Qualitative properties of the non-linear flow in porous media and application in engineering

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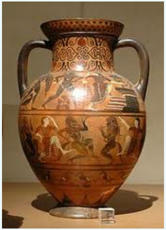
Porous Media



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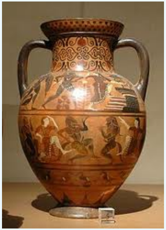
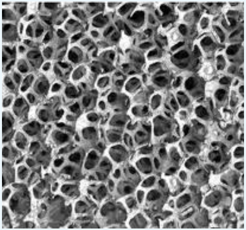
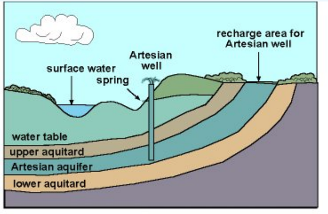
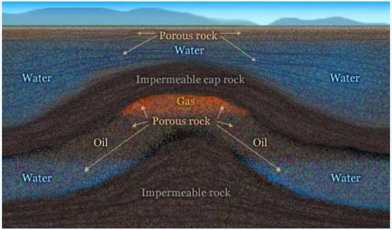


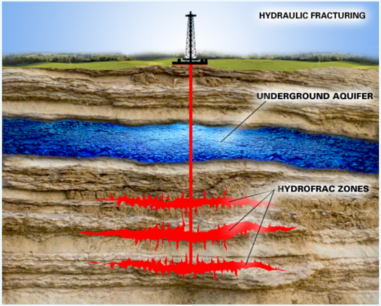


Intended Application - Geoscience



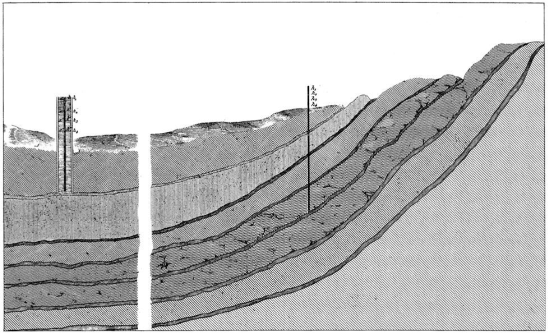
CROSS SECTION OF EARTH SHOWING PARTS OF AN AQUIFER



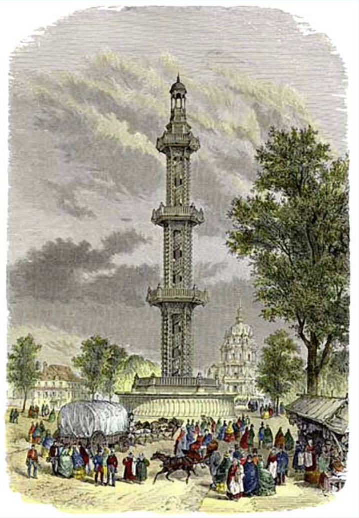


The Beginnings: Darcy's Law

Henry Darcy's experiments (1856)



Cross section of the Paris Basin

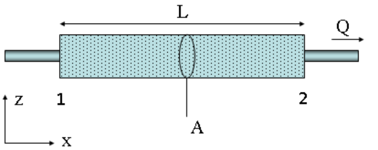


Grenelle artesian well in Paris

R. Ritzi, P. Bobeck, *Comprehensive principles of quantative hydrogeology established by Darcy and Dupuit*, Water Resources Research, V 44, W10402, 2008

Darcy's Law, 1856

Homogeneous, 1D, single-phase flow:



$$Q = -\frac{kA}{\mu}\frac{p_2 - p_1}{L}$$

Q	–	rate of flow
A	–	cross-section
k	–	permeability of porous media
μ	–	viscosity of the fluid
$p_2 - p_1$	–	pressure drop
L	–	flow path length

$$u = -\frac{k}{\mu}\nabla p$$

$u = \dfrac{Q}{\phi}$	–	pore velocity (through pores)
ϕ	–	porosity of porous media ($\phi = V_e/V_t$)
p	–	pressure function

About Darcy's Law

- Darcy's Law: $u = -\alpha \nabla p$ (hydrogeology)

- Fick's Law: $J = -D\nabla\phi$ (population dynamics)

