The Einstein field equations
Part I: the right-hand side

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Contents:

References:

- Book by Wald: “General Relativity”
- Wikipedia

Prerequisites:

- Special relativity (basic results)
- General relativity (vague idea)
- Differential Geometry (basic knowledge) OR
  Classical Fluid Dynamics (basic knowledge)

Aims:

- Preparation for future lectures (Big bang, Schwarzschild solution, Gödel cosmos)
- (Hopefully useful) supplement for the study of Wald’s book

1 Einstein field equations: overview

Let us consider a fixed "space time" (M, g).

We use "abstract index notation":

- We write g_{ab} for the type (2,0) tensor g
- g^{ab} denotes the type (0,2) tensor given by $\sum_b g_{ab}g^{bc} = \delta_a^c$ (where $\delta_c^a := \delta_{ac}$ Kronecker symbol)
- R^d_{abc} denotes the curvature tensor associated to (M, g)
- We set $R_{ab} := \sum_c R^c_{acb}$ ("Ricci tensor")
- We set $R := \sum_{a,b} R_{ab}g^{ab}$ ("scalar curvature")

Remark 1 Elementary reformulation for $M = \mathbb{R}^4$:

Pseudo-Riemannian metric g_{ab} on $M = \mathbb{R}^4$ can be considered as a matrix $(g_{ab})_{1\leq a,b\leq 4}$ of smooth functions $g_{ab} : \mathbb{R}^4 \to \mathbb{R}$ such that for each $x \in \mathbb{R}^4$

- The matrix $(g_{ab}(x))_{a,b}$ is symmetric
- $(g_{ab}(x))_{a,b}$ has three (strictly) positive eigenvalues and one (strictly) negative eigenvalue

$\Rightarrow$ $(g_{ab})_{1\leq a,b\leq 4}$ determined by the 10 functions

$$g_{11}, g_{22}, g_{33}, g_{44}, g_{12}, g_{13}, g_{14}, g_{23}, g_{24}, g_{34}$$

$(g^{ab})_{1\le a,b\le 4}$: Matrix of functions $g^{ab} : \mathbb{R}^4 \to \mathbb{R}$ given by

$$(g^{ab}(x))_{ab} = \big((g_{ab}(x))_{a,b}\big)^{-1} \text{ for each } x \in \mathbb{R}^4$$

Curvature tensor $(R^d_{abc})_{1\le a,b,c,d\le 4}$: family of functions $R^d_{abc} : \mathbb{R}^4 \to \mathbb{R}$ given explicitly as

$$R^d_{abc}(x) := \partial_b \Gamma^d_{ac}(x) - \partial_a \Gamma^d_{bc}(x) + \sum_i \big(\Gamma^i_{ac}(x)\Gamma^d_{ib}(x) - \Gamma^i_{bc}(x)\Gamma^d_{ia}(x)\big)$$

where

$$\Gamma^c_{ab}(x) := \frac{1}{2}\sum_d g^{cd}(x)\big(\partial_a g_{bd}(x) + \partial_b g_{ad}(x) - \partial_d g_{ab}(x)\big)$$

R_{ab}, $a, b \le 4$, and R are functions on $M = \mathbb{R}^4$ given by

$$R_{ab}(x) = \sum_c R^c_{acb}(x), \qquad R(x) = \sum_{a,b} R_{ab}(x) g^{ab}(x)$$

for each $x \in \mathbb{R}^4$.

Convention 1

- Index set $\{1, 2, 3, 4\}$ instead of $\{0, 1, 2, 3\}$
- Einstein sum convention: e.g. we write

$$R_{ab}g^{ac} \text{ instead of } \sum_a R_{ab}g^{ac}$$

$$R_a^a \text{ instead of } \sum_a R_a^a$$

- Normal rules for raising/lowering indices: e.g. we write

$$R_b^c \text{ instead of } R_{ab}g^{ac} = \sum_a R_{ab}g^{ac}$$

$$v_a v^a \text{ instead of } g_{ab}v^b v^a = \sum_{a,b} g_{ab}v^b v^a$$

1.1 Vacuum case

Basic problem: For given M and $\Lambda \in \mathbb{R}$ ("cosmological constant") find g_{ab} such that

$$R_{ab} - \tfrac{1}{2} R g_{ab} + \Lambda g_{ab} = 0 \tag{1}$$

is fulfilled.

