(Non-)Causality in General Relativity
The Gödel universe

Atle Hahn
GFM, Universidade de Lisboa

Lisbon, 12th March 2010

Contents:

References:

- Book by Wald: "General Relativity"
- Hawking/Ellis
- Wikipedia

1 Review

1.1 The general Einstein field equations

Fix 4-dimensional smooth manifold M and $\Lambda \in \mathbb{R}$ ("the cosmological constant").

Let Φ be matter/radiation field on M. We assume that for every Lorentzian metric g on M

- corresponding "stress energy tensor" $T_{ab} = T_{ab}(g, \Phi)$ is known explicitly
- Equations of motions $F(g, \Phi) = 0$ for Φ are known explicitly, i.e. function F given explicitly.

Main problem: For given M and Λ find simultaneous solutions (g, Φ) of

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi T_{ab}(g, \Phi) \quad \text{"Einstein field equations"} \tag{1a}$$

$$F(g, \Phi) = 0 \quad \text{"equations of motion"} \tag{1b}$$

1.2 The Einstein field equations for perfect fluids

Consider fluid in spacetime $M = (M, g)$. The state of fluid described by

- density function $\rho : M \to \mathbb{R}_+$
- 4-velocity field u^a on M
- temperature distribution $T : M \to \mathbb{R}_+$

We assume that equation of state $p = f(\rho, T)$ is known explicitly, e.g.,

$$f(\rho, t) = \begin{cases} C \cdot \rho T & \text{for an ideal gas } (C > 0 \text{ fixed}) \\ 0 & \text{for a pressure-less fluid (=“dust”)} \end{cases}$$

In special case where fluid is **perfect** (i.e. no viscosity and in “thermal equilibrium”, i.e. $\forall x \in M : T(x) = T_0$ for some T_0) the stress energy tensor $T_{ab} = T_{ab}(\rho, u^a)$ is given explicitly by

$$T_{ab} = (\rho + p)u_a u_b + p\, g_{ab} \quad \text{with} \quad p(x,t) = f(\rho(x,t), T_0)$$

and equations of motions are just $\nabla^a T_{ab} = 0$.

Here: ∇^a Levi-Civita connection associated to (M, g).

Observation: Einstein field equations imply $\nabla^a T_{ab} = 0$
$\rightarrow$ in dust situation the system of equations above reduces to

$$R_{ab} - \tfrac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi \rho u_a u_b \tag{2}$$

Let M be a 4-dimensional smooth manifold and $\Lambda \in \mathbb{R}$.

Definition 1 A **dust solution** of the Einstein field equations for M and Λ is a triple (g_{ab}, ρ, u^a) where

- g_{ab} is Lorentzian metric on M
- ρ is smooth positive function on M
- u^a is smooth vector field on M with $g_{ab} u^a u^b = -1$

such that Eq. (2) is fulfilled.

Remark 1

i) If $\Lambda \neq 0$ one often calls such a dust solution a **lambda dust solution**

ii) If (g_{ab}, ρ, u^a) is a dust solution for M and Λ then ρ and u^a are uniquely determined by g_{ab}.

Digression 1 Compare Wikipedia entry for "dust solutions":

- Friedmann(-Robertson-Walker) dust
- Kasner dusts
- Bianchi dust models (homogeneous, generalize first two examples)
- LTB dusts (some of the simplest inhomogeneous cosmological models)
- van Stockum dust (a cylindrically symmetric rotating dust)
- Kantowski-Sachs dusts
- the Neugebauer-Meinel dust

1.3 The Friedmann(-Robertson-Walker) solutions

The Friedmann solutions are special dust solutions. They can be characterized by the following conditions on M, Λ and $g = g_{ab}$:

Condition 1 $\Lambda = 0$

Condition 2

i) $M \cong \mathbb{R} \times \Sigma$

ii) $\Sigma_t \cong \{t\} \times \Sigma$ is orthogonal to $\mathbb{R} \times \{\sigma\}$, $\sigma \in \Sigma$.

iii) $\Sigma_t \cong \{t\} \times \Sigma$, $t \in \mathbb{R}$, is "space-like"
(i.e. restriction g_t of g to Σ_t is a Riemannian metric)

Condition 3 *Each* (Σ_t, g_t) *is homogenous, isotropic, and* $\Sigma_t \cong \Sigma$ *is simply-connected*

Remark 2 Condition 2 above is in fact a "causality condition", the strongest of a "hierarchy" of causality conditions (see below).

