The Einstein field equations

Part II: The Friedmann model of the Universe

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Contents:

References:

- Book by Wald: "General Relativity"
- Wikipedia

1 Geometric background

1.1 Foundations

Recall: Euclidean Geometry is about (straight) lines, planes, lengths, angles, ... in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Modern formulation based on

Definition 1 A Euclidean space is a pair $(V, \langle\cdot,\cdot\rangle)$ where

- V is a finite-dimensional real vector space,
- $\langle\cdot,\cdot\rangle : V \times V \rightarrow \mathbb{R}$ a positive-definite (symmetric) bilinear form ("scalar product")

i) V can have arbitrary dimension $\rightarrow$ already a generalization

ii) Without loss of generality: $V = \mathbb{R}^n$ where $n := \dim(V)$.

iii) lines, planes, ...: definition only uses vector space structure

iv) lengths, angles: definition uses $\langle\cdot,\cdot\rangle$

$$length(v) := \|v\| := \sqrt{\langle v, v\rangle}$$

Angle $\varphi \in [0, \pi]$ between v and w given by

$$\cos(\varphi) = \tfrac{\langle v,w\rangle}{\|v\|\|w\|} \tag{1}$$

Convention 1 *We often write $g : V \times V \rightarrow \mathbb{R}$ instead of $\langle\cdot,\cdot\rangle$ and call $g = g(\cdot,\cdot)$ a "metric".*

Aim: Generalize this to more general spaces where notions like "distance" and "angles" can be defined.

Two of the best generalizations:

- (Pseudo-)Riemannian manifolds
- Metric spaces (with special properties)

Definition 2 A topological manifold M is a topological space which "looks locally like $\mathbb{R}^n$"

(formally: M is Hausdorff and every point has a neighborhood which is homeomorphic to $\mathbb{R}^n$ for some n).

Example 1 *i) Every open subset of* $\mathbb{R}^n$

ii) "Curved surfaces" in $\mathbb{R}^3$

iii) S^n *for arbitrary* n

"Non-example" Most non-open subsets of $\mathbb{R}^n$ are not topological manifolds.

Problem: A general topological manifold has no vector space structure. How can we define analogue of the metric $g(\cdot, \cdot)$?

Solution: Introduce additional structure $\rightarrow$

"Definition" 3 A smooth manifold is a topological manifold M equipped with certain extra-structure, called "differentiable structure". Differentiable structure allows definition of

i) the notion of "smoothness" for maps

ii) a "canonical" finite-dim. vector space T_xM in each $x \in M$.

iii) structure of a smooth manifold on $TM := \bigcup_{x \in M} T_xM$

Example 2 *i) $M = \mathbb{R}^n$. Here T_xM can be canonically identified with $\mathbb{R}^n$.*

ii) M is surface in $\mathbb{R}^3$: Here T_xM can be identified with some 2-dimensional subspace V_x of $\mathbb{R}^3$.

Definition 4 A tensor field of type (p, q) on a smooth manifold M is a "smooth" family $A = (A_x)_{x \in M}$ s. t. each A_x is a multilinear map

$$A_x : T_xM \times \ldots \times T_xM \times T_xM^* \times \ldots \times T_xM^* \rightarrow \mathbb{R}$$

where T_xM appears p times and T_xM^* appears q times.

Definition 5 i) A pseudo-Riemannian metric on a smooth manifold M is a tensor field $g = (g_x)_{x \in M}$ of type $(2, 0)$ on M s. t. each

$$g_x : T_x M \times T_x M \longrightarrow \mathbb{R}$$

is symmetric and non-degenerate.

ii) Let $g = (g_x)_{x \in M}$ be a pseudo-Riemannian metric on M. The "signature of g" is the signature of the bilinear form g_x for any x (independent of x!)

iii) Riemannian/Lorentzian metric on M is a pseudo-Riemannian metric on M with signature $(n, 0)/(n-1, 1)$ where $n = \dim(M)$.

Definition 6 i) A pseudo-Riemannian/Riemannian/Lorentzian manifold is a pair (M, g) where M is a smooth manifold and g is a pseudo-Riemannian/Riemannian/Lorentzian metric on M.

ii) A "spacetime" is a 4-dimensional Lorentzian manifold.