  J - diffusion flux; D - diffusivity; ϕ - concentration

- Fourier's Law: $q = -K\nabla T$ (heat conduction)

  q - local heat flux; K - material conductivity; T - temperature

- Ohm's Law: $J = \sigma E$ (electrodynamics)

  J - current density; σ - conductivity; E - electric field

- Homogenization of Stokes equations in inhomogeneous domain

Linear parabolic equation for pressure only

- The Darcy's Law $\quad u = -\dfrac{k}{\mu}\nabla p$

Linear parabolic equation for pressure only

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- The continuity equation (conservation law)

$$\phi(x)\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho u) \qquad\qquad \rho(p,t) \quad \text{- density}$$

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- The equation of state for slightly compressible fluids [Bear, Muskat]

$$\frac{\partial \rho}{\partial p} = \gamma\rho \qquad \text{or} \qquad \rho(p) = \rho_0 e^{\gamma p} \qquad \left[\gamma \sim 10^{-8}\right]$$

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Obtain

$$\boxed{\frac{\partial p}{\partial t} = \frac{1}{\gamma}\left[\nabla \cdot \left(\frac{k}{\mu} \nabla p\right) + \gamma \cdot \frac{k}{\mu} \left|\nabla p\right|^2\right]}$$

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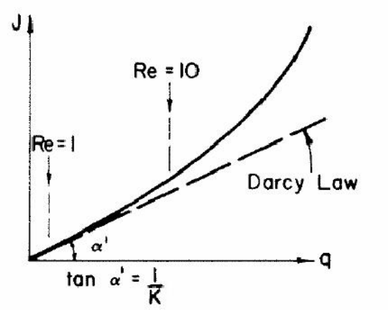
Nonlinearity of the Flow

Range of validity of Darcy law

Darcy: "*below a certain velocity threshold, the pressure drop is linearly proportional to velocity, and above that threshold, the pressure drop is proportional to velocity squared*"



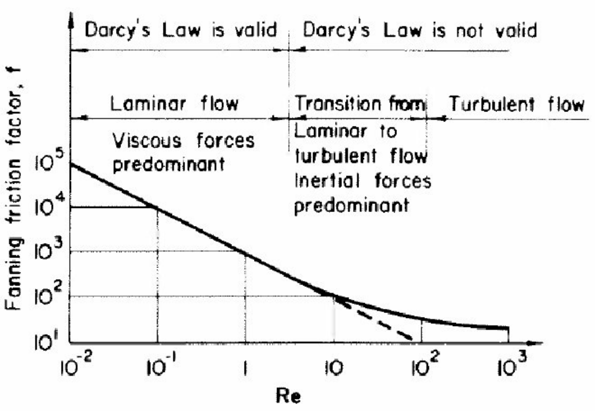
$$-\Delta P = \alpha q \quad \text{and} \quad -\Delta P = \alpha q + \beta q^2$$

Reynolds number

$$\mathrm{Re} = \frac{\text{inertial forces}}{\text{viscous forces}}$$

M. Muskat, *Flow of Homogeneous Fluids Through Porous Media*, Springer, 1982
R. Ritzi, P. Bobeck, *Comprehensive principles of quantitative hydrogeology established by Darcy and Dupuit*, Water Resources Research, V 44, W10402, 2008

Various factors contributing to nonlinearity



- inertial forces
- high velocity in fractures and near wells
- friction in a pipe
- high viscosity of the fluid
- not clear

J. Bear, *Dynamics of Fluids in Porous Media*, Dover, 1988

Experimental observations - Philipp Forchheimer

$\Delta P = p_2 - p_1$ - pressure difference; q - rate of flow

- Darcy's law (1856): $-\Delta P = \alpha q$
- Forchheimer "two terms" law (1901): $-\Delta P = \alpha q + \beta q^2$
- Forchheimer "three terms" law (1901): $-\Delta P = Aq + Bq^2 + Cq^3$
- Forchheimer "power" law (1930): $-\Delta P = \alpha q + \beta q^m$
  $1.6 \leq m \leq 2$

