DIFFERENTIAL GEOMETRY WITH APPLICATION TO DISLOCATION THEORY AND EINSTEIN FIELD EQUATIONS

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LECTURE SERIES, February 19, 2010

Introduction: Einstein field equations

The Einstein field equations are a set of 10 equations in Einstein's theory of general relativity which describe the fundamental interaction of gravitation as a result of spacetime being curved by matter and energy

$$\boxed{R_{ij} - \frac{1}{2} g_{ij} R + g_{ij}\Lambda = \frac{8\pi G}{c^4} T_{ij}}$$

- R_{ij}, R: Ricci and Gauss curvature (of the underlying spacetime manifold)
- g_{ij}: (pseudo-Riemannian) Minkowski metric of spacetime
- T_{ij}: energy-momentum tensor

Main point of this talk: Geometry of the LHS vs physics of the RHS

Other point will be the physics of defects. Why?

- Massive objects of our universe do modify its intrinsic curvature
- Defects in a crystal modify its intrinsic metric, curvature and torsion
- These geometric properties imply dynamical laws

A historical interplay between physics and mathematics

- Newton (1713). Mass, acceleration: velocities at different space points
- Euler (1748). Calculus of variations
- Lagrange (1754). Generalization of CV to arbitrary coordinate systems. Euler-Lagrange equations depend of the velocity and position
- Gauss (1827-1847). Theory of surfaces, Geodesics, Curvature. Theorema Egregium ("remarkable"): curvature as an intrisic property of a surface
- Riemann (1854). Generalization of Gauss work to N-dimensional "manifolds", general metric and curvature
- Christoffel (1869). Relations between differentials of order 2. Covariant derivative. Connexion. Symbols
- Ricci & Levi-Civita (1900) Systematization, theorization. Tool for physics
- Einstein (1905-1912). Special Relativity (Space-Time), General relativity (Gravitation modifies the Geometry)

BASIC NOTIONS

Cartesian space

- A point $\Rightarrow$ real coordinates $\{\alpha^i\}$
- $\neq$ point $\Leftrightarrow$ $\neq$ coordinates
- All n-uples are admissible
- Change of coord. $A^i_j = \frac{\partial \alpha^i}{\partial \alpha'^j}$

Euclidean space

- Euclid. length in a Cart. space
  $l_E^2 = \left(x^i_Q - x^i_P\right)^2$,
  $\alpha^i = x^i =$ length
- Scalar Product $< \xi, \eta >= \xi^i \eta^i$
- Angle φ: $\cos\varphi = \frac{<\xi,\eta>}{|\xi||\eta|}$

Riemannian space

- Rieman. length in Cart. space
  $l_g = \int_a^b \sqrt{g_{ij} \dot{x}^i \dot{x}^j}\, dt,\ \dot{x}^i = \frac{dx^i}{dt}$
- Riemannian metric g_{ij}: smooth positive definite quadratic form
- Scalar Prod. $< \xi, \eta >= g_{ij} \xi^i \eta^j$

Euclidean metric

- If $\exists A^i_j$ s.t. $\forall P:\ g_{ij} = A^k{}_i A^k{}_j$

Pseudo-Riemannian metric

- g_{ij} must not be positive definite:
  ex.: Minkowski metric

EXAMPLES

- Euclidean spherical coordinates (r, θ, φ): $dl^2 = dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2 \right)$,

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{bmatrix}_{ij}$$

- Pseudo-Euclidean Minkowski space $\mathbb{R}^4_{1,3}$: coordinates (ct, x_i), length $dl^2 = dx_0^2 - dx_i^2$. A "world-line" has tangent vector $\xi = (c, \dot{x}_i)$ with $\xi^2 \geq 0$ (light-like ($= 0$ – photon) or time-like (> 0 – massive particle))

