On the inviscid limit for 2D incompressible stochastic Navier-Stokes equations with friction type boundary conditions

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Velocity equation with additive noise

We consider the following stochastic Navier-Stokes equation in dimension 2:

$$\begin{cases} \frac{\partial u^\nu(t)}{\partial t} - \nu \Delta u^\nu(t) + (u^\nu(t) \cdot \nabla) u^\nu(t) + \nabla p(t) = f(t) + \sqrt{\mathcal{Q}}\, \dot{W} & \text{in }]0, T[\times \mathcal{O} \\ \operatorname{div} u^\nu = 0 & \text{in }]0, T[\times \mathcal{O} \\ u^\nu(0) = u_0 & \text{in } \mathcal{O}, \\ u^\nu \cdot \mathbf{n} = 0 \text{ and } 2D(u^\nu)\mathbf{n} \cdot \mathbf{t} + \alpha u^\nu \cdot \mathbf{t} = 0 & \text{on }]0, T[\times \Gamma \end{cases} \quad (1)$$

- $\mathcal{O}$ is a bounded simply connected domain in $\mathbb{R}^2$, Γ is sufficiently regular
- $\alpha \in \mathcal{C}^2(\Gamma, \mathbb{R}^+)$
- $D^{ij}(u^\nu) = \frac{1}{2}\left(\partial_j u^{\nu,i} + \partial_i u^{\nu,j}\right)$ is the rate of strain tensor,
- $\mathbf{n}$ and $\mathbf{t}$ are the unit exterior normal and the unit tangent vector,
- $f(t,x)$ is a given deterministic force
- $\sqrt{\mathcal{Q}}\, \dot{W}$ is the formal derivative of a Gaussian random field in time and correlated in space that will be set later.

Velocity equation with additive noise

We introduce the following Hilbert spaces

$$H = \left\{ v \in \left[L^2(\mathcal{O})\right]^2 : \mathrm{div}\, v = 0 \text{ in } \mathcal{O} \text{ and } v \cdot \mathbf{n} = 0 \text{ on } \Gamma \right\},$$

$$V = \left\{ v \in \left[H^1(\mathcal{O})\right]^2 : \mathrm{div}\, v = 0 \text{ in } \mathcal{O} \text{ and } v \cdot \mathbf{n} = 0 \text{ on } \Gamma \right\},$$

$$\mathcal{W} = \left\{ v \in V \cap \left[H^2(\mathcal{O})\right]^2 : 2D(v)\mathbf{n} \cdot \mathbf{t} + \alpha v \cdot \mathbf{t} = 0 \text{ on } \Gamma \right\}$$

$$= \left\{ v \in V \cap \left[H^2(\mathcal{O})\right]^2 : \mathrm{curl}\, v = (2\kappa - \alpha) v \cdot \mathbf{t} \text{ on } \Gamma \right\},$$

where κ denotes the curvature of Γ. We recall that $\mathrm{curl}\, v = \partial_1 v^2 - \partial_2 v^1$.

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V is endowed with the inner product

$$\langle u, v \rangle_V = \langle \nabla u, \nabla v \rangle$$

and the associated norm $\| \cdot \|_V$.
Let us denote by V' the topological dual of V and by $\langle \cdot, \cdot \rangle_{V',V}$ the corresponding duality. We define the operator $\mathcal{A} : V \to V'$ by

$$\langle \mathcal{A}u, v \rangle_{V',V} = \int_{\mathcal{O}} \nabla u \cdot \nabla v - \int_{\Gamma} (\kappa - \alpha) u \cdot v,$$

for all $u, v \in V$. Since $\left| \langle \mathcal{A}u, v \rangle_{V',V} \right| \leq C \|u\|_V \|v\|_V$, $\mathcal{A}$ is a continuous operator from V to V'. Moreover $\mathcal{A} : \mathcal{W} \to H$ coincides with the Stokes operator $-P_H \Delta$, where P_H denotes the Leray projector. In fact,

$$\langle \mathcal{A}u, v \rangle_{V',V} = \langle -\Delta u, v \rangle, \quad u \in \mathcal{W}, v \in V.$$

We also define $\mathcal{B} : V \to V'$ as $\mathcal{B}(u) = (u \cdot \nabla) u$, that is,

$$\langle \mathcal{B}(u), v \rangle = \int_{\mathcal{O}} (u \cdot \nabla) u \cdot v,$$

for all $u, v \in V$.