P. Forchheimer, *Wasserbewegung durch Boden Zeit*, 45, 1782, Ver. Deut. Ing., 1901

Recent advances on nonlinearity of the flow

- Even for low Re it is possible $|\nabla p| \sim |u|^3$
  (Balhoff&Wheeler, 2010; Wodie&Levy, 1991; Auriault, 1991)
- In a presence of fractures Darcy's law is not valid even for $Re \approx 1$ (Tavera et al, 2006)
- Forchheimer law needs correction (Marušić-Paloka&Mikelić, 2000)
- Generalized Forchheimer equation
  (Aulisa, Bloshanskaya, Hoang, Ibragimov, 2009)

Generalized Forchheimer equation [ABHI09]

$$\boxed{g(|\mathbf{u}|)\mathbf{u} = -\nabla p}$$

Function $g(s)$ is a GPPC:

$$g(s) = a_0 + \sum_{j=1}^{k} a_j(x) s^{\alpha_j} = \overbrace{a_0}^{>0} + \overbrace{a_1}^{\geq 0} s^{\alpha_1} + \overbrace{a_2}^{\geq 0} s^{\alpha_2} + \ldots + \overbrace{a_k}^{\geq 0} s^{\alpha_k}$$

Aulisa, Bloshanskaya, Hoang, Ibragimov, *Analysis of generalized Forchheimer flows of compressible fluids in porous media*, J. Math. Phys. 50, 103102 (2009), 44 pp

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- Darcy: $\alpha u = -\nabla p$: $\quad a_0 = \alpha, a_j = 0, j \geq 1$
- 2Forch: $(\alpha + \beta|u|)u = -\nabla p$: $\quad a_0 = \alpha, a_1 = \beta, \alpha_1 = 1; a_j = 0, j \geq 2$
- Power: $(\alpha + \beta|u|^{m-1})u = -\nabla p$:
  $a_0 = \alpha, a_1 = \beta, \alpha_1 = m - 1, a_j = 0, j \geq 2$

Aulisa, Bloshanskaya, Hoang, Ibragimov, *Analysis of generalized Forchheimer flows of compressible fluids in porous media*, J. Math. Phys. 50, 103102 (2009), 44 pp

Two equivalent forms of g-Forchheimer equation

$$g(|u|)u = -\nabla p \qquad \Longleftrightarrow \qquad u = -K(|\nabla p|)\nabla p$$

g-Forchheimer eq. — generalized (nonlinear) Darcy eq.

Aulisa, Ibragimov, Valko, Walton *Mathematical framework ... for Forchheimer flows in porous media*, WSPC, 2008

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g-Forchheimer eq. generalized (nonlinear) Darcy eq.

- In general $K(\xi)$ is not analytic
- For Forchheimer two-terms law: $\alpha u + \beta |u| u = -\nabla p$

$$K(|\nabla p|) = \frac{2}{\alpha + \sqrt{\alpha^2 + 4\beta|\nabla p|}}$$

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Features of nonlinear permeability $K(|\nabla p|)$

- Degeneracy similar to p-Laplace equations

$$\frac{C_1}{(1+\xi)^a} \le K(\xi) \le \frac{C_2}{(1+\xi)^a} \qquad \text{where } a = a(deg(g)) < 1$$

- Strict weighted monotonicity:

$$\int_U (K(|\nabla u_1|)\nabla u_1 - K(|\nabla u_1|)\nabla u_1) \cdot \nabla(u_1 - u_2)\, dx \ge C_\Phi \|\nabla(u_1 - u_2)\|_H^2$$

$$C_\Phi = C_0(1 + \max(\|\nabla u_1\|_{L^{2-a}}, \|\nabla u_1\|_{L^{2-a}}))^{-a}$$

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Nonlinear parabolic equation for pressure only

- ~~The Darcy's Law~~ Nonlinear Darcy eq. $u = -K(|\nabla p|)\nabla p$
- The continuity equation (conservation law) $\phi(x)\dfrac{\partial \rho}{\partial t} = -\nabla \cdot (\rho u)$
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Obtain

$$\boxed{\gamma \cdot \frac{\partial p}{\partial t} = \nabla \cdot (K(|\nabla p|)\nabla p)}$$