Remark 1 Pseudo-Riemannian metric g on $M = \mathbb{R}^n$ can be considered as a matrix $g = (g_{ab})_{1 \leq a,b \leq n}$ of smooth functions $g_{ab} : \mathbb{R}^n \longrightarrow \mathbb{R}$ s. t. for each $x \in \mathbb{R}^n$

- matrix $(g_{ab}(x))_{a,b}$ is symmetric
- matrix $(g_{ab}(x))_{a,b}$ has no zero eigenvalues

Digression 1 A "metric space" is a pair (X, d) where

- X is any set
- $d : X \times X \longrightarrow \mathbb{R}_+$ ("distance function" or "metric") s.t.

  i) $d(x, y) = 0$ if and only if $x = y$.

  ii) $d(x, y) = d(y, x)$

  iii) $d(x, y) \leq d(x, z) + d(z, y)$

Observation 1: Euclidean space $(V, \langle \cdot, \cdot \rangle) \longrightarrow$ metric space (V, d_V) where

$$d_V(v, w) := \sqrt{\langle v - w, v - w \rangle}$$

Observation 2:

$$\langle v, w \rangle = \tfrac{1}{2}\left[d(v, 0)^2 + d(w, 0)^2 - d(v, w)^2\right]$$

$\Rightarrow$ $\langle \cdot, \cdot \rangle$ can be reconstructed from d_V

$\Rightarrow$

$$\cos(\varphi(v, w)) = \frac{d(0, v)^2 + d(0, w)^2 - d(v, w)^2}{2d(0, v)d(0, w)}$$

$\Rightarrow$ lengths of vectors and angles between them can be defined using only the metric space structure (V, d_V)!

$\Rightarrow$ notions like lengths and angles can be defined in general metric space

1.2 The isometry group

Fix a Riemannian manifold $M = (M, g)$

Definition 7 (Isometry grouop)

$$\mathrm{Isom}(M) := \{\psi : M \to M \mid \ \psi \text{ is "bi-smooth" bijection preserving } g\}$$

(ψ "bi-smooth" = both ψ and ψ^{-1} are smooth).

Definition 8 i) M is "homogeneous" iff for all $x, y \in M$

$$\exists \psi \in \mathrm{Isom}(M) : \quad \psi(x) = y$$

ii) M is "isotropic" in $x \in M$ iff for all unit vectors $v, w \in T_x M$

$$\exists \psi \in \mathrm{Isom}(M) : \quad \psi_*(v) = w$$

where $\psi_* : TM \to TM$ is bi-smooth bijection induced by $\psi : M \to M$

Digression 2 i) $\mathrm{Isom}(M)$ has natural Lie group structure

ii) Every subgroup $\Gamma \subset \mathrm{Isom}(M)$ operates on M. If Γ is discrete and operation on M is "properly-discontinuous" then M/Γ has canonical Riemannian manifold structure.

1.3 Some basic results on curvature

Fix pseudo-Riemannian manifold (M, g).

Recall: we use "abstract index notation"

$\longrightarrow$ we write g_{ab} for the type (2,0) tensor g

- g^{ab} is type (0,2) tensor given by $\sum_b g_{ab} g^{bc} = \delta_a^c$
  (here δ_c^a is type (1,1) tensor given by $\delta_c^a(x) = \delta_{ac}$ for all $x \in M$)
- $R_{abc}{}^d$ denotes the curvature tensor associated to (M, g)
- We set $R_{ab} := \sum_c R_{acb}{}^c$ ("Ricci tensor")
- We set $R := \sum_{a,b} R_{ab} g^{ab}$ ("scalar curvature")

Convention 2 i) Einstein sum convention, i.e. we often drop $\sum$-signs. E.g. we write $R_{ab}g^{ac}$ instead of $\sum_a R_{ab}g^{ac}$.

ii) Normal rules for raising and lowering indices: e.g. we write $R_b{}^c$ instead of $R_{ab}g^{ac}$ and $v_a v^a$ instead of $g_{ab}v^b v^a$.

iii) Replace index set $\{1, 2, \ldots, n\}$ by $\{0, 1, \ldots, n-1\}$.