The most famous solutions of the Einstein field equations which violate even the weakest standard causality condition are the Gödel solution.

2 The Gödel solutions, part I

2.1 Definition

Let us temporarily use the convention of the previous lectures and consider a pseudo-Riemannian metric g on $\mathbb{R}^4$ as a matrix of funcions $(g_{ab}(x))_{ab}$.

Definition 2 The **Gödel solution** with parameter $\omega > 0$ is the following dust solution (g_{ab}, ρ, u^a) for $M = \mathbb{R}^4$ and $\Lambda = -\omega^2 < 0$:

- $g = g_{ab} = (g_{ab}(x))_{ab}$ is given by

$$(g_{ab}(x))_{ab} = \frac{1}{2\omega^2} \begin{pmatrix} -1 & -\exp(x_2) & 0 & 0 \\ -\exp(x_2) & -\frac{1}{2}\exp(2x_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- $\rho = \omega^2/4\pi$
- $u^a = \sqrt{2}\omega(-1, 0, 0, 0)$

Remark 3 u^a looks trivial but $u_a = g_{ab}u^b = \frac{1}{\sqrt{2}\omega}(1, \exp(x_2), 0, 0)$ does not!

Gödel solutions arise naturally

1) Gödel solutions arise natural from the following simple ansatz for finding a (lambda) dust solution:

   - Take the "nicest" of all smooth 4-dimensional manifold, namely $M = \mathbb{R}^4$.
   - Take non-diagonal Lorentz metric as close to trivial case as possible, e.g.,

   $$(g_{ab}(x))_{ab} = \begin{pmatrix} f(x) & h(x) & 0 & 0 \\ h(x) & k(x) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   where f, h, k are unknown functions on M.
   - Assume the simplest situation where f, h, and k (and also ρ) only depends on one of the four variable x_0, x_1, x_2, x_3.
   - For each of these 24 situations write down the Einstein Field equations to obtain a system of differential equations for the unknown functions f, h, k and ρ and u^a.

     Einstein field equations contain 10 sub equations $\longrightarrow$ enough restrictions for determining 8 (or rather 7) unknown functions f, h, k and ρ and u^a and constant Λ.

2) Gödel solutions arise "automatically" within the Bianchi classification of 3-dimensional homogeneous (pseudo-)Riemannian manifolds.

2.2 Important Properties/Features of the Gödel solutions

- $\mathbb{R}^4 \cong \mathbb{R} \times \mathbb{R}^3$ so Condition 2 i) fulfilled.

  However, parts ii) and iii) can not be fulfilled.

- The Gödel solutions have no singularities (as opposed to Friedmann or Schwarzschild solutions)

- Cosmological constant $\Lambda = -\omega^2$ finely balanced to match mass density $\rho = 2\omega^2$ ($\rightarrow$ somewhat "artificial")

- Hubble law not satisfied

- Causality violated in strongest possible way

Last 3 observations $\quad \longrightarrow \quad$ Gödel solutions are highly unphysical.

However: high pedagogical value.

2.3 The Gödel solutions really are solutions

Recall: $\rho = \omega^2/4\pi \quad u^a = \sqrt{2}\omega(-1,0,0,0)$

$$(g_{ab}(x))_{ab} = \frac{1}{2\omega^2}\begin{pmatrix} -1 & -\exp(x_2) & 0 & 0 \\ -\exp(x_2) & -\frac{1}{2}\exp(2x_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus

$$u_a = g_{ab}u^b = \tfrac{1}{\sqrt{2}\omega}(1, \exp(x_2), 0, 0)$$

and therefore

$$(T_{ab})_{ab} = \rho(u_a u_b)_{ab} = \frac{1}{8\pi}\begin{pmatrix} 1 & \exp(x_2) & 0 & 0 \\ \exp(x_2) & \exp(2x_2) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Inverse g^{ab} of g_{ab} given by

$$(g^{ab})_{ab} = 2\omega^2\begin{pmatrix} 1 & -2\exp(-x_2) & 0 & 0 \\ -2\exp(-x_2) & 2\exp(-2x_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Recall:

$$R_{ac} = \partial_b \Gamma^b_{ac} - \partial_a \Gamma^b_{bc} + \Gamma^i_{ac}\Gamma^b_{ib} - \Gamma^i_{bc}\Gamma^b_{ia} \tag{3}$$

with

$$\Gamma^d_{ab} := \tfrac{1}{2} g^{dc}\big(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}\big) \tag{4}$$