Velocity equation with additive noise

We shall also fix in the following $\mathcal{Q} = \mathcal{A}^{-2m}$, where $m \in \mathbb{N}$ will be fixed later and $W(t) = \sum_{k=1}^{\infty} \beta_k(t) v_k$, $t \geq 0$, where $\{v_k\} \subset \mathcal{W}$ is a H-orthonormal system of eigenvectors of $\mathcal{A}$ with the corresponding eigenvalues λ_k which a basis for V, and $\{\beta_k\}$ is a sequence of standard Brownian motion mutually independent of some stochastic basis $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\}_{t \geq 0})$. In fact,

$$\sqrt{\mathcal{Q}}\, W(t) = \sum_{k=1}^{\infty} \beta_k(t) \sqrt{\mathcal{Q}}\, v_k = \sum_{k=1}^{\infty} \lambda_k^{-m} v_k \beta_k(t)$$

is a H-valued centered Wiener process on $(\Omega, \mathcal{F}, \mathbf{P})$, with covariance $\mathcal{Q}$ in H. We take $m \in \mathbb{N}$ such that

$$\sum_{k=1}^{\infty} \lambda_k^{-2m+3} < \infty.$$

Then, with this choice of m we have that $\mathcal{Q}$ is an operator of trace class. We denote the trace of $\mathcal{Q}$ by $\mathrm{tr}(\mathcal{Q}) \doteq \sum_{k=1}^{\infty} \langle \mathcal{Q} v_k, v_k \rangle = \sum_{k=1}^{\infty} \lambda_k^{-2m}$.

Velocity equation with additive noise

In terms of $\mathcal{A}$, $\mathcal{B}$ and f we can write Equation (1) as the following stochastic evolution equation in V':

$$\begin{cases} du^\nu = F(t, u^\nu(t))\,dt + \sqrt{\mathcal{Q}}\,dW(t) & \text{in }]0,T[\times\mathcal{O}, \\ u^\nu(0) = u_0 & \text{in } \mathcal{O}, \end{cases} \tag{2}$$

where $F(t, u^\nu) = f - \nu\mathcal{A}u^\nu - \mathcal{B}(u^\nu)$.

Definition 1.1

Given $u_0 \in L^2(\Omega; H)$, an adapted stochastic process u^ν with sample paths in $\mathcal{C}([0,T];H) \cap L^2(0,T;V)$ is said a weak solution of the stochastic Navier-Stokes equation (2) if

$$\langle u^\nu(t), v\rangle = \langle u_0, v\rangle + \int_0^t \langle F(s, u^\nu(s)), v\rangle\, ds + \int_0^t \left\langle \sqrt{\mathcal{Q}}\,dW(s), v\right\rangle,$$

in $]0,T[$, for all $v \in V$ and a.e. $\omega \in \Omega$.

Velocity equation with additive noise

Our main result is the following:

Theorem 1.1

Let $T > 0$, $\nu_0 > 0$ *and* $p > 2$. *Suppose that* $f \in L^2(0, T; H)$, $\operatorname{curl} f \in L^1(0, T; L^p(\mathcal{O}))$, $u_0 \in L^p(\Omega; H)$ *and* $\operatorname{curl} u_0 \in L^p(\Omega; L^p(\mathcal{O}))$. *Then,*

(i) For any $\nu \in]0, \nu_0]$, *there exists a unique weak solution* u^ν *of the stochastic Navier-Stokes equation* (2) *such that*

$$u^\nu \in L^p(\Omega; \mathcal{C}([0, T]; H)) \cap L^2(\Omega; L^2(0, T; V)) \cap [L^4(]0, T[\times \mathcal{O} \times \Omega)]^2,$$
$$\operatorname{curl} u^\nu \in L^2(\Omega; L^\infty(0, T; L^p(\mathcal{O})).$$