System of 10 non-linear PDEs of second order!

Example 1 A Schwarzschild black hole solution g_{ab} on $M = (\mathbb{R}^3 \backslash \{0\}) \times \mathbb{R}$): Solution of Eq. (1) which is "rotation-invariant" and "static".

It is completely specified by a parameter $m \in \mathbb{R}_+$ (the "mass" of black hole).

Digression 1 If g_{ab} fulfills Eq. (1) then, using $R = R_{ab} g^{ab}$ and $g_{ab} g^{ab} = \delta^a_a = 4$ we obtain

$$R - \tfrac{1}{2} R 4 + \Lambda 4 = 0$$

and therefore

$$R = 4\Lambda$$

Thus R is a constant function and

$$R_{ab} = \Lambda g_{ab}$$

$\Rightarrow$ (M, g) is a "Einstein manifold".

Special case $\Lambda = 0$: $R_{ab} = 0$, i.e. (M, g) is "Ricci flat".

1.2 General case

Real space-time contains matter & radiation. Most important cases:

- A perfect fluid is present ($\rightarrow$ we have 4-velocity field u^a and a density field ρ)
- Electromagnetic radiation is present ($\rightarrow$ we have a 4-potential A_a)

Now consider general situation where (matter or radiation) field Ψ is present, e.g. $\Psi = (u^a, \rho)$ or $\Psi = A_a$.

Assumption 1: Equations of motions for Ψ are known, i.e.

$$F(g, \Psi) = 0$$

for a known function F.

Assumption 2: The tensor

$$T_{ab} = T_{ab}(g, \Psi) \quad (\text{"the stress energy tensor"})$$

is known explicitly (see Sec. 4.2 for a definition)

Example 2 Ψ is a Klein-Gordon field ϕ on (M, g), i.e. a field $\phi : M \rightarrow \mathbb{R}$ with equation of motion

$$F(g, \phi) = (\nabla^a \nabla_a - m^2)\phi = 0, \quad m \in \mathbb{R}$$

The associated stress energy tensor is (see Digression 5 below)

$$T(g, \phi) = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab}(\nabla_c \phi \nabla^c \phi + m^2 \phi^2)$$

(∇^a: Levi-Civita connection)

Basic problem: For given M and $\Lambda \in \mathbb{R}$ find (g, Φ) such that

$$R_{ab} - \tfrac{1}{2}R g_{ab} + \Lambda g_{ab} = \tfrac{8\pi G}{c^4} T_{ab}(g, \Phi) \tag{2a}$$

$$F(g, \Phi) = 0 \tag{2b}$$

are fulfilled where c is the speed of light and G Newton's gravitational constant.

Remark 2 Eqs. (2a), (2b) is considerably more difficult than Eq. (1). We must find g and Φ simultaneously!

Convention 2 For most of the talk (exception: Sec. 2) we use physical units such that $c = 1$ and $G = 1$

Most interesting situations involve matter/radiation

- Friedmann-Robertson-Walker model for the big bang (perfect fluid)
- Schwarzschild solution for a normal star (perfect fluid)
- Gödel cosmos (perfect fluid)
- Black hole with electric charge (electromagnetic field)

2 Special relativity: review

Let $c > 0$ be speed of light (in this section not necessarily $c = 1$)