For the Gödel metric g the non-vanishing Γ^c_{ab} are

$$\begin{aligned}
\Gamma^0_{12}(x) = \Gamma^0_{21}(x) = \Gamma^2_{01}(x) = \Gamma^2_{01}(x) &= \tfrac{1}{2}\exp(x_2) \\
\Gamma^1_{02}(x) = \Gamma^1_{20}(x) &= -\exp(-x_2) \\
\Gamma^2_{11}(x) &= \tfrac{1}{2}\exp(2x_2) \\
\Gamma^0_{02}(x) = \Gamma^0_{20}(x) &= 1
\end{aligned}$$

Thus we obtain:

$$(R_{ab}(x))_{ab} = \begin{pmatrix} 1 & \exp(x_2) & 0 & 0 \\ \exp(x_2) & \exp(2x_2) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which implies

$$R_{ab} = 8\pi T_{ab}$$

On the other hand

$$R = R_{ab} g^{ab} = -2\omega^2$$

Since $\Lambda = -\omega^2$ we have

$$R_{ab} - \tfrac{R}{2} g_{ab} + \Lambda g_{ab} = R_{ab}$$

Thus the assertion follows.

3 Mathematical Intermezzo

3.1 Manifolds and tensors: the formal definitions

Recall:

- A topological manifold M is a topological space which "looks locally like $\mathbb{R}^n$". Examples are

  i) Every open subset of $\mathbb{R}^n$

  ii) "Curved surfaces" in $\mathbb{R}^3$

  iii) S^n for arbitrary n

- A smooth manifold is a topological manifold M equipped with certain extra-structure, called "differentiable structure". The differentiable structure allows definition of

  i) the notion of "smoothness" for maps (cf. the definition below)

  ii) a canonical finite-dimensional real vector space T_xM for each $x \in M$ (cf. the definition below)

  iii) The structure of a smooth manifold on $TM := \bigcup_{x \in M} T_xM$

Here are (most of) the formal definitions:

Definition 3 A **topological manifold** is a (Hausdorff) topological space M with the property that every point has a neighborhood U which is is homeomorphic to $\mathbb{R}^n$ for some n.

Definition 4 Let M be a topological space. A **chart** of M is a pair (U, ψ) where U is an open subset of M and $\psi : U \to V$ a homeomorphism onto an open subset V of $\mathbb{R}^n$.

Definition 5 Let M be a topological manifold.

i) An **atlas** of M is a family $\{(U_i, \psi_i) \mid i \in I\}$ of charts of M such that $M = \bigcup_i U_i$

ii) An atlas of M is **smooth** iff for all (U_i, ψ_i) and (U_j, ψ_j) such that $U := U_i \cap U_j$ is non-empty the map $\psi_i \circ \psi_j^{-1} : \psi_j(U) \to \psi_i(U)$ is smooth

iii) A **smooth structure** on M is a smooth atlas on M which is a maximal

iv) A **differentiable manifold** is a topological manifold equipped with a smooth structure.

Example 1 Let $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$.

Smooth atlas $\mathcal{A} = \{(U_1, \psi_1), (U_2, \psi_2)\}$ where

$$U_1 := S^2 \backslash \{(1,0,0)\} \text{ and } U_2 := S^2 \backslash \{(-1,0,0)\}$$

and where
$\psi_i : U_i \to \mathbb{R}^2$, $i = 1, 2$ is corresponding "stereographical projection"

Digression 2 Very deep mathematics involved!

- Not every topological manifold has a smooth structure
- Many topological manifolds have several different structures, for example S^7 has 15 different smooth structures and $\mathbb{R}^4$ infinitely many
- The question if S^4 has more than one smooth structure is a major open problem (the "smooth Poincare conjecture" in 4 dimensions)

Definition 6 Let M_1, M_2 be two smooth manifolds. A map $f : M_1 \to M_2$ is **smooth** iff for all charts (U_1, ψ_1) resp. (U_2, ψ_2) of M_1 resp. M_2 the map $\psi_2 \circ f \circ \psi_1^{-1} : \psi_1(U_1) \to \psi_2(U_2)$ is smooth.