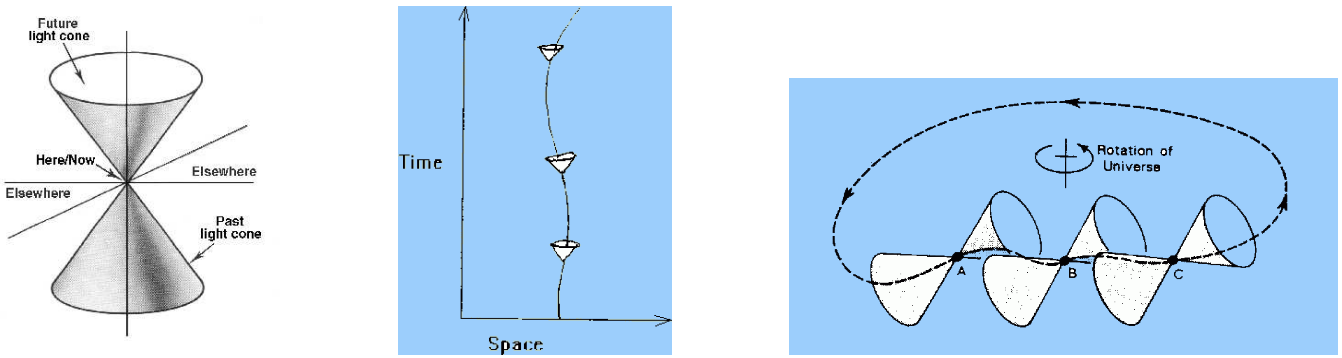

Figure 1: Word-line and light cones (Gribbin 19992)

- Induced metric g_{ij} on a surface. Take a curve $r(t) = (x(t), y(t), z(t))$ on a (x^1, x^2)-surface of $\mathbb{R}^3$: $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = g_{ij}\dot{x}^i\dot{x}^j, \quad 1 \leq i, j \leq 2$

- First fundamental form: $g_{ij}dx^i dx^j = E(dx^1)^2 + 2F(dx^1 dx^2) + G(dx^2)^2$

- Riemannian non-Euclidean metric: $g_{ij} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}_{ij}$

- $2D$ surface in $3D$ Euclidean space: $F(x^1, \cdots, x^n) = 0$ and

$$g_{ij} = \delta_{ij} + \frac{(\partial F/\partial x^i)(\partial F/\partial x^j)}{(\partial F/\partial x^n)^2}$$

- Surface $z = f(x, y)$ and consider $d^2 f = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$. The hessian of f is $H = [f_{ij}]$.

  1. Mean curvature at P is $\text{tr } H$
  2. Gauss curvature at P is $\det\ H$ (intrisic notion)

The elastic metric

- Consider an elastic solid submitted to internal and external loads
- The stress $[\frac{\text{force}}{\text{surface}}]$ is given at every interior point by a matrix $[\sigma_{ij}]$
- In linear elasticity, the strain is defined from the stress by the Lamé relation $\epsilon_{ij} = A^{-1}_{ijkl}\sigma_{kl}$ (diagonal elements mean relative stretch of matter)
- The elastic metric is $g^E_{ij} = \delta_{ij} - 2\epsilon_{ij}$
- The external observer is equiped with the Cartesian metric δ_{ij} and coordinates $\{x^i\}$
- The elastic metric is Euclidean if one finds (holonomic) coordinates $\{\alpha^j\}$
- This will happen if $g^E_{ij} = A^m_i A^m_j$ for some $A^m_i = \frac{\partial x^m}{\partial \alpha^i}$
- Small displacements: $a^m_i = \delta^m_i - \frac{\partial u^m}{\partial x^i}$ IFF the strain is compatible
- In the presence of line-like defects (dislocations & disclinations) it is not compatible

Covariance and contravariance

Contravariant object: the velocity vector

- Change of base: $v^j = \frac{dx^j}{dt}, v'^i = \frac{dx'^i}{dt} \Longrightarrow v'^i = A^i_j v^j$ with $A^i_j = \frac{\partial x^i}{\partial x'^j}$
- Above indice $\Rightarrow$ contravariance: “velocity-like” object (live on the manifold)

Covariant object: the gradient of a scalar

- Change of base: $\nabla_i f = \frac{df}{dx^i}, \nabla_j f = \frac{df}{dx'^j} \Longrightarrow \nabla_i f = A^j_i \nabla_j f$ with A^j_i the inverse of A^i_j
- Below indice $\Rightarrow$ covariance: “gradient-like” object (live on the tangent space)

Most physical quantities: mixed Covariant/Covariant object

- Tensor field of type (or “valence”) (p, q) and order (or “rank”) $p + q$
- ex.: T^{IJKL}_{mn} is of type $(4, 2)$