Minkowski space: $(\mathbb{R}^4, \langle \cdot, \cdot \rangle_L)$ where

$$\langle x, y \rangle_L = -x_0 y_0 + \sum_{i=1}^{3} x_i y_i$$

We set

$$|x| := \sqrt{|\langle x, x \rangle_L|}$$

$O(1,3)$ denotes "Lorentzgroup", i.e. automorphism group of $(\mathbb{R}^4, \langle \cdot, \cdot \rangle_L)$. Explictly:

$$O(1,3) = \{A \in \mathrm{GL}(\mathbb{R}^4) \mid \langle Ax, Ay \rangle_L = \langle x, y \rangle_L \text{ for all } x, y \in \mathbb{R}^4\}$$

Let us now work with a concrete basis of $\mathbb{R}^4$, namely the standard basis $(e_i)_i$ of $\mathbb{R}^4$. Using the "concrete" index notation w.r.t. this basis we have

$$\langle x, y \rangle_L = \eta_{ab} x^a y^b = x^a y_a$$

where

$$(\eta_{ab})_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For a $\vec{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ we define the corresponding 4-vector v_0^a by

$$v_0^a = (c, \vec{v}) = (c, v_1, v_2, v_3)$$

If $|\vec{v}| < c$ we introduce the normalized 4-vector v^a by

$$v^a := c\frac{v_0^a}{|v_0^a|} = \frac{1}{\sqrt{1 - |\vec{v}|^2/c^2}}(c, v_1, v_2, v_3)$$

Clearly, the definitions above imply

$$\langle v^a, v^a \rangle_L = v_a v^a = -c^2 \tag{3}$$

For a particle of "rest mass" m moving with velocity vector $\vec{v}$ (with respect to the inertial system $(e_i)_i$) we call the corresponding normalized 4-vector the 4-velocity and introduce the "momentum 4-vector" P^a by

$$P^a := mv^a \tag{4}$$

Digression 2 From definitions

$$E := cP^0 = \frac{1}{\sqrt{1 - |\vec{v}|^2/c^2}}mc^2$$

If particle at rest (in our inertial system) then $\vec{v} = 0$ so

$$E = mc^2$$

Moreover,

$$P_a P^a = -c^2 m^2$$

or, equivalently,

$$\tfrac{1}{c^2}E^2 = (P^0)^2 = c^2 m^2 + \sum_i P^i P_i$$

3 Classical and relativistic fluid dynamics

3.1 Classical fluid dynamics

Consider fluid (liquid or gas) in domain $D \subset \mathbb{R}^3$.

For simplicity take $D = \mathbb{R}^3$.

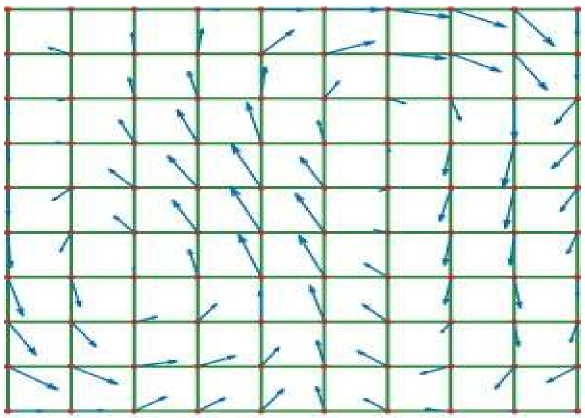


Figure 2 - A velocity field. Each point stores a velocity.

$\rho(x,t)$: mass-density of fluid
$\vec{u}(x,t)$: velocity field of fluid
$T(x,t)$: temperature distribution of fluid
Assume: Equation of state for fluid is known, i.e.

$$p = f(\rho, T) \tag{5}$$

where f is a known function.