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(ii) In addition, if $\operatorname{curl} f \in L^1(0,T;L^\infty(\mathcal{O}))$, *there exists a measurable stochastic process* u *that is a solution of the incompressible 2D stochastic Euler equation* (2) $(\nu = 0)$, *in the sense that*

$$\langle u(t), v\rangle = \langle u_0, v\rangle - \int_0^t \langle \mathcal{B}(u(s)), v\rangle \, ds + \int_0^t \langle f(s), v\rangle \, ds + \int_0^t \left\langle \sqrt{\mathcal{Q}}\, dW(s), v \right\rangle \tag{3}$$

for all $v \in V$ *and* $\mathbb{P}$*-a.e.* $\omega \in \Omega$. *Furthermore, taking* $\operatorname{curl} u_0 \in L^p(\Omega; L^\infty(\mathcal{O}))$, *for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$

$$u^\nu(\omega) \to u(\omega) \quad \text{strongly in} \quad \mathcal{C}([0,T];H), \text{ as } \nu \to 0.$$

L^2 a priori estimates for u^ν independent of ν

We consider the following Faedo-Galerkin approximations of Equation (2).
Let $H_n \doteq \mathrm{span}\,\{v_1, \ldots, v_n\}$, where $\{v_j\}_j$ is the previous fixed basis for V.
Define u_n^ν as the solution of the following stochastic differential equation:
For each $v \in H_n$,

$$d\langle u_n^\nu(t), v\rangle = \langle F(t, u_n^\nu(t)), v\rangle\, dt + \langle \sqrt{\mathcal{Q}}\, dW(t), v\rangle, \tag{4}$$

with $u_n^\nu(0) = \sum_{k=1}^n \langle u_0, v_k\rangle v_k$.

Notice that Equation (4) defines a system of stochastic ordinary differential equations in $\mathbb{R}^n$ with locally Lipschitz coefficients. Therefore, we need some a priori estimate to prove the global existence of a solution $u_n^\nu(t)$ as an adapted process in the space $\mathcal{C}(0, T; H_n)$.

L^2 a priori estimates for u^ν independent of ν

Proposition 2.1

Let $T > 0$ and $\nu_0 > 0$. Suppose that $f \in L^1(0, T; H)$ and $u_0 \in L^2(\Omega; H)$. Let $u_n^\nu(t)$ be an adapted process in the space $\mathcal{C}([0, T]; H_n)$ solution of Equation (4). Then

$$\sup_{0<\nu\leq\nu_0} \sup_n \left\{ \mathbb{E}\left(\sup_{0\leq r\leq T} \|u_n^\nu(r)\|_{L^2}^2 \right) + \nu \int_0^T \mathbb{E}\left(\|u_n^\nu(s)\|_V^2\right) ds \right\} \leq C(f, \mathcal{Q}, \nu_0)\left(\mathbb{E}\left(\|u_0\|_{L^2}^2\right) + 1\right). \tag{5}$$

Furthermore we have

$$\|u_n^\nu(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u_n^\nu(s)\|_{L^2}^2 \, ds = \|u_n^\nu(0)\|_{L^2}^2 + 2\nu \int_0^t \left(\int_\Gamma (\kappa - \alpha) u_n^\nu(s) \cdot u_n^\nu(s) \, d\mathcal{S} \right) ds + 2 \int_0^t \langle f(s), u_n^\nu(s)\rangle \, ds + 2 \int_0^t \langle \sqrt{\mathcal{Q}} \, dW(s), u_n^\nu(s)\rangle + \int_0^t \mathrm{tr}(\mathcal{Q}) \, ds, \tag{6}$$

L^2 a priori estimates for u^ν independent of ν

Corollary 2.1

Assume hypotheses of Proposition 2.1 and $u_0 \in L^p(\Omega; H)$. *Then for any* $p \geq 4$

$$\sup_{0<\nu\leq\nu_0} \sup_n \left\{ \mathbb{E}\left(\sup_{0\leq r\leq T} \|u_n^\nu(r)\|_{L^2}^p \right) + \nu \int_0^T \mathbb{E}\left(\|u_n^\nu(s)\|_{L^2}^{p-2} \|u_n^\nu(s)\|_V^2 \right) ds \right\}$$
$$\leq C(p, f, \mathcal{Q}, \nu_0) \left(\mathbb{E}\left(\|u_0\|_{L^2}^p \right) + 1 \right). \qquad (7)$$

L^2 a priori estimates for u^ν independent of ν

The next lemma gives an important monotonicity property of operator F in order to prove the existence and uniqueness for the solution to Equation (2).

