LDP and the zero viscosity limit for 2D NSE

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Description of the models

D bounded domain of $\mathbb{R}^2$ with "regular" boundary, $\nu > 0$ viscosity

$$\partial_t u - \nu \Delta u + (u.\nabla)u + \nabla p = F \quad \text{in} \quad D,$$

subject to conditions

$$\operatorname{div} u = \nabla . u = 0 \quad \text{in} \quad D$$
$$u.n = 0 \quad \text{and} \quad \operatorname{curl} u = 0 \quad \text{on} \quad \partial D$$

$u = (u^1(x,t), u^2(x,t))$ fluid velocity, $p(x,t)$ pressure, and $F(x,t)$ external force (will be random)
∇u gradient, $\operatorname{curl} u := \partial_1 u_2 - \partial_2 u_1$, $\Delta u = (\sum_{i=1,2} \partial_i^2 u^k, k = 1,2)$
Stokes, $\operatorname{div} u = \sum_{i=1,2} \partial_i u_i$, n outwards normal to ∂D

$$H = \{f \in \left[L^2(D)\right]^2 : \operatorname{div} f = 0 \text{ in } D \text{ and } f.n = 0 \text{ on } \partial D\}$$

$$V = W^{1,2}(D)^2 \cap H$$

Description of the models

The NS equations

• $(H, |\ |)$ and $(V, \|\ \|)$ Hilbert spaces, scalar products $(.,.)$ and $((.,.))$
curvature at the boundary k : for $u, v \in V$

$$\int_{\partial D} k(r)u(r).v(r)dr \le C\|u\|\,\|v\|,$$
$$\int_{\partial D} k(r)|u(r)|^2 dr \le \epsilon\|u\|^2 + C(\epsilon)|u|^2$$

$(au\ ,\ v) = \sum_{j=1,2} \int_D \nabla u^j \cdot \nabla v^j\, dx - \int_{\partial D} k(r)u(r).v(r)dr$
$-\Delta = A$ where $(Au, v) = a(u, v)$ with a bilinear on $Dom(\Delta)$

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- $\mathcal{H} = H^{1/2,2} \subset L^4(D)^2 \cap H$ **Interpolation space**:

$\|u\|_{\mathcal{H}}^2 \leq C|u|\|u\|$

$|\langle B(u,v), w\rangle| \leq C\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}\|w\|$

Noise and diffusion coefficient

- $W(t)$ H-valued Wiener process
  Covariance operator Q symmetric non-negative on H,
  $Trace(Q) < +\infty$, $\quad H_0 = Q^{\frac{1}{2}} H$
  scalar product $(\phi, \psi)_0 = (Q^{-\frac{1}{2}}\phi, Q^{-\frac{1}{2}}\psi)$
  $L_Q = \{S \in L(H_0, H) \, : \, SQ^{\frac{1}{2}} \text{ Hilbert Schmidt from } H \text{ to } H\}$,
  $|M|^2_{L_Q} = Trace(M\,Q\,M^*)$, where M^* is the adjoint of M.

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  $|M|^2_{L_Q} = Trace(M\, Q\, M^*)$, where M^* is the adjoint of M.
- $\sigma \in C([0,T] \times V) \to L_Q := L_Q(H_0, H)$ There exist constants K_i and L_i such that for $t \in [0,T]$, $\phi, \psi \in V$
  $|\sigma(t,\phi)|^2_{L_Q} \le K_0 + K_1|\phi|^2 + K_2\|\phi\|^2$,
  $|\sigma(t,\phi) - \sigma(t,\psi)|^2_{L_Q} \le L_1|\phi - \psi|^2 + L_2\|\phi - \psi\|^2$.
  (in the above conditions, σ may depend on the gradient of the solution)

$\sigma(u)$ estimated in terms of u in V with constants K_2 and L_2

$$d_t u(t) + [\nu A u(t) + B(u(t), u(t))]dt = \sigma(t, u(t))\, dW(t), \;\; u(0) = \zeta \quad (1)$$

Theorem

Let $\nu > 0$ and $E|\zeta|^4 < \infty$. Then for K_2 small enough and $L_2 < 2\nu$, there exists $C = C(K_i, L_i, T)$ such that (1) has a unique solution $u \in X = C([0,T]; H) \cap L^2(0,T;V)$. Furthermore,

$$E\Big(\sup_{0\leq t\leq T} |u(t)|^4 + \int_0^T \|u(t)\|^2\, dt + \int_0^T \|u(t)\|^4_{\mathcal{H}}\, dt \Big) \leq C\,(1 + E|\zeta|^4)$$