Fix n-dim. smooth manifold M, $x \in M$ and chart (U, ψ) with $x \in U$.

- Let $\Gamma_x(M)$ be set of smooth curves $\gamma : \mathbb{R} \to M$ with $\gamma(0) = x$.
- Let $\sim$ be equivalence relation $\sim$ on $\Gamma_x(M)$ by

$$\gamma_1 \sim \gamma_2 \quad \Leftrightarrow \quad (\psi \circ \gamma_1)'(0) = (\psi \circ \gamma_2)'(0) \quad \forall \gamma_1, \gamma_2 \in \Gamma_x(M)$$

Definition 7 Set

$$T_xM := \{[\gamma] \mid \gamma \in \Gamma_x(M)\}$$

$\mathbb{R}$-vector space structure on T_xM obtained from the one on $\mathbb{R}^n$ by transport of structure using bijection

$$\theta : T_xM \to \mathbb{R}^n \quad \text{given by} \quad \theta([\gamma]) = (\psi \circ \gamma)'(0)$$

Observation 1 Relation $\sim$ and space T_xM do not (!) depend on (U, ψ).

Convention 1 Let $\gamma \in \Gamma_x(M)$ and $s \in \mathbb{R}$. We write

$$\gamma'(0) \text{ instead of } [\gamma], \quad \text{and}$$

$$\gamma'(s) \text{ or } \tfrac{d}{ds}\gamma(s) \text{ instead of } [\gamma(\cdot + s)]$$

Recall:

- A **vector field** on M is a "smooth" family $(X_x)_{x \in M}$ where $X_x \in T_xM$ for each $x \in M$.
- A **co-vector field (or 1-form)** on M is a "smooth" family $(\alpha_x)_{x \in M}$ where $\alpha_x : T_xM \to \mathbb{R}$ is linear.
- A **pseudo-Riemannian metric** on M is a "smooth" family $(g_x)_x$ where $g_x : T_xM \times T_xM \to \mathbb{R}$ is bilinear and non-degenerate.
- A pseudo Riemannian metric on M with signature (n,0) (resp. (n-1,1)) is called a **Riemannian metric** resp. **Lorentzian metric**
- A **tensor field** on M of type (p, q) is a "smooth" family $(A_x)_x$ where $A_x : T_xM^* \times \ldots \times T_xM^* \times T_xM \times \ldots \times T_xM \to \mathbb{R}$ is multilinear

Remark 4 Observe that a tensor field of type $(1, 0)$ can be considered as a vector field in the obvious way.

Definition 8 A **spacetime** is a 4-dimensional smooth manifold equipped with a Lorentzian metric.

3.2 The (abstract) local coordinate formalism

Fix n-dimensional smooth manifold M.

Definition 9 A **system of local coordinates** on M is an n-tuple of smooth functions $f_1, f_2, ..., f_n$ of the form $f_i : V_i \to \mathbb{R}$ where $V_i \subset M$ are open, such that there is a chart (U, ψ) of M with $U = \bigcap_i V_i$ and $f_i = \psi_i$ on U.

Example 2 For every chart (U, ψ) of M the corresponding components $(\psi_1, \psi_2, \ldots, \psi_n)$ form a system of local coordinates.

Example 3 The **polar coordinates** are/is the system (r, ϕ) of local coordinates on $\mathbb{R}^2$ where

$$r : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}, \qquad \phi : \{x \in \mathbb{R}^2 \mid x_1 \neq 0\} \to \mathbb{R}$$

and

$$r(x) = \|x\|$$

$$\phi(x) = \begin{cases} \arctan(\frac{x_2}{x_1}) & \text{if } x_1 > 0 \\ \arctan(\frac{x_2}{x_1}) + \pi/2 & \text{if } x_1 < 0 \text{ and } x_2 \geq 0 \\ \arctan(\frac{x_2}{x_1}) - \pi/2 & \text{if } x_1 < 0 \text{ and } x_2 < 0 \end{cases}$$

Fix a system $(x_1, x_2, \ldots, x_n)$ of local coordinates on M with (joint) domain U.