Lemma 2.1

For a given $r > 0$ we consider the following (closed) L^4-ball B_r in the space V:

$$B_r \doteq \left\{ v \in V : \|v\|_{[L^4(\mathcal{O})]^2} \leq r \right\}.$$

Then the nonlinear operator $u \mapsto F(t,u)$, $t \in [0,T]$, is monotone in the convex ball B_r, that is, for any $u \in V$, $v \in B_r$, there exists a positive constant $C \doteq C(\nu_0, \mathcal{O}, \alpha)$, depending on ν_0, the domain $\mathcal{O}$ and α such that

$$\langle F(t,u) - F(t,v), u - v \rangle \leq C \left(1 + \frac{r^4}{\nu^3} \right) \|u - v\|_{L^2}^2. \tag{8}$$

L^2 a priori estimates for u^ν independent of ν

The following proposition give the pathwise uniquenes of Equation (2).

Proposition 2.2

Assume the hypotheses of Proposition 2.1. Let u^ν *be a solution of Equation (2), that is, an adapted stochastic process* $u^\nu(t,x,\omega)$ *satisfying (2) and such that*

$$u^\nu \in L^2\left(\Omega; \mathcal{C}(0,T;H) \cap L^2(0,T;V)\right) \cap [L^4\left(]0,T[\times \mathcal{O} \times \Omega\right)]^2.$$

If v^ν *is another solution of Equation (2) as an adapted stochastic process in the space* $\mathcal{C}(0,T;H) \cap L^2(0,T;V)$*, then*

$$\|u^\nu(t) - v^\nu(t)\|_{L^2}^2 \exp\left\{-2C\int_0^t \left(1 + \frac{1}{\nu^3}\|u^\nu(s)\|_{[L^4(\mathcal{O})]^2}^4\right) ds\right\}$$
$$\leq \|u^\nu(0) - v^\nu(0)\|_{L^2}^2,$$

with probability 1*, for any* $0 \leq t \leq T$*, where* C *is the positive constant that appears in Lemma 2.1. In particular* $u^\nu = v^\nu$*, if* v^ν *satisfies the same initial condition as* u^ν*.*

L^2 a priori estimates for u^ν independent of ν

The existence of solution to Equation (2) is given in the following proposition

Proposition 2.3

Suppose the hypotheses of Corollary 2.1. Then there exists an adapted process $u^\nu(t,x,\omega)$ *such that*

$$u^\nu \in L^p\left(\Omega; \mathcal{C}(0,T;H)\right) \cap L^2\left(\Omega; L^2(0,T;V)\right) \cap [L^4\left(]0,T[\times\mathcal{O}\times\Omega\right)]^2,$$

and verifying Equation (2). Furthermore,

$$\sup_{0<\nu\le\nu_0} \mathbb{E}\left\{\sup_{0\le r\le T}\|u^\nu(r)\|_{L^2}^p + \nu\int_0^T \|u^\nu(s)\|_V^2\,ds + \nu\int_0^T \|u^\nu(s)\|_{L^2}^{p-2}\|u^\nu(s)\|_V^2\,ds\right\} \le C(p,f,\mathcal{Q},\nu_0)\left(\mathbb{E}\left(\|u_0\|_{L^2}^p\right)+1\right). \quad (9)$$

Vorticity equation with additive noise

Set $\xi^\nu = \mathrm{curl}\, u^\nu$. We apply the operator curl to our equation (1), obtaining the following vorticity equation:

$$\begin{cases} \frac{\partial \xi^\nu(t)}{\partial t} - \nu \Delta \xi^\nu(t) + (u^\nu(t)\cdot\nabla)\xi^\nu(t) \\ \qquad\qquad = \mathrm{curl}\, f(t) + \mathrm{curl}(\sqrt{\mathcal{Q}}\ \dot{W}(t)) & \text{in }]0,T[\times\mathcal{O}, \\ \xi^\nu(0) = \mathrm{curl}\, u_0 & \text{in } \mathcal{O}, \\ \xi^\nu = (2\kappa-\alpha)u^\nu\cdot\mathbf{t} & \text{on }]0,T[\times\Gamma \end{cases}$$

Notice that

$$\mathrm{curl}(\sqrt{\mathcal{Q}}\, dW) = \sum_{k=1}^{\infty} \lambda_k^{-m} \mathrm{curl}\, v_k\, d\beta_k.$$