Sritharan-Sundar (Navier Stokes), Chueshov-M. (general hydrodynamical models),

Inviscid LDP - Aim

Let the positive viscosity coefficient $\nu \to 0$ and

$$d_t u^\nu(t) + \left[\nu A u^\nu(t) + B(u^\nu(t), u^\nu(t))\right] dt = \sqrt{\nu}\, \sigma_\nu(u^\nu(t))\, dW(t)$$

with initial condition $u^\nu(0) = \zeta$.
Prove exponential decay of $P(u^\nu(.) \in \Gamma)$ as $\nu \to 0$ for $\Gamma \subset \mathcal{X} \subset X = C([0,T]; H) \cap L^2(0,T;V)$ that is

$$\text{if } \sigma_\nu \to \sigma_0 \text{ as } \nu \to 0, \quad \text{then} \quad \text{"} \lim_{\nu \to 0} \text{"} \nu \ln P(u^\nu \in \Gamma)$$

in terms of some rate function and interior (resp. closure) of Γ in a $\mathcal{X}$ with some "appropriate topology"
So far the LDP was proved for fixed ν when the noise is multiplied by $\epsilon \to 0$ (Shritharan-Sundar for the 2D NSE ; Chueshov-M. for more general models ; Röckner-Zhang for the 3D tamed equation)

LDP

Definition

The random family (u^ν) is said to satisfy a large deviation principle on $\mathcal{X}$ with the good rate function I if the following conditions hold:
I is a good rate function. The function $I : \mathcal{X} \to [0, \infty]$ is such that for each $M \in [0, \infty[$ the level set $\{\phi \in \mathcal{X} : I(\phi) \leq M\}$ is a compact subset of $\mathcal{X}$.
For $A \in \mathcal{B}$, set $I(A) = \inf_{u \in A} I(u)$.
Large deviation upper bound. For each closed subset F of $\mathcal{X}$:

$$\limsup_{\nu \to 0} \nu \log P(u^\nu \in F) \leq -I(F).$$

Large deviation lower bound. For each open subset G of $\mathcal{X}$:

$$\liminf_{\nu \to 0} \nu \log P(u^\nu \in G) \geq -I(G).$$

Transfert of the LDP - small perturbation

- $\sqrt{\nu}W$ satisfies a LDP in $C([0,T],H)$ with good rate function $J(\phi)=\frac{1}{2}\int_0^T |\dot{\phi}(t)|_0^2 dt$ if $\phi \in \mathbb{H}$ and $J(\phi)=+\infty$ otherwise

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- Let $\mathcal{G}^\nu : C([0,T],H)\to X$ be defined by $\mathcal{G}^\nu(\sqrt{\nu}W)=u^\nu$ solution.

  Multiplicative noise and $\mathcal{G}^\nu$ is NOT continuous

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  Multiplicative noise and $\mathcal{G}^\nu$ is NOT continuous
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Similar result proved by:
• Mariani for conservation laws
• Bessaih-M. for shell models of turbulence.
Make **stronger assumptions** on $\sigma_\nu\to\sigma_0$ (no gradient in σ_ν; raise the regularity; control $H^{1,q}$ norms):
The topology on $\mathcal{X}$ in NOT optimal
Why ? The rate function is formulated in terms of the unique solution to the "irregular" inviscid case, for $h\in L^2(0,T;H_0)$

$$du_h^0(t)+B(u_h^0(t),u_h^0(t))\,dt=\sigma_0(u_h^0(t))\,h(t)\,dt\,,\quad u_h^0(0)=\zeta$$

Existence of the controlled Euler equation

$$\text{Let } |\sigma_0(t,u)|^2_{L_Q} \le K_0 + K_1 |u|^2, \; |\text{curl } \sigma_0(t,u)|^2_{L_Q} \le K_0 + K_1 \|u\|^2$$

$$|\sigma_0(t,u) - \sigma_0(t,v)|^2_{L_Q} \le L_1 |u-v|^2,$$

$$|\text{curl } [\sigma_0(t,u) - \sigma_0(t,v)]|^2_{L_Q} \le L_1 \|u-v\|^2$$

Proposition

Existence of the solution *Under the previous assumptions, for* $h \in \mathcal{A}_M$ *(i.e. a.s.* $\int_0^T \|h(s)\|_0^2 ds \le M$*) the controlled Euler equation*

$$du_h^0(t) + B(u_h^0(t), u_h^0(t))dt = \sigma_0(u^0(t))h(t)dt, \quad u_h^0(0) = \zeta \in V$$

has a solution $u_h^0 \in C([0,T];H) \cap L^\infty(0,T;V)$ *and a.s.*

$$\sup\{\|u_h^0(t)\| \; : \; h \in \mathcal{A}_M, \; t \in [0,T]\} \le C(T,M)(1 + \|\zeta\|)$$