Definition 10 i) Define dx_i, for $i \leq n$, as the unique smooth co-vector field on U given by

$$dx_i(\gamma'(0)) = (x_i \circ \gamma)'(0) \quad \text{for every smooth curve } \gamma \text{ in } U \tag{5}$$

ii) Define $\frac{\partial}{\partial x_i}$, for $i \leq n$, as the unique smooth vector field on U given by

$$dx_j(\tfrac{\partial}{\partial x_i}) = \delta_{ij} \text{ for all } j \leq n. \tag{6}$$

Observation 2 From the definitions it easily follows that for every smooth curve γ in U we have

$$\gamma'(s) = \sum_i x_i'(s) \tfrac{\partial}{\partial x_i}(\gamma(s)) \tag{7}$$

where $x_i'(s)$ is a short notation for $(x_i \circ \gamma)'(s)$.

Observation 3 Let g be pseudo-Riemannian metric on M. The restriction $g_{|U}$ of g onto U can be uniquely written as

$$g_{|U} = \sum_{i,j} g_{ij} dx_i dx_j \tag{8}$$

where $(g_{ij})_{ij}$ is a symmetric matrix of smooth functions $g_{ij} : U \to \mathbb{R}$ and where $dx_i dx_j$ is the type $(0,2)$-tensor field on U given by

$$(dx_i dx_j)(X,Y) = dx_i(X) \cdot dx_j(Y) \tag{9}$$

for all vector fields X and Y on U.

We can replace the last equation by $g_{|U} = \sum_{i \leq j} \tilde{g}_{ij} dx_i dx_j$ if we redefine $dx_i dx_j$ by

$$(dx_i dx_j)(X,Y) = \tfrac{1}{2}[dx_i(X) \cdot dx_j(Y) + dx_i(Y) \cdot dx_j(X)] \tag{10}$$

Example 4 The standard Euclidean metric on $\mathbb{R}^n$ is given by

$$g = dx_1^2 + dx_2^2 + \ldots + dx_n^2$$

if $(x_1, x_2, \ldots, x_n)$ are the standard coordinates.

It is convenient to generalize the definition of dx_i above:

Definition 11 For every function $f \in C^\infty(U, \mathbb{R})$, where $U \subset M$ is open, we introduce the covector field df on U by the condition

$$df(\gamma'(0)) = (f \circ \gamma)'(0) \quad \text{for every smooth curve } \gamma \text{ in } U$$

df is called the total derivative (or exterior derivative) of f.

Calculation rules: For $f, g \in C^\infty(M, \mathbb{R})$ and $\Phi \in C^\infty(\mathbb{R}, \mathbb{R})$ we have

$$d(f+g) = df + dg$$

$$d(fg) = (df)g + f dg$$

$$d(\Phi(f)) = \Phi'(f) df$$

Example 5 Euclidean metric $g = dx_1^2 + dx_2^2$ on $\mathbb{R}^2$ in polar coordinates (r, ϕ):

$$x_1 = r\cos(\phi) \text{ and } x_2 = r\sin(\phi) \text{ on } U$$

Calculation rules above $\quad \Rightarrow$

$$\begin{aligned} dx_1 &= (dr)\cos(\phi) + r d(\cos(\phi)) = \cos(\phi) dr - r\sin(\phi) d\phi \\ dx_2 &= (dr)\sin(\phi) + r d(\sin(\phi)) = \sin(\phi) dr + r\cos(\phi) d\phi \end{aligned}$$

Thus on U we have

$$dx_1^2 + dx_2^2 = \big(\cos(\phi)dr - r\sin(\phi)d\phi\big)^2 + \big(\sin(\phi)dr + r\cos(\phi)d\phi\big)^2 = dr^2 + r^2 d\phi^2$$

3.3 Spacetime curves

Let M be a space-time and let $\gamma : \mathbb{R} \longrightarrow M$ be a smooth curve in M.

Definition 12 i) γ is **time-like** iff $g(\gamma'(s), \gamma'(s)) < 0$ for all $s \in \mathbb{R}$.

ii) γ is **null** (or **light-like**) iff $g(\gamma'(s), \gamma'(s)) = 0$ for all $s \in \mathbb{R}$.

iii) γ is **space-like** iff $g(\gamma'(s), \gamma'(s)) > 0$ for all $s \in \mathbb{R}$.

iv) γ is **causal** iff $g(\gamma'(s), \gamma'(s)) \leq 0$ for all $s \in \mathbb{R}$.