Tensor fields

- Every physical property is represented by means of a tensor field (of some given type and order)
- A tensor is defined relatively to a system of coordinates
- In this system a tensor is given by its components $T^{IJKL\cdots}_{mn\cdots}$
- Main property of tensors: law w.r.t. change of coordinate system:

$$\boxed{T'^{BCDE}_{pq} = \left(A^B_I A^C_J A^D_K A^E_L A^m_p A^n_q \cdots\right) T^{IJKL\cdots}_{mn\cdots}}$$

- Examples: velocity or normal vector n to a surface is a $(1,0)$-tensor, temperature gradient is a $(0,1)$-tensor, the stress tensor σ is a $(1,1)$-tensor. The metric g is a $(0,2)$-tensor
- Take a solid with an internal infinitesimal facet of normal n^i. Then $\sigma^m_i n^i = f^m$ with f^m the (contravariant) $m - th$ component of the applied local force on the facet (clearly f^m depends of the coordinate system, but represents the same physical quantity SINCE it transforms as a tensor)

Objectivity (frame indifference)

- Objectivity means invariance w.r.t. change of observer
- An objective quantity is represented by a tensor
- HOWEVER: most physical properties are not objective.
  Example. Let $x'(t) = A(t)(x - x_0(t))$ by a change of origin and a rotation of the axis (Euclidean coordinate change).
  The velocity is not objective: $v' = Av + \dot{x}_0(t) + (\dot{A}A^T)(x - x_0(t))$ except for a Galilean (or inertial) change of axis: $x'(t) = A(x - x_0)$
- BUT: the divergence of the velocity is objective: it is the scalar (0-order tensor) $\nabla \cdot v = \partial_i v^i$ where $\partial_i = \frac{\partial}{\partial x^i}$ (Euclidean coordinates)
- An objective physical quantity $\mathbf{u}$ is written

$$\boxed{\mathbf{u} = u^i \mathbf{e_i} \text{ where } \mathbf{e_i} = \frac{1}{|\frac{\partial \mathbf{r}}{\partial \alpha^i}|}\frac{\partial \mathbf{r}}{\partial \alpha^i} \text{ and } \mathbf{r} = x - x_0 \text{ is the position vector}}$$

Towards Christoffel symbols (1)

- For a Euclidean change of base: $\nabla \cdot v = \partial_i v^i = \partial'_j v'^j$

$$\boxed{\text{What happens for a general change of base?}}$$

- Partial answer.
  $\nabla \cdot u = \frac{1}{\sqrt{|g|}} \partial_i(\sqrt{|g|} u^i)$ where $g = \det[g_{ij}]$
- More general question.
  How does $\nabla \mathbf{u} = \partial_j u^i (\mathbf{e_i} \mathbf{e^j})$ transform under general change of base?
- Theorem 1.
  The quantity $\nabla_k T^{IJKL\cdots}_{mn\cdots} := \partial_k T^{IJKL\cdots}_{mn\cdots}$ transform as a tensor if A =constant (linear coordinate change)

Towards Christoffel symbols (2)

- Theorem 2.
  Given the vector field v^I and the quantity $\nabla_k v^I$ writing as $\partial_k v^I$ in Euclidean coordinates. Then $\nabla_k v^I$ transform as a $(1,1)$-tensor w.r.t. to arbitrary Riemannian coordinates change $x^i \to \alpha^j$ iff the transformed components are $\nabla'_l v'^J = \frac{\partial v'^J}{\partial \alpha^l} + \Gamma^J_{pl} v'^p$ (Γ^J_{pl} depending on the coordinates)
- In particular: $\nabla \cdot v = \frac{\partial v'^l}{\partial \alpha^l} + \Gamma^J_{pJ} v'^p$ with $\Gamma^J_{pJ} := \partial_p \ln(\sqrt{g})$

- Theorem 3.
  Given the co-vector field u_i and the quantity $\nabla_k u_i$ writing as $\partial_k u_i$ in Euclidean coordinates. Then $\nabla_k u_i$ transform as a $(0,2)$-tensor w.r.t. to arbitrary Riemannian coordinates change $x^i \to \alpha^j$ iff the transformed components are $\nabla'_l u'_j = \frac{\partial u'_j}{\partial \alpha^l} - \Gamma^p{}_{jl} u'_p$

Towards Christoffel symbols (3)

- Einstein: "To take into account gravitation, we assume the existence of Riemannian metrics. But in nature we also have electromagnetic fields, which cannot be described by Riemannian metrics. The question arises: How can we add to our Riemannian spaces in a logically natural way an additional structure that provides all this with a uniform character ?"