Uniqueness of the solution to the controlled Euler equation

Condition (Cq) $\mathrm{curl}\ \sigma_0 \in C([0,T] \times H^{1,q}; L(H_0, H^{1,q}))$

$$|\mathrm{curl}\ \sigma_0(t,u)|_{L(H_0,L^q)} \leq K_0 + K_1\|u\|_q + K_2\|\mathrm{curl}\ u\|_q$$

Theorem

Suppose furthermore that $\mathrm{curl}\ \zeta \in L^\infty(D)$ *and that Condition* **(Cq)** *holds for every* $q \in [2,\infty)$. *Then for* $h \in \mathcal{A}_M$

$$du_h^0(t) + B(u_h^0(t), u_h^0(t))dt = \sigma_0(t, u^0(t))h(t)dt, \quad u_h^0(0) = \zeta$$

has a **unique solution in** $C([0,T],H) \cap L^\infty(0,T;H^{1,q})$ *; there exists a constant* $C(T,M)$ *such that for every* $h \in \mathcal{A}_M$ *(*$\int_0^T |h(s)|_0^2 ds \leq M$ *a.s.) and* $q \in (2,\infty)$

$$\sup_{t\in[0,T]} \|u_h^0(t)\|_{H^{1,q}} \leq q\, C(T,M)[1 + \|\zeta\| + \|\mathrm{curl}\ \zeta\|_q] \ a.s.$$

Stochastic integrals for Radonifyng operators

Definition

Let $\mathcal{Y}$ be a Banach space; a linear operator $K : H_0 \to \mathcal{Y}$ is Radonifying if for any ONB (e_k) of H_0 and any sequence (β_n) of iid N(0,1) random variables, the series $\sum_{n\geq 1} \beta_n K e_n$ converges in $L^2(\Omega; \mathcal{Y})$ (or a.s.) and $\|K\|^2_{R(H_0,\mathcal{Y})} := \mathbb{E}|\sum_n \beta_n K e_n|^2_{\mathcal{Y}}$

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Then if $\mathcal{Y} = W^{k,q}(D)$ for $k = 0, 1, \ldots$ and $q \in [2, \infty)$ and if (X_t) is predictable with $X \in L^2(0, T; R(H_0, \mathcal{Y}))$, the stochastic integral $\int_0^t X_s dW(s)$ can be defined as an element of $L^2(0, T; W^{k,q})$ (extended from step processes to $L^2(0, T; R(H_0, \mathcal{Y}))$)

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Following results from Dettweiler, Neidhardt, Brzezniak, Peszat, ... classical results of stochastic calculus (Itô's formula, Burkholder-Davies-Gundy inequality, ...) extend for integrands which are Radonifying.

Conditions on σ_ν

- **Growth and Lipschitz conditions** for $\nu \in [0, \nu_0]$,

$\sigma_\nu : [0, T] \times Dom(\Delta) \to L_Q(H_0; V)$

$$\sigma_\nu : [0, T] \times H^{2,q} \to R(H_0, H^{1,q}), \forall q \in (2, \infty)$$

where for every $u, v \in V$ or $H \cap H^{2,q}$, $q \in [2, \infty)$,

$$|\sigma_\nu(t,u)|^2_{L_Q} \le \bar{K}_0 + \bar{K}_1 \, |u|^2, \quad \left|\mathrm{curl}\ \sigma_\nu(t,u)\right|^2_{L_Q} \le \bar{K}_0 + \bar{K}_1 \, \|u\|^2,$$

$$\|\mathrm{curl}\ \sigma_\nu(t,u)\|^2_{R(H_0,L^q)} \le \bar{K}_2 + \bar{K}_3 \|u\|^2_q + \bar{K}_4 \|\mathrm{curl}\ u\|^2_q,$$

$$|\sigma_\nu(t,u) - \sigma_\nu(t,v)|^2_{L_Q} \le \bar{L}_1 \, |u - v|^2,$$

$$\left|\mathrm{curl}\ \sigma_\nu(t,u) - \mathrm{curl}\ \sigma_\nu(t,v)\right|^2_{L_Q} \le \bar{L}_1 \, \|u - v\|^2$$

with **constants** $\bar{K}_i$ and $\bar{L}_1$ **independent of** $\nu \in [0, \nu_0]$.