Remark 2 Elementary reformulation in special case $M = \mathbb{R}^n$:

Curvature tensor $({R_{abc}}^d)_{1\leq a,b,c,d\leq n}$ can be considered as a family of functions ${R_{abc}}^d : \mathbb{R}^n \to \mathbb{R}$ given explicitly as

$$ {R_{abc}}^d(x) \quad = \quad \partial_b \Gamma^d_{ac}(x) \;-\; \partial_a \Gamma^d_{bc}(x) \;+\; \sum_i \big(\Gamma^i_{ac}(x)\Gamma^d_{ib}(x) \;-\; \Gamma^i_{bc}(x)\Gamma^d_{ia}(x)\big) \quad (2) $$

where

$$ \Gamma^c_{ab}(x) := \frac{1}{2}\sum_d g^{cd}(x)\big(\partial_a g_{bd}(x) + \partial_b g_{ad}(x) - \partial_d g_{ab}(x)\big) \qquad (3) $$

Similarly, $(R_{ac})_{ac}$, and R can be considered as (matrix of) functions on $M = \mathbb{R}^n$. Explicitly:

$$ R_{ac}(x) \quad = \quad \partial_b \Gamma^b_{ac}(x) \;-\; \partial_a \Gamma^b_{bc}(x) \;+\; \sum_i \big(\Gamma^i_{ac}(x)\Gamma^b_{ib}(x) \;-\; \Gamma^i_{bc}(x)\Gamma^b_{ia}(x)\big) \quad (4) $$

Symmetry properties of $R_{abc}{}^d$, R_{abcd}, $R_{ab}{}^{cd}$ and R_{ab}

Proposition 1

i) $R_{abcd} = -R_{bacd}$

ii) $R_{abcd} = -R_{abdc}$

iii) $R_{abcd} + R_{bcad} + R_{cabd} = 0$ *("1. Bianchi identity")*

iv) $R_{abcd} = R_{cdab}$

v) $R_{ab} = R_{ba}$

(Similar but not totally analogous statements hold for $R_{abc}{}^d$ and $R_{ab}{}^{cd}$)

Proof: i) follows immediately from abstract definition of R_{abcd} or, for $M = \mathbb{R}^n$, from Eqs. (2) and (3) above.

ii) and iii): somewhat more difficult to prove

iv) follows from i)–iii)

v) follows immediately from iv)

Digression 3 i) For $d = 2$ all the information in $R_{abc}{}^d$ is already contained in the scalar curvature R.

ii) For $d = 3$ all the information in $R_{abc}{}^d$ is already contained in the Ricci tensor R_{ab}.

1.4 Spaces of constant curvature

Fix Riemannian manifold $M = (M, g)$

Definition 9 $M = (M, g)$ has constant curvature iff

$$R_{ab}{}^{cd} = K\delta_{ab}{}^{cd}$$

for some constant $K \in \mathbb{R}$ where $\delta_{ab}{}^{cd}$ is tensor field of type $(2, 2)$ given by

$$\delta_{ab}{}^{cd} := \delta_a{}^c \delta_b{}^d - \delta_a{}^d \delta_b{}^c$$

Let us assume now that $\dim(M) = 3$.

Theorem 1 *If $M = (M, g)$ is homogenous and isotropic in some point $x_0 \in M$ then M has constant curvature.*

Sketch of proof:

- View $R_{ab}{}^{cd}(x)$ and $\delta_{ab}{}^{cd}(x)$, for $x \in M$, as linear maps

$$T_xM \wedge T_xM \to T_xM \wedge T_xM$$

- $\delta_{ab}{}^{cd}(x)$ is identity on $T_xM \wedge T_xM$
- $R_{ab}{}^{cd}(x)$ is symmetric (w.r.t.obvious scalar product) and hence diagonalizable.
- Isotropy of M in x_0 implies that all eigenvalues of $R_{ab}{}^{cd}(x_0)$ must be the same, so $R_{ab}{}^{cd}(x_0) = K\delta_{ab}{}^{cd}(x_0)$ for some $K \in \mathbb{R}$.

  (rigorous treatment uses irreducibility argument, which is straightforward for $\dim(M) = 3$)
- Homogeneity of M implies that $R_{ab}{}^{cd}(x) = K\delta_{ab}{}^{cd}(x)$ for all $x \in M$.

Let

$$
\begin{aligned}
\mathbb{E}^n &:= \text{ standard n-dimensional Euclidean space} \\
S^n &:= \{x \in \mathbb{R}^{n+1} \mid \textstyle\sum_{i=1}^{n} x_i^2 + x_{n+1}^2 = 1\} \subset \mathbb{E}^{n+1} \\
\mathbb{H}^n &:= \{x \in \mathbb{R}^{n+1} \mid -\textstyle\sum_{i=1}^{n} x_i^2 + x_{n+1}^2 = 1\} \subset \mathbb{E}^{n+1}
\end{aligned}
$$

(S^n and $\mathbb{H}^n$ equipped with metric induced by $\mathbb{E}^{n+1}$).