- This additional notion is the "Columbus connexion": for Columbus, navigating straight right meant going westwards, that is, on a sphere, to keep a fixed angle with respect to the lines of constant latitude

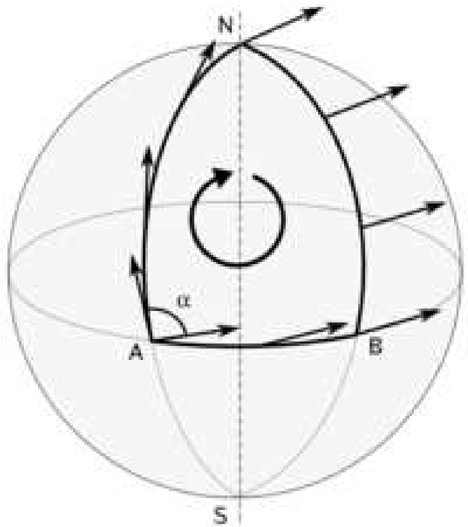

Figure 2: Vectors end up with an angle as parallelly transported along 2 curves

- The connexion is the differential geometric property which governs the law of parallel transport of vectors generalising Euclidean parallelism
- In Euclidean geometry, the parallelism of two vectors means equaling their components. In Riemannian geometry this is no longer true and the parallelism of two vectors depends on the vector origin positions, the choice of a curve joining these two points and of the space connexion

Christoffel symbols

- Definition in terms of Euclidean/general coordinates:

$$\boxed{\Gamma^n_{lj} = -\frac{\partial x^p}{\partial \alpha^l}\frac{\partial x^q}{\partial \alpha^j}\frac{\partial^2 \alpha^n}{\partial x^p \partial x^q}}$$

- $\nabla \mathbf{v} = \left(\frac{\partial v'^J}{\partial \alpha^k} + \Gamma^J_{pk} v'^p\right)\mathbf{e_J e^k}$, $\nabla \mathbf{u} = \left(\frac{\partial u'_j}{\partial \alpha^k} - \Gamma^p_{jk} u'_p\right)\mathbf{e^j e^k}$
- For a tensor T of type $(0,2)$: $\nabla \mathbf{T} = \left(\frac{\partial T_{ij}}{\partial \alpha^k} - \Gamma^l_{ik} T_{lj} - \Gamma^l_{jk} T_{il}\right)\mathbf{e^i e^j e^k}$
- How do the Christoffel symbols transform under arbitrary coordinate change?

$$\boxed{\Gamma'^m_{ki} = \frac{\partial \alpha'^m}{\partial \alpha^n}\left(\Gamma^n_{lj}\frac{\partial \alpha^l}{\partial \alpha'^k}\frac{\partial \alpha^j}{\partial \alpha'^i} - \frac{\partial^2 \alpha^m}{\partial \alpha'^k \partial \alpha'^i}\right) \quad \text{(connexion)}}$$

- The Christoffel symbols transform as tensors only under affine coordinate change
- An object which transforms under arbitrary coordinate change according to the law above is called a CONNEXION

Parallel transport of a vector field

- Consider a curve $x^i(s)$ and two points P and Q of this curve. Consider a vector field $\xi(x)$
- In a Euclidean space, two tensors $\xi(P)$ and $\xi(Q)$ are parallel if $\frac{dx^i}{ds}\frac{\partial\xi}{\partial x^i}(s) = \frac{d\xi}{ds} = 0$ (have equal tensor components along the curve)
- In a (general) Riemannian space, $\xi(P)$ and $\xi(Q)$ are parallel along a curve of tangent vector τ^i if $\tau^i \nabla_i \xi^J = 0$
- A geodesic w.r.t. a given connexion is a curve with tangent vector τ_i satisfying

$$\boxed{\partial_\tau \tau = \tau^i \nabla_i \tau^J = \frac{\partial \tau^J}{\partial \alpha^i}\tau^i + \Gamma^J_{pi}\tau^p\tau^i = 0}$$

  (a curve whose velocity is parallelly transported)
- Curved space. If the tensor components after parallel transport are not conserved