Example 3 For an ideal gas we have $f(\rho, T) = c \cdot \rho\, T$ where c is a constant

3.1.1 Special case: perfect fluid (situation)

Special case: perfect fluid (situation)

- fluid is inviscid (= has vanishing viscosity)
- fluid in "thermal equilibrium", i.e. we have $T(x,t) = T_0$ where $T(x,t)$ is temperature distribution in fluid and T_0 is a constant

Mass conservation & momentum conservation $\Rightarrow$

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}(\rho\vec{u}) = 0 \quad \text{“continuity equation”} \tag{6}$$

$$\rho\Big(\frac{\partial}{\partial t}\vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{u}\Big) = -\vec{\nabla}p \quad \text{“Euler equation”} \tag{7}$$

where

$$p(x,t) := f(\rho(x,t), T_0) \text{ for all } x \text{ and } t \tag{8}$$

4 PDEs of first order in t for 4 unknown functions ρ, u_1, u_2, u_3

One can expect that there exists a unique solution $(\rho, \vec{u})$ for every "nice" initial configurations $\rho(x,0)$ and $\vec{u}(x,0)$

Digression 3

Relation $p = f(\rho, T)$ is invertible for fixed T, i.e.

$$\rho = g(p, T)$$

for a suitable function g.

$\Rightarrow$ We can take p as unknown function (instead of ρ) and use Euler and continuity equations with Eq. (8) replaced by

$$\rho(x, t) := g(p(x, t), T_0) \text{ for all } x \text{ and } t$$

Useful when fluid is "almost incompressible", i.e.

$\rho = g(p, T_0) \approx \rho_0$ where ρ_0 is a constant.

Then use idealization:

$$\rho = g(p, T_0) = \rho_0$$

(fluid "totally incompressible").

The continuity equation and Euler equation simplify:

$$\vec{\nabla} \cdot \vec{u} = 0 \tag{9}$$

$$\frac{\partial}{\partial t}\vec{u} + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\vec{\nabla} p / \rho_0 \tag{10}$$

4 PDEs of first order in t for 4 unknown functions p, u_1, u_2, u_3.

3.1.2 The general case

Drop condition that fluid is inviscid $\Rightarrow$

Generalization of Euler equation

$$\rho\Big(\frac{\partial}{\partial t}\vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{u}\Big) = -\vec{\nabla}\mathbb{T}(\vec{u},\rho) \tag{11}$$

with

$$\vec{\nabla}\mathbb{T}(\vec{u},\rho) := \sum_{i,j} \partial_i T_{ij}(\vec{u},\rho) e_j$$

where $T_{ij}(\vec{u},\rho)$ is the corresponding stress tensor, cf. Sec. 4 below. (if p is considered to be the free variable one uses $\mathbb{T}(\vec{u},p)$ instead of $\mathbb{T}(\vec{u},\rho)$).

Special case: fluid is "Newtonian" and incompressible (with constant density $\rho = \rho_0$) $\Rightarrow$

$$\vec{\nabla}\mathbb{T}(\vec{u},p) = \vec{\nabla}p - \nu\triangle\vec{u}$$

where $\nu > 0$ is viscosity (cf. Sec. 4.1).

$\Rightarrow$ Eq. (11) reads

$$\rho_0\Big(\frac{\partial}{\partial t}\vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{u}\Big) = -\vec{\nabla}p + \nu\triangle\vec{u} \quad \text{“Navier Stokes equation”} \tag{12}$$

One expects that there exists a unique solution $(\vec{u},p)$ for every "nice" initial configurations $\vec{u}(x,0)$ and $p(x,0)$

Digression 4

Most general situation: Drop condition of thermal equilibrium (i.e. condition $T(x,t) = T_0$).

$\Rightarrow$ temperature distribution $T(x,t)$ will be additional unknown function.

$\Rightarrow$ we need 5 equations, namely the 4 equations above + an additional equation.

This additional equation is obtained from energy conservation

(This equation will contain additional material constants like specific heat capacity and the thermal conductivity of fluid)

3.2 Relativistic fluid dynamics

Consider perfect fluid situation above: velocity field $\vec{u}$, density ρ, constant temperature T_0, equation of state $p = f(\rho, T_0)$.

Question: What is the relativistic modification of the continuity and Euler equation?