Aulisa, Bloshanskaya, Hoang, Ibragimov, *Analysis of generalized Forchheimer flows of compressible fluids in porous media*, J. Math. Phys. 50, 103102 (2009), 44 pp

IBVP(Prescribed Pointwise Flux or Pressure on the Boundary)

$$\frac{\partial p}{\partial t} = \nabla \cdot (K(|\nabla p|)\nabla p) \text{ in} U$$

on Γ_i

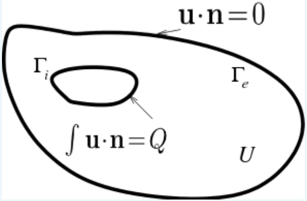
$$-K(|\nabla p|)\nabla p \cdot N = q(x,t) \text{ Neumann BC}$$

or

$$p(x,t) = \Psi(x,t) \text{ Dirichlet BC}$$

$$\left.\frac{\partial p}{\partial N}\right|_{\Gamma_e} = 0 \text{ Non-flow BC}$$

$$p(x,0) = p_0(x)$$



In case of Dirchlet we are interesting in the BD which growing to ∞ at time-infinity. It is expected that if $\Psi(x,t)$ is stabilizing to "regular function" problem is reducing to Darcy case.

Diffusive Capacity/Productivity Index

Engineering parameters of well capacity *Productivity Index (PI) at time of observation*

$$\mathrm{PI} = \frac{\text{Hydrocarbon Production}}{\text{Reservoir Pressure} - \text{Well Pressure}}$$

Averages over the domain and on the boundary

$$\overline{p}_U(t) = \frac{1}{|U|}\int_U p(x,t)\,dx \qquad \overline{p}_{\Gamma_i}(t) = \frac{1}{|\Gamma_i|}\int_{\Gamma_i} p(x,t)\,ds$$

$$PDD(t) = \overline{p}_U(t) - \overline{p}_{\Gamma_i}(t)$$ *pressure drawdown* in the domain U

$$Q(t) = \int_{\Gamma_i} u \cdot N\,ds$$ *total flux* through Γ_i

Diffusive Capacity $$\boxed{J_g(\Gamma_i)(t) = \frac{Q(t)}{PDD(t)}} \qquad \left(PI = \frac{d}{dt} ln\left(\int p dx\right)\right)$$

Important feature of diffusive capacity: *in case of PSS regime $J_g(\Gamma_i)(t)$ is time invariant, a.e. PI is constant and for this reason is used by engineers to evaluate well capacity*

Time Independent Diffusive Capacity/Productivity Index, Linear Darcy Case

Productivity Index (PI) in long time observation

Prescribed constant pressure on the well

$$\mathrm{PI}_1 = \lambda |U|$$

Prescribed constant rate

$$\mathrm{PI}_2 = \frac{|U|}{\int W(x)dx}$$

Here λ first eigenvalue for Laplace equation with corresponding boundary conditions, and

$$-A = \frac{Q_s}{|U|} = \Delta W, \quad W(x)|_{\Gamma_i} = 0, \quad \frac{\partial W}{\partial N}\Big|_{\Gamma_e} = 0$$

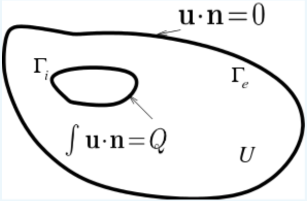
Mathematical inequality (originally observed by engineers)

$$PI_2 > PI_1$$

A.I., D. Khalmanova, P.Valko, J.Walton, 2005, SIAM, JAM,, 65(6).

IBVP-I(Given Total Flux on the Boundary)

$$\frac{\partial p}{\partial t} = \nabla \cdot (K(|\nabla p|)\nabla p)$$
$$-\int_{\Gamma_i} K(|\nabla p|)\nabla p \cdot N\, ds = Q(t)$$
$$\left.\frac{\partial p}{\partial N}\right|_{\Gamma_e} = 0$$
$$p(x,0) = p_0(x)$$



No uniqueness $\Longrightarrow$ additional constraints on the boundary data

Constraint on the Boundary Data

Solution $p(x,t)$ of IBVP-I:

Define trace $\quad p(x,t)|_{\Gamma_i} = \psi_0(x,t) = \gamma(t)+\psi(x,t)$

Assume

$$\int_{\Gamma_i} \psi(x,t)\, ds = 0,$$

Uniqueness:

If $p_1(x,t)$ and $p_2(x,t)$ are solut-s of IBVP-I with the same $\psi(x,t)$, then

$$p_1(x,t) = p_2(x,t)$$

PSS Profile

Our intention is qualitative analysis of the solution of IBVP-I in case when total flux $Q(t) \to Q_s \equiv const$:
PSS regime characterizes by two parameters on the boundary:
I. Profile of the pressure $\varphi_0(x)$, and II. Total flux Q_s.
These two parameters defines the PSS profile/regime:

$$p_s(x,t) = -\frac{Q_s}{|U|}t + W(x)$$

where *basic profile* $W(x)$ is a solution of BVP

$$\begin{aligned} &-A = \frac{Q_s}{|U|} = \nabla \cdot (K(|\nabla W|)\nabla W) \\ &W(x)|_{\Gamma_i} = \varphi_0(x) \\ &\frac{\partial W}{\partial N}\Big|_{\Gamma_e} = 0 \end{aligned}$$

for known function $\varphi_0(x)$

Boundary Data Comparison

$$Q(t) \qquad \text{vs} \qquad Q_s$$

and

$$\psi(x,t) \qquad \text{vs} \qquad \varphi(x) = \varphi_0(x) - \frac{1}{|\Gamma_i|}\int_{\Gamma_i} \varphi_0(s)\,ds$$

Extensions $\Psi(x,t)$, $\Phi(x)$ defined on the whole domain U such that:

$$\Psi(x,t)|_{\Gamma_i} = \psi(x,t) \qquad\qquad \Phi(x)|_{\Gamma_i} = \varphi(x)$$

$$\Psi(x,t), \Psi_t(x,t), \Phi(x) \in W_2^1(U)$$

!!! No conditions on the average

$$\gamma(t) = \frac{1}{|\Gamma_i|} \int_{\Gamma_i} \psi_0(x,t)\, ds$$

Results are obtained under some constraints on two parameters

$$\Delta_Q(t) = Q(t) - Q_s \qquad \Delta_\Psi(x,t) = \Psi(x,t) - \Phi(x)$$

Aim:

the comparison of fully transient solution $p(x,t)$ with PSS solution $p_s(x,t)$ in terms of Diffusive Capacity $J_g(t)$

Main result:

$$J_g(t) - J_{g,PSS} \to 0 \quad \text{as} \quad t \to \infty$$

under certain assumptions on the differences in boundary data

$$\Delta_Q(t) \quad \text{and} \quad \Delta_\Psi(x,t)$$

Proof of Asymptotic Convergence of Transient Diffusive Capacity based on Qualitative Property of the Solution

Long term dynamics of the characteristics of the solution as $t \to \infty$

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- $$\int_U |p|^\alpha \, dx \xrightarrow{?}$$

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Long term dynamics of the characteristics of the solution as $t \to \infty$

- $\|\nabla p\|_{L^{2-a}(U)} \overset{?}{\leq} C$
- $\int_U |p|^\alpha \, dx \xrightarrow{?}$
- $\int_U |p_t|^2 dx \xrightarrow{?}$
- $\int_U |\nabla(p)|^{2-a} \, dx \xrightarrow{?}$
- $\cdots \xrightarrow{?}$

Lemma

The difference between transient and PSS PI's

$$J_g(t) - J_{g,PSS} \to 0 \quad \textit{if} \quad \|\nabla(p - p_s)\|_{L^{2-a}} \to 0 \quad \textit{and} \quad \Delta_Q(t) \to 0 \qquad (1)$$

Proof: Observe

$$\Delta_p(t) = \frac{1}{|\Gamma_i|}\int_{\Gamma_i}(p - p_s)\, ds - \frac{1}{|U|}\int_U (p - p_s)\, dx = \frac{1}{|\Gamma_i|}\int_{\Gamma_i} \widetilde{z}\, ds,$$

where

$$\widetilde{z}(x,t) = p - p_s - \frac{1}{|U|}\int_U (p - p_s)\, dx$$

Applying Trace theorem and Poincaré inequality, as $\int_U \widetilde{z}\, dx = 0$, one has

$$|\Delta_p(t)|^{2-a} \le C \int_U |\nabla(p - p_s)|^{2-a}\, dx.$$

Lemma

Let

$$F_1(t) = -A_0 + \frac{1}{|U|}\int_0^t \Delta_Q(\tau)\,d\tau + \frac{1}{|U|}\int_U \Delta_\Psi\,dx.$$