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$$\left|\text{curl } \sigma_\nu(t, u) - \text{curl } \sigma_\nu(t, v)\right|^2_{L_Q} \le \bar{L}_1 \, \|u - v\|^2$$

with **constants** $\bar{K}_i$ and $\bar{L}_1$ **independent of** $\nu \in [0, \nu_0]$.

• **Convergence as** $\nu \to 0$

$\sup_{0 \le t \le T} |\sigma_\nu(t, u) - \sigma_0(t, u)|_{L(H_0, H)} \le C(\nu)(1 + |u|_H)$

where $\nu \in (0, \infty) \to C(\nu) \in [0, \infty)$ converges to 0 as $\nu \to 0$

The inviscid LDP for NS equations

Theorem

(Bessaih-M.) Let $\zeta \in V$ *be such that* $\mathrm{curl}\ \zeta \in L^\infty(D)$*,*
σ_ν, $\nu > 0$ *and* σ_0 *be as above.*
Then as $\nu \to 0$*, the distribution of the solution* u^ν *to*
$du^\nu_t + [\nu A u^\nu_t + B(u^\nu_t, u^\nu_t)]\, dt = \sqrt{\nu}\sigma_\nu(t, u^\nu_t)\, dW(t)$ *with* $u^\nu_0 = \zeta$
satisfies in $\mathcal{X} = C\big([0,T]; H\big) \bigcap L^\infty\big(0,T; H^{1,q} \cap V\big) \bigcap L^2\big(0,T; \mathcal{H}\big)$
a LDP with the good rate function

$$I(u) = \inf\{\|h\|^2_{L^2([0,T],H_0)}/2 \;:\; u = u^0_h,\; h \in L^2(0,T;H_0)\} \quad \textit{and}$$

$$du^0_h(t) + B(u^0_h(t), u^0_h(t))\, dt = \sigma_0(t, u^0_h(t))\, h(t)\, dt,\; u^0_h(0) = \xi$$

Remark: $u^\nu \in C([0,T]; V) \cap L^\infty(0,T; H^{1,q})$ for any $q \in (2,\infty)$
and $\nu \geq 0$; not "optimal" topology
Example of σ_ν**:** Nemytski operators. For $h = (h_1, h_2) \in H_0$,
$i = 1, 2$, $\sigma_\nu(t,u)h(x) = \sum_j g_j(t, x, u(x)) h_j(x)$

Weak convergence approach to transfer the LDP

• Introduced by Dupuis & Ellis (equivalent to the Laplace principle); Budhiraja & Dupuis infinite dimensional mild solutions
• small perturbations in various SPDE contexts with small perturbation: Sritharan & Sundar, Duan & M., Manna, Sritharan & Sundar, Chueshov& M., Liu, Ren & Zhang, Röckner & Zhang, Sanz-Solé, ...
• LDP as $\nu \to 0$ for shell models of turbulence: Bessaih-M.

The Stochastic controlled equation

$$\mathcal{S}_M = \Big\{ h \in L^2([0,T], H_0) : \int_0^T |h(s)|_0^2 \, ds \le M \Big\}$$

$$\mathcal{A}_M = \{ h \quad (\mathcal{F}_t) \, \text{predictable} \;\; : h(\omega) \in \mathcal{S}_M \text{ a.s.} \}$$

For $\nu > 0$, $h_\nu \in \mathcal{A}_M$, let $u^\nu_{h_\nu} = G^\nu(\sqrt{\nu} W, \int_0^{\cdot} h_\nu(s) ds)$ be the solution to the stochastic controlled equation: $u^\nu_{h_\nu}(0) = \xi$

$$du^\nu_{h_\nu} + [\nu A u^\nu_{h_\nu} + B(u^\nu_{h_\nu}, u^\nu_{h_\nu})] dt = \sigma_\nu(t, u^\nu_{h_\nu}) h_\nu dt + \sqrt{\nu} \sigma_\nu(t, u^\nu_{h_\nu}) dW(t)$$