Connexion compatible with the metric

- Main point. To conserve the scalar product $< \xi, \eta >$ w.r.t. parallel transport along the curve $x(t)$
- Result. If the connexion is COMPATIBLE with the metric: $\nabla_k g_{ij} = 0$
- Proof. $0 = \frac{d}{dt}\left(g_{ij}\xi^i\eta^j\right) = \dot{x}^k \left(\nabla_k g_{ij}\right)\left(\xi^i\eta^j\right) + \dot{x}^k g_{ij} \overbrace{\left(\nabla_k \xi^i \eta^j + \xi^i \nabla_k \eta^j\right)}^{=0}$
- Operations of lowering indexes and of covariant differentaition commute

Christoffel symbols of a compatible connexion

- $\exists!$ SYMMETRIC compatible connexion (Riemannian connexion):

$$\boxed{\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{lj}}{\partial \alpha^i} + \frac{\partial g_{il}}{\partial \alpha^j} - \frac{\partial g_{ij}}{\partial \alpha^l}\right)}$$

- Affine connexion ($\Gamma = 0$): $\partial_i \partial_j \alpha^K(x^q) = \Gamma^l_{ij}\partial_l \alpha^K + \Gamma^K_{mn}\partial_i\alpha^m\partial_j\alpha^n$
  (“affine connexion” Γ means more exactly that $\exists \Gamma' = 0$)

Curvature & Riemann tensor

- In a Euclidean space for a smooth enough function f: $(\partial_i\partial_j - \partial_j\partial_i)\, f = 0$
- For a symmetric connexion and any vector field ξ:

$$\boxed{(\nabla_k\nabla_l - \nabla_l\nabla_k)\,\xi^i = -R^i_{qkl}\xi^q (+\text{term if not sym. })}$$

$$\boxed{\text{Riemann tensor: } R^i_{qkl} = -\left(\frac{\partial\Gamma^i_{ql}}{\partial\alpha^k} - \frac{\partial\Gamma^i_{qk}}{\partial\alpha^l} + \Gamma^i_{pk}\Gamma^p_{ql} - \Gamma^i_{pl}\Gamma^p_{qk}\right)}$$

- "Order 1 property":
  $g_{ij}dx^i dx^j(P) = g_{ij}dx^i dx^j(O_{geod}) - \frac{1}{6}R_{ikjl}(P^k dx^i - P^i dx^k)(P^l dx^j - P^j dx^l)$
- If $R^i_{qkl} \neq 0$ then the connexion is not Euclidean (the space is said curved)
- Definition 1. The Ricci curvature is the the $(0,2)$-tensor $R_{ql} = R^i_{qil}$
- Definition 2. The scalar curvature is the the scalar $R = g^{ql}R^i_{qil}$
- Gauss' "Theorema Egregium". For a $2D$ surface in a $3D$ space with a Riemannian metric, the scalar curvature is twice the Gauss curvature, i.e. it is an intrisic invariant of the surface

Main properties of the Riemann tensor

- We always have: $R^i_{qkl} = R^i_{qlk}$. If the connexion is
- symmetric: $R^i_{qkl} + R^i_{klq} + R^i_{lqk} = 0$
- compatible with the metric we have: $g_{ip}R^p_{qkl} = R_{iqkl} = R_{qikl}$
- symmetric and compatible (i.e. Riemannian) we have: $R_{iqkl} = R_{kliq}$

All above propeties hold true for Riemannian metrics

- in $2D$ the Riemann tensor is given by the scalar curvature R: $R_{1212}(= \det[\partial_i\partial_j f]) = K = \frac{g}{2}R$
- in $3D$ the Riemann tensor is given by the Ricci curvature R_{ik}: $R_{ijkl} = R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il} + \frac{R}{2}\left(g_{il}g_{jk} - g_{ik}g_{jl}\right)$
- in space-time, the metric must solve Einstein's field equations: $G_{ij} := R_{ij} - \frac{1}{2}Rg_{ij} = \lambda T_{ij}$ ("energy-momentum tensor" on the RHS, cf. Atle lectures – Rem. $\nabla_k G_{ij} = 0$)