Answer: If u^a is normalized 4-vector of $\vec{u}$, i.e.

$$u^a = \frac{1}{\sqrt{1 - |\vec{u}|^2}}(1, u_1, u_2, u_3), \qquad (\text{so } u_a u^a = -1)$$

then

$$u_a \partial^a \rho + (\rho + p)\partial^a u_a = 0 \tag{13}$$

$$(\rho + p) u_a \partial^a u_b + (\eta_{ab} + u_a u_b)\partial^a p = 0 \tag{14}$$

Exercise 1 Show that $\frac{\partial}{\partial t}\rho + \vec{\nabla}(\rho \vec{u}) = 0$ is non-relativistic limit of Eq. (13) (note $p \ll \rho$ in non-relativistic limit).

"Miracle": Eqs. (13) and (14) can be rewritten in amazingly short and symmetric form

$$\partial^a S_{ab} = 0, \text{ where} \tag{15}$$

$$S_{ab} := (\rho + p)(u_a u_b) + p\, \eta_{ab}$$

Exercise 2 Show that Eq. (13) and Eq. (14) are equivalent to

$$u^c \partial^a S_{ac} = 0, \qquad \text{and}$$

$$\partial^a S_{ab} - \left(u^c \partial^a S_{ac}\right) u_b = 0$$

Later: we explain "miracle" with the help of "stress energy tensor"

4 The stress energy tensor

4.1 The classical stress tensor

The "stress tensor" is a concept describing the forces inside a continuous body, like a solid body, a liquid or a gas which are caused by outside forces (like gravity or forces on the surface) and movements inside the body (in the case of a liquid or gas).

Following general features:

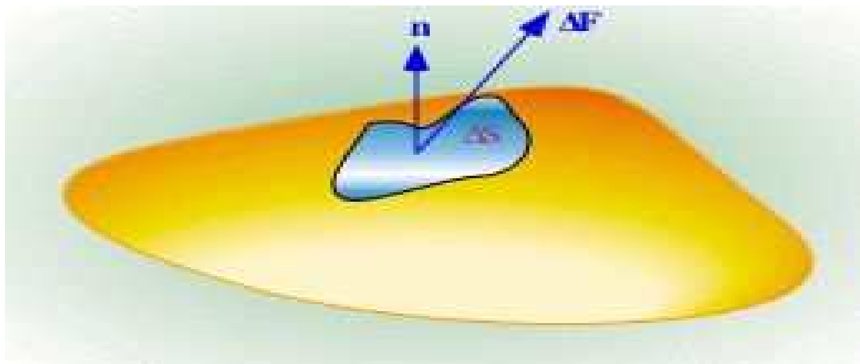
- Let $D \subset \mathbb{R}^3$ be the space taken by the body. The corresponding stress tensor is a tensor field T_{ij} on D.
- T_{ij} is in general time-dependent and depends on the variables describing of the body. For example, in the case of a liquid or gas $T_{ij} = T_{ij}(\vec{u}, \rho, t)$.
- Definition of T_{ij} is based on choosing suitable planes H in $\mathbb{R}^3$ (reason, see below)

Question: Why do we have to work with planes H?

Answer: Assume for simplicity that solid body or liquid considered is in equilibrium

Body in equilibrium $\Rightarrow$ in each point $x \in D$ the net force is zero.

Now let us assume that, for some plane H through x, we would suddenly remove the part of the body "to the right" (or "to the left") of H. In this moment the forces would no longer be in equilibrium. We would immediately, obtain a non-zero force $\triangle \vec{F}$ acting on some area element $\triangle S$ in H containing x



Observation 1 $\triangle \vec{F}$ will depend on the area of $\triangle S$.

Observation 2 H can be described uniquely by its normal vector $\vec{n}$ in x.

Convention 3 The length (resp. area resp. volume) of a time interval $\triangle t$ (resp. area element $\triangle S$ resp. volume element $\triangle V$) will also be denoted by $\triangle t$ (resp. $\triangle S$ resp. $\triangle V$).