Remark 3 $\mathbb{H}^n$ ("n-dimensional hyperbolic space") is homeomorphic to $\mathbb{E}^n$ but not isometric!

Theorem 2 *Let $M = (M, g)$ be simply-connected (!) n-dimensional (complete) Riemannian manifold with constant curvature $K \in \mathbb{R}$.*

Then (M, g) is isometric to suitable rescaling of

$$
(N, g_N) := \begin{cases} \mathbb{E}^n & \text{if } K = 0 \\ S^n & \text{if } K > 0 \\ \mathbb{H}^n & \text{if } K < 0 \end{cases}
$$

More precisely:

$$
(M, g) \cong (N, a \cdot g_N) \quad \text{for suitable } a > 0
$$

($a = \sqrt{|K|}$ in last two cases; in the first case a can be arbitrary).

Digression 4 If $M = (M, g)$ is a general n-dimensional (complete) Riemannian manifold of constant curvature then

$$M \cong N/\Gamma$$

where

$$N \in \{\mathbb{E}^n, S^n, \mathbb{H}^n\}, \quad \text{and}$$

Γ is suitable discrete subgroup of $\mathrm{Isom}(N)$.

Spaces of constant curvature play major role in 2-dim. and 3-dim. Topology/Geometry:

- $d = 2$: Classification of Riemannian surfaces can be reduced to classification of all discrete subgroups Γ of $\mathrm{Isom}(S^2)$, $\mathrm{Isom}(\mathbb{E}^2)$, and $\mathrm{Isom}(\mathbb{H}^2)$ which operate properly discontinuously.
- $d = 3$: Spaces of constant curvature play a major role in classification of compact 3-dimensional topological/smooth manifolds

2 Einstein field equations for perfect fluids

2.1 Review: The general Einstein field equations

Fix 4-dimensional smooth manifold M and $\Lambda \in \mathbb{R}$ ("the cosmological constant").

Let Φ be matter/radiation field on M. We assume that for every Lorentzian metric g on M

- corresponding "stress energy tensor" $T_{ab} = T_{ab}(g, \Phi)$ is known explicitly
- Equations of motions $F(g, \Phi) = 0$ for Φ are known explicitly, i.e. function F given explicitly.

Basic problem: Find (g, Φ) such that

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi T_{ab}(g, \Phi) \tag{5a}$$

$$F(g, \Phi) = 0 \tag{5b}$$

2.2 Review: Perfect fluid in Minkowski space

Recall: (relativistic or non-relativistic) fluid in $\mathbb{R}^3$ described by

- mass density distribution $\rho(x,t)$
- temperature distribution $T(x,t)$
- velocity field $\vec{u}(x,t)$

We assume that equation of state $p = f(\rho, T)$ is given explicitly.

In relativistic case introduce "4-velocity field" (= vector field in Minkowski space $(M,g) = (\mathbb{R}^4, \eta)$ where $\eta_{ab} = \pm\delta_{ab}$; $-$ only for η_{00})

$$u^a = \frac{1}{\sqrt{1-|\vec{u}|^2}}(1, u_1, u_2, u_3)$$

Observe that

$$u_a u^a = -1 \tag{6}$$

If fluid is a "perfect fluid" (i.e. is "inviscid" and in thermal equilibrium, i.e. $T(x,t) = T_0$ for a constant T_0) then:

"Stress energy tensor" given by

$$T_{ab} = (\rho + p)u_a u_b + p\ \eta_{ab} \tag{7}$$

where $p(x,t) = f(\rho(x,t), T_0)$ and equation of motions are

$$\partial^a T_{ab} = 0$$

2.3 Perfect fluids in a general space time

Fluid in general space time (M, g) described by

- mass density distribution $\rho(x, t)$
- temperature distribution $T(x, t)$
- abstract "4-velocity field" $u^a(x, t)$, i.e. arbitrary vector field (=tensor field of type (0,1)) with

$$u_a u^a = -1$$

Again assume that equation of state $p = f(\rho, T)$ given explicitly.

In "perfect fluid situation" (where fluid is "inviscid" and in thermal equilibrium at temperature T_0) stress-energy tensor is given by

$$T_{ab} = (\rho + p)u_a u_b + p\ g_{ab} \tag{8}$$

where $p(x, t) = f(\rho(x, t), T_0)$ and equations of motion are

$$\nabla^a T_{ab} = 0$$

where ∇^a is the Levi-Civita connection of (M, g_{ab}).