Vorticity equation with additive noise

In the following we shall denote by $\tilde{H}$ the space $L^2(\mathcal{O})$ endowed with the L^2-norm. We consider the operator $\tilde{A}: D(\tilde{A}) \subset \tilde{H} \to \tilde{H}$ with domain $D(\tilde{A}) = \{\zeta \in L^2(\mathcal{O}) : \quad \Delta\zeta \in L^2(\mathcal{O})\}$, defined as $\tilde{A}\zeta = -\Delta\zeta$. Set

$$\zeta_k = \frac{\mathrm{curl}\, v_k}{\|\mathrm{curl}\, v_k\|_{L^2}},$$

where $\{v_k\}$ is the previous fixed H-orthonormal basis for V. In addition, $\{\zeta_k\}$ is an $L^2(\mathcal{O})$-orthonormal basis for $\tilde{H}$ that verifies $\tilde{A}\zeta_k = \lambda_k \zeta_k$. Thus,

$$\mathrm{curl}(\sqrt{\mathcal{Q}}\, dW) = \sum_{k=1}^{\infty} \lambda_k^{-m} \mathrm{curl}\, v_k\, d\beta_k = \sum_{k=1}^{\infty} \lambda_k^{-m} \|\mathrm{curl}\, v_k\|_{L^2} \zeta_k\, d\beta_k.$$

We define $\tilde{\mathcal{Q}} \in L(\tilde{H}, \tilde{H})$ by

$$\tilde{\mathcal{Q}}\zeta_k = \lambda_k^{-2m} \mu_k^2 \zeta_k,$$

where $\mu_k = \|\mathrm{curl}\, v_k\|_{L^2}$, and $\tilde{W} = \sum_{k=1}^{\infty} \zeta_k \beta_k$ is a new Wiener process.

Vorticity equation with additive noise

In order to obtain L^p a priori estimates for the vorticity, we shall need to estimate $\mathbb{E}\left(\left\|\tilde{\mathcal{Q}}^{1/2}\tilde{W}(1)\right\|_{L^p}^2\right)$.

Using the Sobolev Imbedding Theorem

$$\mathbb{E}\left(\left\|\sum_{k=1}^{\infty}\beta_k(1)\tilde{\mathcal{Q}}^{1/2}\zeta_k\right\|_{L^p}^2\right) \leq \mathbb{E}\left(\left\|\sum_{k=1}^{\infty}\beta_k(1)\tilde{\mathcal{Q}}^{1/2}\zeta_k\right\|_{H^1}^2\right).$$

Using that $\left[H^1(\mathcal{O})\right]^2$ is a Hilbert space and the independence of the elements of the sequence $\{\beta_k(1)\}_k$, we only need to show that

$$\sum_{k=1}^{\infty}\lambda_k^{-2m}\|\mathrm{curl}\, v_k\|_{H^1} < +\infty.$$

Vorticity equation with additive noise

In fact, it is possible to prove that

$$\|\mathrm{curl}\, v_k\|^2_{H^1(\mathcal{O})} \leq C(1+\lambda_k)^2 \|\mathrm{curl}\, v_k\|^2_{L^2} \leq C(1+\lambda_k)^3 \|v_k\|^2_{L^2} .$$

Thus, we shall take $m \in \mathbb{N}$ such that

$$\sum_{k=1}^{\infty} \lambda^{-2m+3} := \mathcal{M} < +\infty.$$

In addition $\tilde{\mathcal{Q}}^{1/2}\tilde{W}$ is an $\tilde{H}$-valued centered Wiener process on $(\Omega, \mathcal{F}, \mathbf{P})$, with covariance $\tilde{\mathcal{Q}}$ in $\tilde{H}$, and $\tilde{\mathcal{Q}}$ is a trace class operator.

Vorticity equation with additive noise

In terms of $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{Q}}^{1/2}\tilde{W}$ we can write the vorticity equation as

$$\begin{cases} d\xi^\nu(t) + \left\{\nu\tilde{\mathcal{A}}\xi^\nu(t) + (u^\nu(t)\cdot\nabla)\xi^\nu(t)\right\}dt \\ \qquad\qquad = \operatorname{curl} f(t)\,dt + \tilde{\mathcal{Q}}^{1/2}\,d\tilde{W}(t) & \text{in }]0,T[\times\mathcal{O}, \\ \xi^\nu(0) = \operatorname{curl} u_0 & \text{in } \mathcal{O}, \\ \xi^\nu = (2\kappa-\alpha)u^\nu\cdot\mathbf{t} & \text{on }]0,T[\times\Gamma \end{cases} \tag{10}$$