Definition 1 i) For each unit vector $\vec{n} \in \mathbb{R}^3$ we define:

$$\vec{T}(\vec{n}) := \lim_{\triangle S \to 0} \frac{\triangle \vec{F}}{\triangle S}$$

More precisely: we set

$$\vec{T}(\vec{n}) := \lim_{\triangle S \to 0} \frac{\triangle \vec{F}(\vec{n}, \triangle S)}{\triangle S}$$

where $\triangle \vec{F}(\vec{n}, \triangle S)$ is the force acting on the area element $\triangle S \subset H$ after removing the part of the body on the "positive" side of the plane H (which is the plane through x and orthogonal to $\vec{n}$).

ii) We generalize this by setting, for arbitrary $\vec{v}$,

$$\vec{T}(\vec{v}) := \vec{T}(\vec{v}/|\vec{v}|) \cdot |\vec{v}|$$

if $|\vec{v}| \neq 0$ and $\vec{T}(\vec{v}) = 0$ otherwise.

Observation 3 The map $\mathbb{R}^3 \ni \vec{v} \mapsto \vec{T}(\vec{v}) \in \mathbb{R}^3$ is linear. Accordingly, this map is a tensor of type (1,1) and will be denoted by T^i_j or by $T^i_j(x)$.

Proof is non-trivial, cf. Wikipedia entry "stress tensor"

Definition 2 The "stress tensor" of the body considered is the tensor field T_{ij} on D, which is given by

$$T_{ij}(x) := g_{ik} T^k_j(x)$$

(If T_{ij} are the concrete components w.r.t to standard basis (e_i) of $\mathbb{R}^3$ we have

$$T_{ij}(x) = T^i_j(x) = (\vec{T}(e_j))_i$$

Observation 4

1. T_{ij} is symmetric.

2. If system in equilibrium then $\sum_i \partial_i T_{ij} = 0$

3. Let $A \in GL(3, \mathbb{R})$ and let $(e'_j)_j$ be the basis of $\mathbb{R}^3$ given by $e'_j = Ae_j$. Then if $(T'_{ij})_{ij}$ are the components of T_{ij} w.r.t. the new basis we have

$$T'_{ij} = A_i^k A_j^l T_{kl}$$

(this follows immediately from the tensor property of T_{ij}).

Proofs of first two statements: see again Wikipedia article.

Consider fluid situation above: velocity field $\vec{u}$, density ρ, constant temperature T_0, equation of state $p = f(\rho, T_0)$.

Example 4 (Inviscid fluid)

$$T_{ij}(\vec{u}, \rho) = p\,\delta_{ij} = p \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $p = f(\rho, T_0)$. (This equation is just the definition of "inviscid")

Example 5 (Incompressible Newtonian fluid)

$$T_{ij}(\vec{u}, p) = p\,\delta_{ij} - \nu(\partial_i u_j + \partial_j u_i)$$

where ν is the viscosity.

4.2 The stress energy tensor

Aim: Find tensor field $\tilde{T}_{ab}$, $a, b \in \{0,1,2,3\}$ on $(\mathbb{R}^4, \langle\cdot,\cdot\rangle_L)$, defined in an analogous way as the classical stress tensor T_{ij} such that "subtensor" field $\tilde{T}_{ij}$, $i, j \in \{1,2,3\}$ coincides with T_{ij} if system is at rest.

More precisely: If $T^{(t)}_{ij}(x)$ is classical stress tensor in $x \in \mathbb{R}^3$ at time t

$$\tilde{T}_{ij}(t,x) = T^{(t)}_{ij}(x) \quad \text{for } t \in \mathbb{R},\ i,j \in \{1,2,3\} \tag{16}$$

must hold for all $\vec{x} \in D$ in which no movement of the body is present at time t.

