{\mathbf{A}} \nabla \psi) \cdot \nabla \psi.$$

We can choose $s = C_3(1 + R)$ and

$$\psi(x, t) = \kappa_2 \int_\ell^{|x|} (r - \ell) \phi(r) dr.$$

Lemma

Given any $R > 0$ and $\ell \geq R + r_0$. Then the function $\mathcal{W}(\mathbf{x}, t)$ in (29) belongs to $C^{2,1}_{\mathbf{x},t}(D) \cap C(\bar{D})$ and satisfies $\mathcal{LW} \leq 0$ on $\mathcal{O}_R(\ell) \times (0, \infty)$.

Lemma (Lemma of growth in spatial variables)

Given $T > 0$, *let*

$$R = R(T) = C_4(1+T),$$
$$\eta_0 = \eta_0(T) = \Big(1 - \frac{1}{2^{C_5(T+1)}}\Big)\frac{1}{(T+1)^{2C_5(T+1)}},$$

where $C_4 = \max\{1, \frac{8C_3}{\kappa_2 e C_0}\}$ *and* $C_5 = C_3 C_4$. *Suppose* $w(\mathbf{x},t) \in C^{2,1}_{\mathbf{x},t}(U_T) \cap C(\bar{U}_T)$ *satisfies* $\mathcal{L}w \le 0$ *on* U_T *and* $w(\mathbf{x},0) \le 0$ *on* $\bar{U}$. *Let* ℓ *be any number such that* $\ell \ge R + r_0$, *then*

$$\max\Big\{0, \sup_{\mathcal{S}_\ell \times [0,T]} w(\mathbf{x},t)\Big\} \le \frac{1}{1+\eta_0} \max\Big\{0, \sup_{\bar{\mathcal{O}}_R(\ell)\times[0,T]} w(\mathbf{x},t)\Big\}.$$

Dichotomies

Lemma

Let $T > 0$ and R, η_0 and $w(\mathbf{x}, t)$ be as in Lemma 14. For $i \geq 1$, let

$$\bar{m}_i = \max\Big\{0, \sup_{\mathcal{S}_{r_0+iR}\times[0,T]} w(\mathbf{x}, t)\Big\}.$$

Part A (Dichotomy for one cylinder). Then for any $i \geq 1$, we have either of the following cases.

(a) *If $\bar{m}_{i+1} \geq \bar{m}_{i-1}$, then $\bar{m}_{i+1} \geq (1+\eta_0)\bar{m}_i$.*

(b) *If $\bar{m}_{i-1} \geq \bar{m}_{i+1}$, then $\bar{m}_{i-1} \geq (1+\eta_0)\bar{m}_i$.*

Part B (Dichotomy for many cylinders). For any $k \geq 0$, we have the following two possibilities:

(i) *There is $i_0 \geq k+1$ such that $\bar{m}_{i_0+j} \geq (1+\eta_0)^j \bar{m}_{i_0}$ for all $j \geq 0$.*

(ii) *For all $j \geq 0$, $\bar{m}_{k+j} \leq (1+\eta_0)^{-j} \bar{m}_k$.*

Theorem

Let $w \in C^{2,1}_{\mathbf{x},t}(U_T) \cap C(\bar{U}_T)$ be a bounded solution of the IBVP on U_T with $f_0 \in C(\bar{U}_T)$. If

$$\lim_{|\mathbf{x}|\to\infty} w_0(\mathbf{x}) = 0,$$

$$\lim_{|\mathbf{x}|\to\infty} \sup_{0\le t\le T} |f_0(\mathbf{x}, t))| = 0,$$

then

$$\lim_{r\to\infty} \Big(\sup_{\mathcal{S}_r\times[0,T]} |w(\mathbf{x}, t)| \Big) = 0.$$

Corollary

Let $w(\mathbf{x}, t) \in C^{2,1}_{\mathbf{x},t}(D) \cap C(\bar{D})$ *be a bounded solution of (19) on* D *with* $f_0 \in C(\bar{D})$. *Assume* $w_0 \in C(\bar{U})$, $G \in C(\Gamma \times [0, \infty))$ *are bounded, satisfy same conditions as above for each* $T > 0$. *Then there exists an increasing, continuous function* $r(t) > 0$ *satisfying* $\lim_{t\to\infty} r(t) = \infty$ *such that*

$$\lim_{t\to\infty} \Big(\sup_{\mathbf{x}\in\bar{\mathcal{O}}_{r(t)}} |w(\mathbf{x}, t)| \Big) = 0.$$

Dealing with weight $e^{\Lambda(x)}$

From $w(x,t)$, we return to $\sigma(x,t) = we^{\Lambda(x)}$.