Then

$$\|\nabla(p - p_s)\|_{L^{2-a}} \to 0 \tag{2}$$

If

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If

- $\lim\limits_{t\to\infty} \left(\Delta_Q(t) + \|\nabla(\Delta_\Psi)\|_{L^2} + |F_1(t)\Delta_Q(t)|\right) = 0$

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- $\int_U |p_t - p_{s,t}|^2 dx \to 0 \quad as \quad t \to \infty$

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If

- $\lim_{t\to\infty} (\Delta_Q(t) + \|\nabla(\Delta_\Psi)\|_{L^2} + |F_1(t)\Delta_Q(t)|) = 0$
- $\|\nabla p_s\|_{L^{2-a}(U)} \leq C$
- $\int_U |p_t - p_{s,t}|^2 dx \to 0 \quad as \quad t \to \infty$
- $\int_U |p - p_s - B.C.|^2\,dx \to 0 \quad as \quad t \to \infty$

Theorem

If $Q(t)$ and $\Psi(x,t)$ satisfy assumptions $A4$

$$\limsup_{t\to\infty}\int_U |\overline{p}_t-\overline{p}_{s,t}|^2\,dx=\limsup_{t\to\infty}\int_U |\overline{p}_t+A|^2\,dx=0 \qquad (3)$$

Proof.

Proof follows from the following differential inequality, which we obtained for the solution of our IBVP for the functional $y(t)=\int_U |\overline{p}_t+A|^2\,dx$. If $p(x,t)$ is a solution of IBVP with given total fluxes, then under constraint on structure of the trace on the boundary

$$y'(t)\leq -Cf(y(t))+CA_3(t) \qquad (4)$$

where

$$f(s)=s^{\theta}-\varepsilon_p s^{2\theta}-\varepsilon_2 s \qquad (5)$$

Assumption 1:

$$\left(|\Delta_Q(t)| + |Q'(t)|\right) \cdot \int_0^t |\Delta_Q(\tau)|\, d\tau + |\left(\Delta_Q(t)\right)'| \leq C$$

$$\int_U \left(|\Delta_\Psi| + |\nabla(\Delta_\Psi)|^{2-a} + |(\Delta_\Psi(x,t))_t|^b + |\nabla\left(\Delta_\Psi(x,t)\right)_t|^2\right) dx \leq C$$

Assumption 2:

$$\lim_{t\to\infty} |\Delta_Q(t)| \cdot \int_0^t |\Delta_Q(\tau)|\, d\tau = 0$$

$$\lim_{t\to\infty} \int_U \left(|\Delta_\Psi|^2 + |\nabla(\Delta_\Psi)|^2\, dx + |(\Delta_\Psi(x,t))_t|^2\right) dx = 0$$

Assumption 3:

$$\lim_{t\to\infty} \left[\int_U \left(|\nabla(\Delta_\Psi(x,t))_t|^2 + |(\Delta_\Psi(x,t))_{tt}|^2\right) dx + \left(\Delta_Q(t)\right)'\right] = 0$$

What is Next?

1. Ideal Gas $\rho = Mp$

$$\frac{\partial p}{\partial t} = \nabla \cdot (K(p, |\nabla p|) p \nabla p)$$

$$-\int_{\Gamma_i} K(|\nabla p|) p \nabla p \cdot N \, ds = Q(t)$$

$$\left. \frac{\partial p}{\partial N} \right|_{\Gamma_e} = 0$$

$$p(x, 0) = p_0(x)$$

$$PI = \frac{d}{dt} ln(\int p^2 dx)$$

2. Dynamic Boundary Conditions

$$\frac{\partial p}{\partial N} = Ap_t + B\frac{p}{t} - q_0 t^{-a}$$

3. Fluctuations in boundary data, which naturally lead to equations of stochastic type

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Thank You!