Definition 13 γ is a **geodesic** in M if for every $t \in \mathbb{R}$ there is a chart (U, ψ) around $\gamma(t)$ such that

$$\tfrac{d^2}{ds^2}x^i(s) + \sum_{j,k} \Gamma^i_{jk}(\gamma(s))\tfrac{d}{ds}x^j(s)\tfrac{d}{ds}x^k(s) = 0$$

for all s sufficiently close to t.

Here: $x^i(s) := x_i(\gamma(s)) = \psi_i(\gamma(s)))$ and $\Gamma^i_{jk} : U \to \mathbb{R}$ are as in Sec. 2 above with $(g_{ij})_{ij}$ given by Observation 3 in Sec. 3.2.

Observation 4 If γ is a geodesic than either γ is time-like or null or space-like.

Physical relevance:

- Light rays "travel" on null geodesics.
- Massive point particles travel on time-like curves (not necessarily geodesics)

4 The Gödel solutions, part II

4.1 The Gödel metric in the abstract local coordinates

We can rewrite Gödel metric g on $M = \mathbb{R}^4$ with parameter $\omega > 0$ as

$$g = \frac{1}{2\omega^2}\left(-dx_0^2 - 2\exp(x_2)dx_0dx_1 - \tfrac{1}{2}\exp(2x_2)dx_1^2 + dx_2^2 + dx_3^2\right) \tag{11}$$

Set $\mathcal{M} := (\mathbb{R}^4, g)$

4.2 The (reduced) Gödel metric

The coordinate x_3 above is inessential can therefore often be ignored. More precisely:

$$\mathcal{M} = \mathcal{M}' \times \mathcal{M}''$$

with $\mathcal{M}' := (\mathbb{R}^3, g')$ and $\mathcal{M}'' := (\mathbb{R}, g'')$ where

$$g' = \frac{1}{2\omega^2}\left(-dx_0^2 - 2\exp(x_2)dx_0dx_1 - \tfrac{1}{2}\exp(2x_2)dx_1^2 + dx_2^2\right)$$

$$g'' = \tfrac{1}{2\omega^2}dx_3^2$$

(x_0, x_1, x_2 are the standard coordinates of $\mathbb{R}^3$ and x_3 standard coordinate of $\mathbb{R}$)

4.3 The (reduced) Gödel metric in "cylindrical-type" coordinates

Let (t, r, ϕ) be the system of coordinates $t : U \to \mathbb{R}$, $r : U \to (0, \infty)$, $\phi : U \to (-\pi, \pi)\backslash\{0\}$ with $U = \mathbb{R} \times (\mathbb{R}\backslash\{0\}) \times \mathbb{R} \subset \mathbb{R}^3$ which is given uniquely by

$$x_0 = \sqrt{2}\Big(\sqrt{2}t - \phi + 2\arctan\big(e^{-2r}\tan(\phi/2)\big)\Big)$$
$$x_1 = \sqrt{2}\frac{\sin(\phi)\sinh(2r)}{\cosh(2r) + \cos(\phi)\sinh(2r)}$$
$$x_2 = \ln\big(\cosh(2r) + \cos(\phi)\sinh(2r)\big)$$

Using the computation rules above we find that

$$dx_0 = \sqrt{2}\Bigg(\sqrt{2}dt - d\phi + \tfrac{2}{(e^{-2r}\tan(\phi/2))^2+1}\Big[\big(d(e^{-2r})\big)\tan(\phi/2) + e^{-2r}\big(d(\tan(\phi/2))\big)\Big]\Bigg)$$
$$= 2dt + \sqrt{2}\big[-1 + \tfrac{e^{-2r}(1+\tan(\phi/2)^2)}{(e^{-2r}\tan(\phi/2))^2+1}\big]d\phi + \sqrt{2}\big[\tfrac{-4e^{-2r}\tan(\phi/2)}{(e^{-2r}\tan(\phi/2))^2+1}\big]dr$$