• In the case $n \geq 3$,

$$0 < C_7^{-1} \leq e^{\Lambda(\mathbf{x})} \leq C_7 \quad \forall |\mathbf{x}| \geq r_0.$$

• In the case $n = 2$,

$$e^{\Lambda(\mathbf{x})} \leq C_8 \quad \forall |\mathbf{x}| \geq r_0.$$

Theorem

Let $n \geq 3$*. Assume* **(E1)** *and*

$$\Delta_{10} \stackrel{\text{def}}{=\!=} \max\{\sup_U |\sigma_0(\mathbf{x})|, \sup_{\Gamma\times[0,\infty)} |g(\mathbf{x},t)|\} < \infty,$$

$$\Delta_{11} \stackrel{\text{def}}{=\!=} \sup_D |\nabla\cdot(\underline{\mathbf{A}}(\mathbf{x})\mathbf{c}(\mathbf{x},t))| < \infty.$$

Then,

(i) *There exists a solution* $\sigma(\mathbf{x},t) \in C^{2,1}_{\mathbf{x},t}(D) \cap C(\bar{D})$*. This solution is unique in class of solutions* $\sigma(\mathbf{x},t)$ *that satisfy*

$$\sup_{U\times[0,T]} |\sigma(\mathbf{x},t)| < \infty \quad \textit{for any } T > 0.$$

(ii) *There is* $C > 0$ *such that for* $(\mathbf{x},t) \in D$*,*

$$|\sigma(\mathbf{x},t)| \leq C\big[\Delta_{10} + \Delta_{11}(t+1)\big].$$

Theorem (continued)

(iii) *In addition, if*

$$\lim_{|\mathbf{x}|\to\infty} \sigma_0(\mathbf{x}) = 0 \quad \textit{and} \quad \lim_{|\mathbf{x}|\to\infty} \sup_{0\le t\le T} |\nabla\cdot(\underline{\mathbf{A}}(\mathbf{x})\mathbf{c}(\mathbf{x},t))| = 0 \ \textit{for each } T>0,$$

then

$$\lim_{r\to\infty}\Big(\sup_{\mathcal{S}_r\times[0,T]}|\sigma(\mathbf{x},t)|\Big) = 0 \quad \textit{for any } T>0,$$

and furthermore, there is a continuous, increasing function $r(t)>0$ *with* $\lim_{t\to\infty} r(t)=\infty$ *such that*

$$\lim_{t\to\infty}\Big(\sup_{\mathbf{x}\in\bar{\mathcal{O}}_{r(t)}}|\sigma(\mathbf{x},t)|\Big) = 0.$$

Theorem

Let $n = 2$ *and* $\hat{S}(r)$ *be a solution (for the steady state) with* $c_1, c_2 < 0$. *Assume* **(E1)** *and*

$$\Delta_{12} \overset{\text{def}}{=\!=} \max\{\sup_U e^{-\Lambda(\mathbf{x})}|\sigma_0(\mathbf{x})|, \sup_{\Gamma\times[0,\infty)} |g(\mathbf{x},t)|\} < \infty,$$

$$\Delta_{13} \overset{\text{def}}{=\!=} \sup_D e^{-\Lambda(\mathbf{x})}|\nabla\cdot(\underline{\mathbf{A}}(\mathbf{x})\mathbf{c}(\mathbf{x},t))| < \infty.$$

Then the following statements hold true.
(i) *There exists a solution* $\sigma(\mathbf{x},t) \in C^{2,1}_{\mathbf{x},t}(D)\cap C(\bar{D})$. *This solution is unique in class of solutions* $\sigma(\mathbf{x},t)$ *that satisfy*

$$\sup_{U\times[0,T]} e^{-\Lambda(\mathbf{x})}|\sigma(\mathbf{x},t)| < \infty \quad \text{for any } T > 0.$$

(ii) *There is* $C > 0$ *such that for* $(\mathbf{x},t)\in D$,

$$|\sigma(\mathbf{x},t)| \le C\big[\Delta_{12} + \Delta_{13}(t+1)\big].$$

Theorem (continued)

(iii) *In addition, if*

$$\lim_{|\mathbf{x}|\to\infty} e^{-\Lambda(\mathbf{x})}\sigma_0(\mathbf{x}) = 0 \quad \textit{and} \quad \lim_{|\mathbf{x}|\to\infty} \sup_{0\le t\le T} e^{-\Lambda(\mathbf{x})}|\nabla\cdot(\underline{\mathbf{A}}(\mathbf{x})\mathbf{c}(\mathbf{x},t))| = 0$$

for each $T>0$, *then*

$$\lim_{r\to\infty}\Big(\sup_{\mathcal{S}_r\times[0,T]}|\sigma(\mathbf{x},t)|\Big) = 0 \quad \textit{for any } T>0,$$

and furthermore, there is a continuous, increasing function $r(t)>0$ *with* $\lim_{t\to\infty} r(t) = \infty$ *such that*

$$\lim_{t\to\infty}\Big(\sup_{\mathbf{x}\in\bar{\mathcal{O}}_{r(t)}}|\sigma(\mathbf{x},t)|\Big) = 0.$$

THANK YOU FOR YOUR ATTENTION!