1. Velocity equation with additive noise
2. L^2 a priori estimates for u^ν independent of ν
3. Vorticity equation with additive noise
4. **L^p a priori estimates for ξ^ν independent of ν**
5. Vanishing viscosity limit

L^p a priori estimates for ξ^ν independent of ν

The improvement on the a priori estimates (9) for the solution of Equation (2) is given in the following result:

Proposition 4.1

Suppose hypotheses of Corollary 2.1. Assume also that $p > 2$, $\operatorname{curl} f \in L^1(0, T; L^p(\mathcal{O}))$ *and* $\operatorname{curl} u_0 \in L^p(\Omega; L^p(\mathcal{O}))$. *Let* ξ^ν *be the vorticity of* u^ν, *then we have*

$$\mathbb{E}\left(\sup_{0\le r\le T} \|\xi^\nu(r)\|_{L^p}^p\right) \le C(\operatorname{curl} f, \tilde{\mathcal{Q}}, T, p, \mathcal{O}, \alpha)\left\{\mathbb{E}\left(\|u_0\|_{L^2}^p\right) + \mathbb{E}\left(\|\operatorname{curl} u_0\|_{L^p}^p\right) + 1\right\}. \quad (11)$$

L^p a priori estimates for ξ^ν independent of ν

The idea is: Let us denote by w the solution of the following linear equation

$$\begin{cases} dw(t) + \left\{\nu\tilde{\mathcal{A}}w(t) + (u^\nu(t)\cdot\nabla)w(t)\right\}dt = 0 & \text{in }]0,T[\times\mathcal{O}, \\ w(0) = 0 & \text{in } \mathcal{O}, \\ w = (2\kappa-\alpha)u^\nu\cdot\mathbf{t} & \text{on }]0,T[\times\Gamma \end{cases}$$

We introduce the process $\rho = \xi^\nu - w$ that verifies following s.d.e.:

$$\begin{cases} d\rho(t) + \left\{\nu\tilde{\mathcal{A}}\rho(t) + (u^\nu(t)\cdot\nabla)\rho(t)\right\}dt & \\ \qquad\qquad = \operatorname{curl} f(t)\,dt + \tilde{\mathcal{Q}}^{1/2}\,d\tilde{W}(t) & \text{in }]0,T[\times\mathcal{O}, \\ \rho(0) = \operatorname{curl} u_0 & \text{in } \mathcal{O}, \\ \rho = 0 & \text{on }]0,T[\times\Gamma \end{cases}$$

L^p a priori estimates for ξ^ν independent of ν

Using Propositions 2.1 and 4.1, we can deduce the following result:

Proposition 4.2

Assume the hypotheses of Proposition 4.1. Then

$$\mathbb{E}\left(\|u^\nu\|^p_{L^\infty(0,T;[W^{1,p}(\mathcal{O})]^2)}\right) \leq C, \tag{12}$$

with a constant $C > 0$ independent of viscosity.

Vanishing viscosity limit

In the next two Lemmas the Wiener process $\sqrt{\mathcal{Q}}\, W(t)$ has covariance $\mathcal{Q} = \mathcal{A}^{-2m}$, $m > 4$.

Lemma 5.1

Assume that for a.e. $\omega \in \Omega$, $u_0 \in L^p(\mathcal{O})$ *and* $\operatorname{curl} f \in L^1(0, T; L^\infty(\mathcal{O}))$. *Let* u^ν *be the weak solution of* (2), *then we have*

$$\|u^\nu(\omega)\|_{L^\infty(0,T;W^{1,p}(\mathcal{O}))} \leq C(\omega), \tag{13}$$

where $C(\omega)$ *does not depend on the viscosity* ν, *for a.e.* ω *in* Ω *but depends on* ω. *Moreover, if we assume for a.e.* $\omega \in \Omega$, $u_0 \in L^\infty(\mathcal{O})$, *the estimate* (13) *holds for* $p = \infty$.

Lemma 5.2

Under the assumptions of Lema 5.1. Then exists a stochastic process u *with sample paths in* $C([0, T]; H) \cap L^\infty(0, T; W^{1,p}(\mathcal{O}))$, $p > 2$ *that is solution of the Euler equation in the sense of (3). Moreover, in the case* $p = \infty$, *such solution is unique.*

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Thank you for your attention!!!