Making similar computations for

$$dx_1 = ...dr + ...d\phi$$

$$dx_2 = ...dr + ...d\phi$$

we obtain

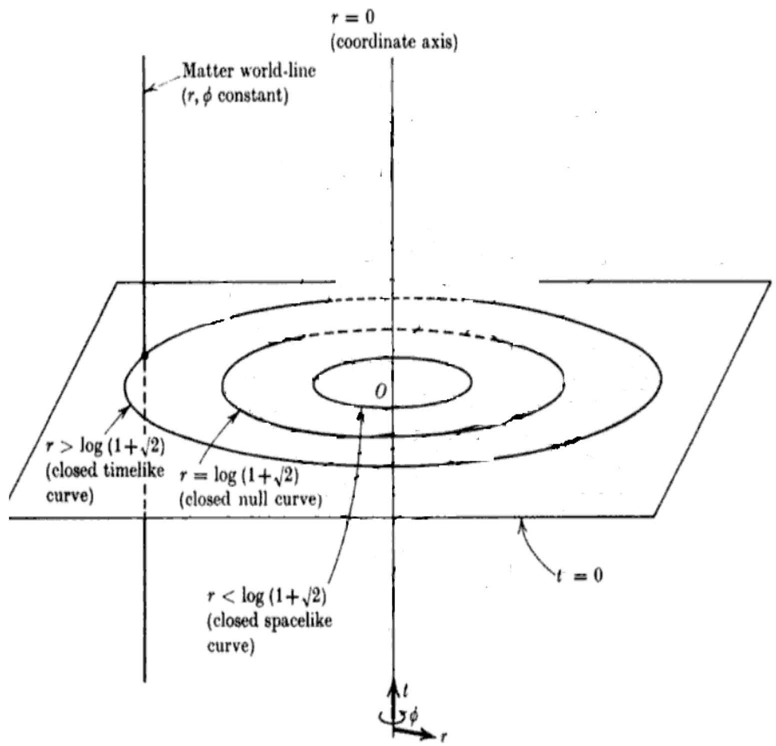
$$g' = \tfrac{2}{\omega^2}\big(-dt^2 + dr^2 - (\sinh^4(r) - \sinh^2(r))d\phi^2 + 2\sqrt{2}\sinh^2(r)dtd\phi\big)$$

4.4 Existence of closed time-like curves

For fixed $r_0 \in \mathbb{R}_+$ consider the 2π-periodic (and therefore closed) smooth curve in $\mathcal{M}' = (\mathbb{R}^3, g')$ which is given by

$$t(s) = 0, \quad r(s) = r_0, \quad \phi(s) = s, \qquad s \in (-\pi, \pi)\backslash\{0\}$$

where $t(s)$, $r(s)$, $\phi(s)$ is a short notation for $t(\gamma(s))$, $r(\gamma(s))$, $\phi(\gamma(s))$.



Observation 2 $\Rightarrow$ $\gamma'(s) = \frac{\partial}{\partial\phi}(\gamma(s))$

Setting $\frac{\partial}{\partial\phi} := \frac{\partial}{\partial\phi}(\gamma(s))$ we have

$$\begin{aligned} d\phi^2(\tfrac{\partial}{\partial\phi}, \tfrac{\partial}{\partial\phi}) &= d\phi(\tfrac{\partial}{\partial\phi}) \cdot d\phi(\tfrac{\partial}{\partial\phi}) = 1 \cdot 1 = 1 \\ dtd\phi(\tfrac{\partial}{\partial\phi}, \tfrac{\partial}{\partial\phi}) &= dt(\tfrac{\partial}{\partial\phi}) \cdot d\phi(\tfrac{\partial}{\partial\phi}) = 0 \cdot 1 = 0 \\ dt^2(\tfrac{\partial}{\partial\phi}, \tfrac{\partial}{\partial\phi}) &= ... = 0 \\ dr^2(\tfrac{\partial}{\partial\phi}, \tfrac{\partial}{\partial\phi}) &= ... = 0 \end{aligned}$$

Recalling that

$$g' = \tfrac{2}{\omega^2}\big(-dt^2 + dr^2 - (\sinh^4(r) - \sinh^2(r))d\phi^2 + 2\sqrt{2}\sinh^2(r)d\phi dt\big)$$

we see that

$$g'(\gamma'(s), \gamma'(s)) = g'(\tfrac{\partial}{\partial\phi}, \tfrac{\partial}{\partial\phi}) = -\tfrac{2}{\omega^2}\big(\sinh^4(r_0) - \sinh^2(r_0)\big)$$

and therefore

$$\begin{aligned} g'(\gamma'(s), \gamma'(s)) < 0 \text{ for all } s \quad &\Leftrightarrow \quad \sinh^4(r_0) - \sinh^2(r_0) > 0 \\ &\Leftrightarrow \quad \sinh^2(r_0) > 1 \\ &\Leftrightarrow \quad (e^{r_0} - e^{-r_0})/2 > 1 \\ &\Leftrightarrow \quad r_0 > \log(1 + \sqrt{2}) \end{aligned}$$