2.4 The Einstein field equations for perfect fluids

Taking $\Phi = (u^a, \rho)$ in Eqs. (5a) and (5b) above we see that for a perfect fluid in M (with equation of state $p = f(\rho, T)$ at temperature T_0) the corresponding Einstein field equations read

$$R_{ab} - \tfrac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \tag{9a}$$

$$\nabla^a T_{ab} = 0, \text{ with} \tag{9b}$$

where $T_{ab} = (\rho + p) u_a u_b + p\ g_{ab}$ and $p(x,t) = f(\rho(x,t), T_0)$.

Observation: We always have

$$\nabla^a \big(R_{ab} - \tfrac{1}{2} R g_{ab} \big) = 0, \qquad \nabla^a g_{ab} = 0,$$

$\Rightarrow$ Eq. (9a) implies Eq. (9b)!

$\Rightarrow$ Einstein field equations in perfect fluid situation

$$R_{ab} - \tfrac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi \big((\rho + p) u_a u_b + p\ g_{ab} \big) \tag{10a}$$

$$p(x,t) = f(\rho(x,t), T_0) \tag{10b}$$

Special case: Fluid has vanishing pressure, i.e $p = f(\rho, T) = 0$ ("Dust situation"):

$\Rightarrow$ Eqs. (10) reduce to

$$R_{ab} - \tfrac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi \rho u_a u_b \tag{11}$$

3 The Friedmann(-Robertson-Walker) model

3.1 Assumptions

Consider spacetime $M = (M, g)$ fulfilling:

Assumption 1 (Product Ansatz)

i) $M \cong \mathbb{R} \times \Sigma$

ii) $\Sigma_t \cong \{t\} \times \Sigma$ is orthogonal to $\mathbb{R} \times \{\sigma\}$, $\sigma \in \Sigma$.

iii) $\Sigma_t \cong \{t\} \times \Sigma$, $t \in \mathbb{R}$, is "space-like"
(i.e. restriction g_t of g to Σ_t is a Riemannian metric)

Assumption 2 Each (Σ_t, g_t) is homogenous.

Assumption 3 Each (Σ_t, g_t) is isotropic in each $x \in \Sigma_t$.

Assumption 4 M is simply-connected

Assumption 5 Only one matter field, namely a perfect fluid

For simplicity:

Assumption 6 i) Perfect fluid is "dust"

ii) Cosmological constant $\Lambda = 0$

3.2 The Robertson-Walker metric

Assumption 1 $\Rightarrow$ $T_{(t,x)}M \cong T_t\mathbb{R} \oplus T_x\Sigma$.

For fixed $(t,x) \in \mathbb{R} \times \Sigma \cong M$ we can choose basis $(e_i)_{i=0,1,2,3}$ of $T_{(t,x)}M$ such that

$$\begin{cases} e_0 & \in T_t\mathbb{R} \subset T_t\mathbb{R} \oplus T_x\Sigma \\ e_i & \in T_x\Sigma \subset T_t\mathbb{R} \oplus T_x\Sigma, \ i = 1,2,3 \end{cases}$$

Conclusion 1 In basis above we have

$$(g_{ij})_{ij} = (g_{ij}(t,x))_{ij} = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & g_{12} & g_{13} \\ 0 & g_{21} & g_{22} & g_{23} \\ 0 & g_{31} & g_{32} & g_{33} \end{pmatrix} \tag{12}$$

where $g_{00} < 0$. Moreover, by a suitable reparametrization of t we can achieve that $g_{00} = -1$. Finally, $g_t = (g_{ij})_{i,j=1,2,3}$.

Assumptions 2–4 and Theorem 1
$\Rightarrow$ (Σ_t, g_t) is simply-connected Riem. manifold of const. curvature $\Rightarrow$ (cf. Theorem 2)

Conclusion 2 (Σ_t, g_t) is isometric to $(N, g_N) \in \{\mathbb{E}^3, S^3, \mathbb{H}^3\}$ after rescaling with suitable $a(t) \in \mathbb{R}_+$ (i.e. $g_t = a(t) \cdot g_N$)

Remark 4 If $(N, g_N) = \mathbb{E}^3$ then (cf. Remark 1)

$$(g_{ij})_{ij} = (g_{ij}(t,x))_{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & a(t)^2 & 0 \\ 0 & 0 & 0 & a(t)^2 \end{pmatrix} \tag{13}$$

Remark 5 Metric g written in "standard" local coordinates:

- $N = \mathbb{E}^3$: $g = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)$
- $N = S^3$: $g = -dt^2 + a(t)^2\big(d\psi^2 + \sin^2(\psi)(d\theta^2 + \sin^2(\theta)d\varphi^2)\big)$
- $N = \mathbb{H}^3$: $g = -dt^2 + a(t)^2\big(d\psi^2 + \sinh^2(\psi)(d\theta^2 + \sin^2(\theta)d\varphi^2)\big)$

Conclusion 3 $u^a(t,x) = (1,0,0,0)$ and $\rho(t,x) = \rho(t)$. Thus

$$T_{ab} = T_{ab}(t,x) = \begin{pmatrix} \rho(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{14}$$

(recall $T_{ab} = \rho u_a u_b$ in dust situation).

- Intuitively, Conclusion 3 is "clear"
- Formal proof in general case not too difficult
- In the special case $N = \mathbb{E}^3$ it follows easily from computations below

3.3 Reduction of the Einstein field equations

Aim: Simplify Eq. (11) if Assumptions 1 – 6 are fulfilled.

For simplicity: consider only $N = \mathbb{E}^3$ where

$$(g_{ij})_{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & a(t)^2 & 0 \\ 0 & 0 & 0 & a(t)^2 \end{pmatrix} \tag{15}$$

$\Rightarrow$ non-vanishing components of Γ^k_{ij} are (cf. Eq. (3))

$$\Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = a'a, \tag{16}$$

$$\Gamma^1_{10} = \Gamma^1_{01} = \Gamma^2_{20} = \Gamma^2_{02} = \Gamma^3_{30} = \Gamma^3_{03} = a'/a, \tag{17}$$

$\Rightarrow$ (cf. Eq. (4))

$$\big(R_{ij}\big)_{ij} = \begin{pmatrix} -3\frac{a''}{a} & 0 & 0 & 0 \\ 0 & a''a + 2(a')^2 & 0 & 0 \\ 0 & 0 & a''a + 2(a')^2 & 0 \\ 0 & 0 & 0 & a''a + 2(a')^2 \end{pmatrix}$$

and therefore $R = 6\frac{a''a+(a')^2}{a^2} \Rightarrow$

$$\big(R_{ij} - \tfrac{1}{2}Rg_{ij}\big)_{ij} = \begin{pmatrix} 3\frac{(a')^2}{a^2} & 0 & 0 & 0 \\ 0 & F(a) & 0 & 0 \\ 0 & 0 & F(a) & 0 \\ 0 & 0 & 0 & F(a) \end{pmatrix} \overset{!}{=} 8\pi\big(T_{ij}\big)_{ij}$$

where $F(a) := -2a''a - (a')^2$

$$\left(R_{ij} - \tfrac{1}{2}Rg_{ij}\right)_{ij} = \begin{pmatrix} 3\frac{(a')^2}{a^2} & 0 & 0 & 0 \\ 0 & F(a) & 0 & 0 \\ 0 & 0 & F(a) & 0 \\ 0 & 0 & 0 & F(a) \end{pmatrix} \stackrel{!}{=} 8\pi\left(T_{ij}\right)_{ij}$$

$\Rightarrow$ reduces to system of two ODEs for $a = a(t)$ and $\rho = \rho(t)$,

$$3\tfrac{(a')^2}{a^2} = 8\pi\rho, \qquad -2a''a - (a')^2 = 0$$

or, equivalently,

$$3\frac{(a')^2}{a^2} = 8\pi\rho, \qquad 3\frac{a''}{a} = -4\pi\rho \tag{18}$$

Similar computation for $N \in \{S^3, \mathbb{H}^3\} \quad \Rightarrow$

$$3\frac{(a')^2}{a^2} = 8\pi\rho - \frac{3k}{a^2}, \qquad 3\frac{a''}{a} = -4\pi\rho \tag{19}$$

where $k = 1$ for $N = S^3$ and $k = -1$ for $N = \mathbb{H}^3$

Problem: For $k \in \{-1, 0, 1\}$ find solutions $(a, \rho) = (a(t), \rho(t))$ $a(t) : I \to \mathbb{R}_+$ and $\rho(t) : I \to \mathbb{R}_+$ on interval $I \subset \mathbb{R}$

$$3\frac{(a')^2}{a^2} = 8\pi\rho - \frac{3k}{a^2}, \qquad 3\frac{a''}{a} = -4\pi\rho$$

(a must be C^2 and ρ must be C^1)