With the above assumption, for $\nu > 0$ this equation has a unique solution $u^\nu_{h_\nu} \in C([0,T]; V) \cap L^\infty(0,T; H^{1,q})$

$$\sup_{0<\nu\le\nu_0} \sup_{h_\nu \in \mathcal{A}_M} E\Big(\sup_{0\le t\le T} \|u^\nu_{h_\nu}(t)\|^{2p} \Big) \le C(p,M) \big(1 + E\|\zeta\|^{2p}\big)$$

$$\sup_{0<\nu\le\nu_0} \sup_{h_\nu \in \mathcal{A}_M} E\Big(\sup_{0\le t\le T} \|u^\nu_{h_\nu}(t)\|^q_{H^{1,q}} \Big) \le C(M,q) \big(1 + E\|\zeta\|^q_{H^{1,q}}\big)$$

Weak convergence approach

The stochastic controlled equation

Given $h \in \mathcal{A}_M$ let $G^0(\int_0^\cdot h(s)ds) := u_h^0$ be the solution to

$$du_h^0(t) + B(u_h^0(t), u_h^0(t))\, dt = \sigma_0(t, u_h^0(t))\, h(t)\, dt,\ \ u_h^0(0) = \xi$$

For $\nu > 0$, $h_\nu \in \mathcal{A}_M$, let $u_{h_\nu}^\nu = G^\nu(\sqrt{\nu}W, \int_0^\cdot h_\nu(s)ds)$ be the solution to the stochastic controlled equation: $u_{h_\nu}^\nu(0) = \xi$

$$du_{h_\nu}^\nu + [\nu A u_{h_\nu}^\nu + B(u_{h_\nu}^\nu, u_{h_\nu}^\nu)]dt = \sigma_\nu(t, u_{h_\nu}^\nu)h_\nu dt + \sqrt{\nu}\sigma_\nu(t, u_{h_\nu}^\nu)dW(t)$$

Step 1 (weak convergence) Let $(h_\nu), h \in \mathcal{A}_M$ be such that as $\nu \to 0$, $h_\nu \to h$ **in distribution** (for the **weak topology** on $\mathcal{S}_M$)
Then $G^\nu(\sqrt{\nu}W, \int_0^\cdot h_\nu(s)ds) \to G^0(\int_0^\cdot h(s)ds)$ in distribution in $\mathcal{X}$.

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For $\nu > 0$, $h_\nu \in \mathcal{A}_M$, let $u_{h_\nu}^\nu = G^\nu(\sqrt{\nu}W, \int_0^\cdot h_\nu(s)ds)$ be the solution to the stochastic controlled equation: $u_{h_\nu}^\nu(0) = \xi$

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Then $G^\nu(\sqrt{\nu}W, \int_0^\cdot h_\nu(s)ds) \to G^0(\int_0^\cdot h(s)ds)$ in distribution in $\mathcal{X}$.
Step 2 (compactness) For $M > 0$,
$\Gamma_M = \{G^0(\int_0^\cdot h(s)ds) \,:\, h \in \mathcal{S}_M\} \subset\subset \mathcal{X}$

Weak convergence of the stochastic controlled equation

Proposition

Assume that σ_ν *satisfies the previous conditions Let* ζ *be* $\mathcal{F}_0$*-measurable such that for some* $q > 4$, $\mathbb{E}\big(\|\zeta\|^q_{H^{1,q}} + \|\zeta\|_V^{4q/(q-4)}\big) < \infty$, *and let* h_ν *converge to* h *in distribution as random elements taking values in* $\mathcal{A}_M$ *endowed with the weak topology of the space* $L_2(0,T;H_0)$. *Then, as* $\nu \to 0$, *the solution* $u^\nu_{h_\nu}$ *of the stochastic controlled NS equation*

$$du^\nu_{h_\nu} + \big[\nu A u^\nu_{h_\nu} + B(u^\nu_{h_\nu}, u^\nu_{h_\nu})\big]dt = \sqrt{\nu}\sigma_\nu(t, u^\nu_{h_\nu})dW(t) + \sigma_\nu(t, u^\nu_{h_\nu})h_\nu(t)dt$$

converges in distribution in $\mathcal{X}$ *to a solution* u^0_h *of the controlled Euler equation.*
That is, if $\zeta \in V$ *and* $\mathrm{curl}\,\zeta \in L^\infty$, *as* $\nu \to 0$, *the process* $\mathcal{G}^\nu_\zeta\Big(\sqrt{\nu}\big(W_\cdot + \frac{1}{\sqrt{\nu}}\int_0^\cdot h_\nu(s)ds\big)\Big)$ *converges* **in distribution** *to* $\mathcal{G}^0_\xi\big(\int_0^\cdot h(s)ds\big)$ *in* $\mathcal{X}$