Non-Riemannian spaces

The torsion of a connexion: $T^k_{ij} := \Gamma^k_{ij} - \Gamma^k_{ji}$

- A connexion is said non-Riemannian if its torsion does not vanish (and non-Euclidean if its curvature does not vanish)
- We have $\partial^2_{[ij]}\alpha^K = T^l_{ij}\partial_l\alpha^K - T^K_{mn}\partial_i\alpha^m\partial_j\alpha^n$ (change of coord. is not $\mathcal{C}^2$)
- Can a connexion be metric-compatible in a non-Riemannian space?
- Let $\tilde{\Gamma}^k_{ij}$ be the symmetric Christoffel symbols defined by the metric
- Then the following connexion is compatible with the metric:

$$\Gamma^k_{ij} = \tilde{\Gamma}^k_{ij} + \Delta\Gamma^k_{ij} (+\text{non-metric terms})$$

"Contortion" of a metric connexion: $\Delta\Gamma^k_{ij} = -\frac{1}{2}\left(T^j_{ik} + T^i_{jk} - T^k_{ji}\right)$

- Christoffel symbols are not tensor-like, but curvature, torsion and contortion are tensor quantities (i.e. have physical meaning)

Line Defects in crystals

In the **perfect** **crystal** the atoms form, in a stress-free configuration, a regular pattern proper to the prescribed nature of the matter

The **defective** **crystal** is, by contrast, an aggregation of an immense number of small pieces of perfect crystals that cannot be connected with one another so as to form a finite lump of perfect crystals as an organic unity” (Kondo (1954))

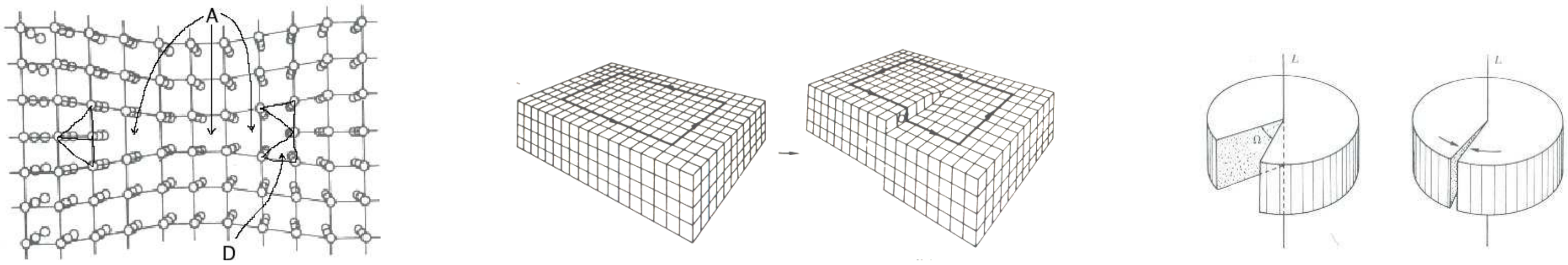

Figure 3: Dislocations and Disclinations

The internal observer. "In our universe we are internal observers who do not possess the ability to realize external actions on the universe, if there are such actions at all. Here we think of the possibility that the universe could be deformed from outside by higher beings. A crystal, on the other hand, is an object which certainly can deform from outside. We can also see the amount of deformation just by looking inside it, eg, by means of an electron microscope. Imagine some crystal being who has just the ability to recognize crystallographic directions and to count lattice steps along them. Such an *internal observer* will not realize deformations from outside, and therefore will be in a situation analogous to that of the physicist exploring the world. The physicist clearly has the status of an internal observer" (Kröner (1990)).

The Bravais metric (of an internal observer counting atomic steps): is for instance in *fcc* crystals give n by $[g^B_{ij}] = \frac{1}{4}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

The geometry of a defective crystal

- The elastic strain (ϵ) is incompatible $\Rightarrow$ the crystal is non-Euclidean (w.r.t. the metric of an internal observer)
- In the presence of pure disclinations, the crystal is Riemannian and the disclination density tensor (Θ) $\Leftrightarrow$ the curvature tensor
- In the presence of pure dislocations, the crystal is non-Riemannian and the dislocation density tensor $\Leftrightarrow$ the (connexion's torsion tensor $\Leftrightarrow$ the connexion's contortion tensor κ)
- In the presence of general line defects, the incompatibility $:= \nabla \times \epsilon \times \nabla = \Theta + \kappa \times \nabla$
- In the presence of point defects, the metric is not compatible, and $\nabla_k g_{ij}^B = \nabla_k (1 - N_V - N_I)^2 g_{ij}^B \Leftrightarrow$ point-defect scalar densities (intertitial N_I and vacancies N_V)
- BUT: If we have line and point defects, the crystal might be flat again $\cdots$

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