Obvious Replacements:

- point $x \in \mathbb{R}^3 \longrightarrow \tilde{x} = (t,x) \in \mathbb{R}^4$
- plane H through $x \longrightarrow$ hyperplane $\tilde{H}$ through $\tilde{x}$
- area element $\triangle S$ $(\ni x) \longrightarrow$ volume element $\triangle V$ $(\ni \tilde{x})$
- unit vectors $\vec{n} \longrightarrow$ normalized 4-vector n^a (i.e. $|n^a| = 1$).

Non-obvious Replacement:

- Force vector $\triangle\vec{F} \longrightarrow$ momentum 4-vector $\triangle P^a$.

  As a motivation observe that for $\triangle V = \triangle S \times \triangle t$ and $\triangle\vec{F} := \triangle\vec{P}/\triangle t$ we have

  $$\frac{\triangle\vec{P}}{\triangle V} = \frac{\triangle\vec{P}}{\triangle t}\frac{1}{\triangle S} = \frac{\triangle\vec{F}}{\triangle S}$$

Naive ansatz:

$$T^b(n^a) := \lim_{\triangle V \to 0} \frac{\triangle P^b}{\triangle V} \tag{17}$$

However: certain difficulties in interpretation of

$$\triangle P^b = \triangle P^b(n^a, \triangle V)$$

(Not all n^a can be treated analogously; sometimes "removal argument" is necessary and sometimes not)

$\Rightarrow$ We work with standard basis $(e_i)_i$ of $\mathbb{R}^4$ and consider only special case $n^a \in \{e_0, e_1, e_2, e_3\}$.

- We define $\triangle P^i(e_0, \triangle V)$, $i = 0, 1, 2, 3$, to be the "amount" of P^i contained in the (spacial) volume element $\triangle V$.
- We define $\triangle P^i(e_j, \triangle V)$ for $j = 1, 2, 3$ and $\triangle V = \triangle S \times \triangle t$ as the "amount" of P^i which is created in the time interval $\triangle t$ inside the spacial volume element

$$\triangle S \times (\mathbb{R}_+ e_i) \subset \mathbb{R}^2 \times (\mathbb{R}_+ \cdot e_i) \cong \mathbb{R}^2 \times \mathbb{R}_+$$

provided that at time t we have first removed all of the "amount" of P^i in $\mathbb{R}^2 \times (\mathbb{R}_+ \cdot e_i)$.

(here we assume that $\triangle t$ was chosen such that t is its left endpoint)

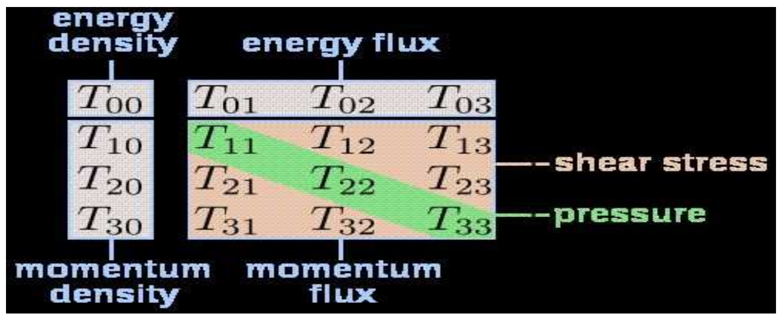
Define $T_{ij}(x)$ by

$$T_{ij}(x) := T^i(e_j) := \lim_{\triangle V \to 0} \frac{\triangle P^i(e_j, \triangle V)}{\triangle V} \tag{18}$$

One can see that

- T_{ij} for $i, j = 1, 2, 3$ indeed coincides with the classical stress tensor
- T_{i0} is the P^i-density
- T_{ij} for $j = 1, 2, 3$ is P^i-flux (density) in the e_j-direction

We can summarize this in the following picture:



Observation 5

1. T_{ab} is symmetric
2. We have $\partial^a T_{ab} = 0$

   (this follows from energy momentum conservation)
3. Let $A \in O(1,3)$ and let $(e'_i)_i$ be the basis of $\mathbb{R}^4$ given by $e'_i = Ae_i$ where $(e_i)_i$ is the standard basis of $\mathbb{R}^4$. Let $(T'_{ij})_{ij}$ denote the family of numbers which we would have got if we had defined the stress energy "tensor" using $(e'_i)_i$ as our inertial system instead of $(e_i)_i$. Then

$$T'_{ij} = A_i^k A_j^l T_{kl}$$

   (This implies that T_{ij} really is a tensor)

Remark 3 Definition can be generalized to arbitrary space times (M, g). Generalized T_{ab} will have analogous properties. But one exception: the generalization of $\partial^a T_{ab} = 0$ will not hold in general.

Example 6 (Perfect fluid) Reconsider situation of Sec. 3.2 (u^a is 4-velocity and ρ density function of perfect fluid on (M, g_{ab}) at temperature T_0 and equation of state $p = f(\rho, T_0)$):

i) Special case $(M, g_{ab}) = (\mathbb{R}^4, \eta_{ab})$ and fluid at rest, i.e. $u^a = (1, 0, 0, 0)$:

$$T_{ab} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

(cf. the last figure). Observe that we can rewrite T_{ab} as

$$T_{ab} = (\rho + p)u_a u_b + p\ \eta_{ab}$$

iii) Special case $(M, g_{ab}) = (\mathbb{R}^4, \eta_{ab})$ but u^a arbitrary constant field:

$$T_{ab} := (\rho + p)u_a u_b + p\ \eta_{ab}$$

This follows from i) by applying the principle of relativity and using the behavior of T_{ab} under a change of the inertial system.

iii) Special case $(M, g_{ab}) = (\mathbb{R}^4, \eta_{ab})$ but u^a arbitrary:

$$T_{ab} := (\rho + p)u_a u_b + p\ \eta_{ab}$$

This follows from ii) by a locality argument

iv) General case: (M, g_{ab}) and u^a arbitrary:

$$T_{ab} := (\rho + p)u_a u_b + p\ g_{ab}$$

Example 7 (Electromagnetic field) Let (M, g_{ab}) and let A_a be the 4-potential of a given electromagnetic field.

i) Special case $(M, g_{ab}) = (\mathbb{R}^4, \eta_{ab})$:

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b^{\ c} - \tfrac{1}{4}\eta_{ab}F_{cd}F^{cd}\right)$$

where

$$F_{ab} := \partial_a A_b - \partial_b A_a$$

ii) General (M, g_{ab}):

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b^{\ c} - \tfrac{1}{4}g_{ab}F_{cd}F^{cd}\right)$$

where now

$$F_{ab} := \nabla_a A_b - \nabla_b A_a$$

(where ∇_a is the Levi-Civita connection associated to (M, g_{ab})).

Digression 5 In fact, there is a heuristic functional derivative formula for an arbitrary field Ψ, for which a Lagrangian $\mathcal{L}(g, \Psi)$ is given explicitly:

$$T(g, \Psi) = -\frac{1}{8\pi} \frac{1}{\sqrt{-g}} \frac{\delta S(g, \Psi)}{\delta g}$$

with

$$S(g, \Psi) = \int_M \mathcal{L}(g, \Psi) dvol_g$$

where $g = \det((g_{ab})_{ab})$.

For example, the Klein-Gordon field of mass m has the Lagrangian

$$\mathcal{L}(g, \phi) = -\tfrac{1}{2}(\nabla_c \phi \nabla^c \phi + m^2 \phi^2)$$

and heuristically we obtain

$$T(g, \phi) = \nabla_a \phi \nabla_b \phi - \tfrac{1}{2} g_{ab}(\nabla_c \phi \nabla^c \phi + m^2 \phi^2)$$

where ∇_a is the Levi-Civita connection associated to (M, g_{ab}).