Weak convergence - Sketch of proof

Step 1 for $\alpha \in (0, \frac{1}{2})$ and any $p \in [2, \infty)$,
$\mathbb{E}\|u^\nu_{h_\nu}\|^2_{W^{\alpha,2}(0,T;H)} + \mathbb{E}\|u^\nu_{h_\nu}\|^p_{W^{\alpha,p}(0,T;V')} + \mathbb{E}\int_0^T \|u^\nu_{h_\nu}(t)\|^2_V dt < \infty$
$F_\nu(.) = \int_0^. h_\nu(s)ds$ converges to $F(.) = \int_0^. h(s)ds$ in $H^{1,2}(0,T;H_0)_w$
For $\beta > \frac{1}{2}$ and $p\alpha > 1$, $(F_\nu, u^\nu_{h_\nu})$ is tight in
$C^\alpha([0,T], H_0) \times \left[L^2(0,T;H^{\frac{1}{2},2}) \times C([0,T], DomA^{-\beta})\right]$

Weak convergence - Sketch of proof

Step 1 for $\alpha \in (0, \frac{1}{2})$ and any $p \in [2, \infty)$,
$\mathbb{E}\|u^\nu_{h_\nu}\|^2_{W^{\alpha,2}(0,T;H)} + \mathbb{E}\|u^\nu_{h_\nu}\|^p_{W^{\alpha,p}(0,T;V')} + \mathbb{E}\int_0^T \|u^\nu_{h_\nu}(t)\|^2_V dt < \infty$
$F_\nu(.) = \int_0^\cdot h_\nu(s)ds$ converges to $F(.) = \int_0^\cdot h(s)ds$ in $H^{1,2}(0,T;H_0)_w$
For $\beta > \frac{1}{2}$ and $p\alpha > 1$, $(F_\nu, u^\nu_{h_\nu})$ is tight in
$C^\alpha([0,T],H_0) \times \left[L^2(0,T;H^{\frac{1}{2},2}) \times C([0,T],DomA^{-\beta})\right]$
Step 2 Change probability space and extract subsequence denoted again $(h_\nu, u^\nu_{h_\nu})$ which converges a.s. to (h,u) in
$L^2(0,T;H_0)_w \times \left[L^2(0,T;H^{\frac{1}{2},2}) \times C([0,T],DomA^{-\beta})\right]$;
furthermore, $u \in C([0,T];H) \cap L^\infty(0,T;H^{1,q})$ for $q \in [2,\infty)$

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furthermore, $u \in C([0,T]; H) \cap L^\infty(0,T; H^{1,q})$ for $q \in [2,\infty)$
Step 3 Prove that $du(t) + B(u(t),u(t))dt = \sigma(t,u(s))h(s)ds$; let $\varphi \in Dom(A^\beta)$; as $\nu \to 0$
$\mathbb{E}\langle u^\nu_{h_\nu}) - \zeta - \int_0^t \big[-B(u^\nu_{h_\nu}(s), u^\nu_{h_\nu}(s)) + \sigma(s, u^\nu_{h_\nu}(s)) h_\nu(s) \big] ds, \varphi\rangle \to 0$
For $t \in (0,T]$, a.s. $\langle u^\nu_{h_\nu}(t) - u(t), \varphi\rangle \to 0$
$u \in C([0,T], H)$

Compactness of Euler controlled equation

$h_n \to h$ in $L^2(0,T;H_0)_w$ and u_n solutions to Euler equation

$$du_n(t) + B(u_n(t),u_n(t))dt = \sigma(t,u_n(t))h_n(t)dt,\ \ u_n(0)=\zeta$$

• Similar argument; u_n is bounded in
$W^{1,2}(0,T;L^q) \cap W^{\alpha,p}(0,T;L^q) \cap L^2(0,T;H^{1,q})$
for $\alpha \in (0,\frac{1}{2})$ and $\alpha p > 1$, $q \in [2,\infty)$
(no smoothing) and (u_n) is not bounded in $L^2(0,T;H^{2,2})$.

• (u_n) is relatively compact in
$C([0,T],Dom(A^{-\beta})) \cap L^2(0,T;H^{\frac{1}{2},2})$
• Similar (easier) argument to identify the limit u as solution to
$du(t) + B(u(t),u(t))dt = \sigma(t,u(t))h(t)dt,\ u(0)=\zeta$