Temporary assumption: $a'(t) \geq 0$ on I

3.4 Explicit solution of the Einstein field equations

We want to solve

$$3\frac{(a')^2}{a^2} = 8\pi\rho - \frac{3k}{a^2}, \qquad 3\frac{a''}{a} = -4\pi\rho$$

First note that

$$\rho' + 3\rho\frac{a'}{a} = 0$$

and therefore

$$(\rho a^3)' = (\rho' + 3\rho\frac{a'}{a})a^3 = 0$$

so

$$\rho = \frac{C}{a^3}, \qquad \text{for some } C > 0$$

Thus

$$3(a')^2 = 8\pi\frac{C}{a} - 3k$$

and therefore (recall assumption $a' \geq 0$ on I)

$$\frac{da}{dt} = a' = \sqrt{\frac{C'}{a} - k} \quad \text{with } C' := 8\pi C/3 \tag{20}$$

so

$$dt = \frac{da}{\sqrt{\frac{C'}{a} - k}}$$

so

$$t(a) = \int \frac{1}{\sqrt{\frac{C'}{a} - k}}\, da + const$$

Problem: Find explicit formula for

$$t(a) = \int \frac{1}{\sqrt{\frac{C'}{a} - k}} da, \qquad \text{defined on } \begin{cases} (0, \infty) & \text{if } k = 0, -1 \\ (0, C'] & \text{if } k = 1 \end{cases}$$

Solution:

$$t(a) = \begin{cases} \frac{1}{\sqrt{C'}} \, \frac{2}{3} \, a^{3/2} + const & \text{if } k = 0 \\ \frac{C'}{2}(\sinh(x) - x)_{|x = \operatorname{arccosh}(\frac{2a}{C'} + 1)} + const & \text{if } k = -1 \\ \frac{C'}{2}(x - \sin(x))_{|x = \arccos(1 - \frac{2a}{C'})} + const & \text{if } k = 1 \end{cases}$$

Derivation:

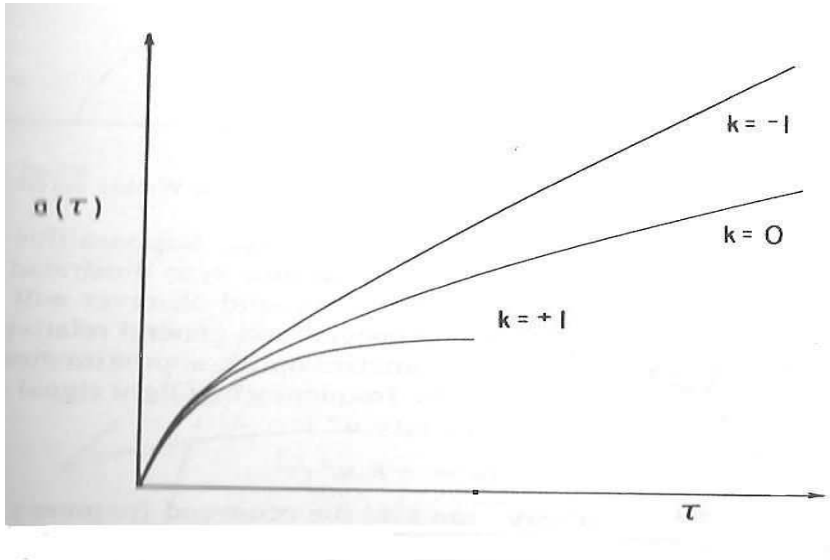
- $k = 0$: easy
- $k = -1$: similar to case $k = 1$
- $k = 1$: Use substitution $x = \arccos(1 - \frac{2a}{C'}) \quad \rightarrow$

$$\int \frac{1}{\sqrt{\frac{C'}{a} - k}} \, da \quad \text{transformed into}$$

$$\int \frac{\frac{C'}{2} \sin(x) dx}{\sqrt{\frac{2}{1 - \cos(x)} - 1}} = \frac{C'}{2} \int (1 - \cos(x)) dx = \frac{C'}{2}(x - \sin(x)) + const$$

Remark 6 We obtain $a(t)$ by inverting $t(a)$, e.g. for $k = 0$

$$a(t) = c \cdot (t - t_0)^{2/3}, \qquad c := \Big(\frac{3\sqrt{C'}}{2}\Big)^{2/3}, \quad t_0 := const.$$

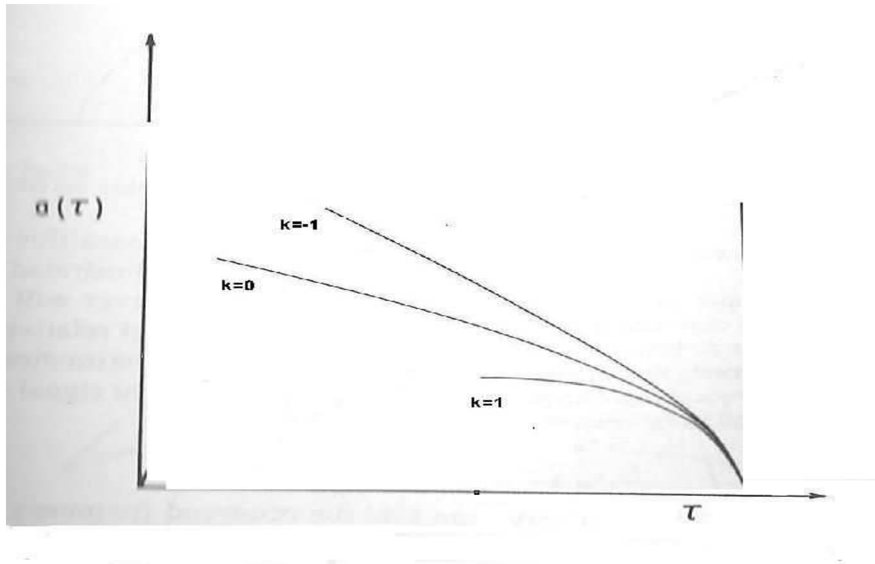


Observation: Solutions for $k = 0, -1$ are "maximal",

Solution for $k = 1$ is not maximal.

Recall: We assumed above that $a' \geq 0$ on interval I.

Situation $a' \leq 0$ can be treated similarly. We obtain



Again the solutions for $k = 0, -1$ are maximal but the solution for $k = 1$ is not maximal.

However, the two solutions for $k = 1$ (the one with $a' \geq 0$ and the one with $a' \leq 0$) can be "joined" to give a maximal solution.

Full Solutions:

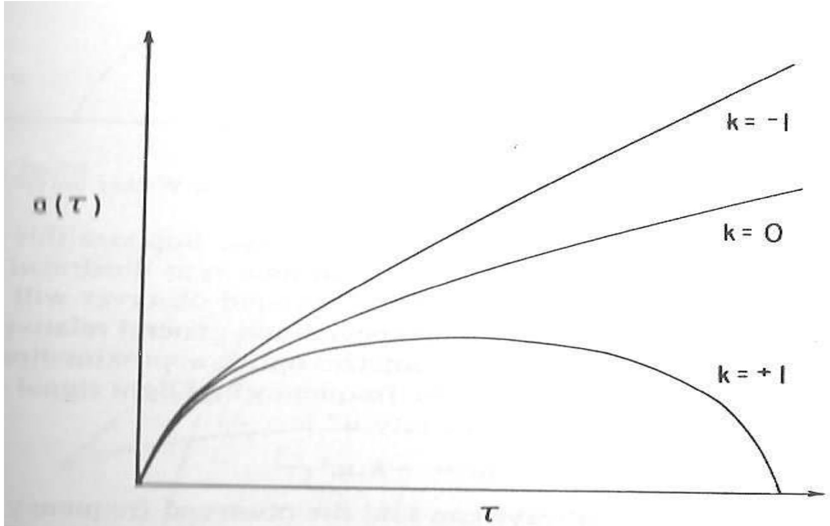


Fig. 5.3. The dynamics of dust-filled Robertson-Walker universes.

Remark 7 Our universe is expanding at the moment.

$$\text{“Hubble's constant”} \quad H(t_0) := a'(t_0)/a(t_0), \qquad t_0 = \text{present time}$$

can be determined experimentally by measuring the “redshift” in the spectral lines of the light coming from distant galaxies. One finds $H(t_0) > 0$.

Summary:

- “Big bang” singularity
- Eternal expansion for $k = 0, -1$; recollapse (=“big crunch”) for $k = 1$.
- For $k \in \{-1, 0\}$: $M \cong \mathbb{R} \times \mathbb{R}^3 \cong \mathbb{R}^4$ and each (Σ_t, g_t) has infinite volume.
- For $k = 1$: $M \cong \mathbb{R} \times S^3$ and each (Σ_t, g_t) has finite volume.

Open problem: $k = -1$ or $k = 0$ or $k = 1$ for our universe?