Clearly, if we consider the curve γ in $\mathcal{M}'$ as a curve in $\mathcal{M}$ in the obvious way, we have

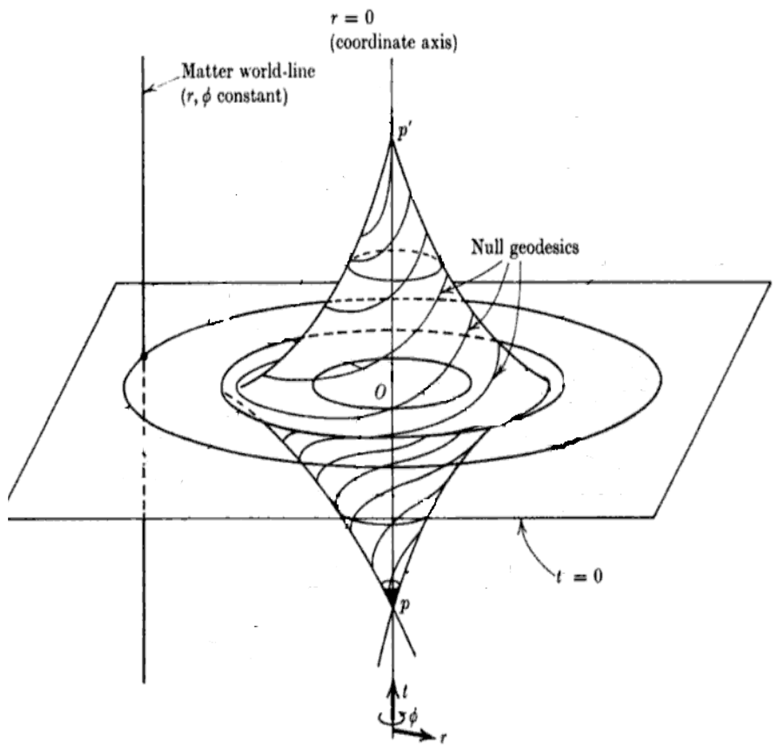
$$g(\gamma'(s), \gamma'(s)) = g'(\gamma'(s), \gamma'(s)) < 0 \quad \text{for all } s$$

so γ is time-like (and closed).

4.5 Rejoining of light-rays

The cylindrical-type coordinates (t, r, ϕ) introduced above are also very useful for studying another rather counterintuitive property of the Gödel solutions.

Fix point P in $\mathcal{M}' \subset \mathcal{M}$ on axis $r = 0$. It turns out that all the light rays through P which stay in $\mathcal{M}' \subset \mathcal{M}' \times \mathcal{M}'' = \mathcal{M}$ refocus in one point P' in $\mathcal{M}'$.



The light-rays "travel" on null geodesics so in order to prove this one will have to write down and solve the equations for the geodesics in the coordinates (t, r, ϕ).

4.6 The isometry group

Recall:

- $\mathrm{Isom}(M,g) := \{\psi \in \mathrm{Diff}(M) \mid \psi \text{ leaves } g \text{ fixed }\}$ where

$$\mathrm{Diff}(M) := \{\psi : M \to M \mid \ \psi \text{ is bijective and } \psi \text{ and } \psi^{-1} \text{ are smooth }\}$$

- (M,g) is "homogeneous" iff $\mathrm{Isom}(M,g)$ operates transitively on M
- $\mathrm{Isom}(M,g)$ has a natural Liegroup structure

Convention: If M is fixed then we can write $\mathrm{Isom}(g)$ instead of $\mathrm{Isom}(M,g)$ and use the notion "homogenous" for the metric g.

Observation 5 Both the original Gödel metric and the reduced Gödel metric are homogeneous.

Observation 6 i) The isometry group of the original Gödel metric (resp. the reduced Gödel metric) is 5 dimensional (resp. 4-dimensional).

ii) The Gödel solutions are the only dust solutions with a simply-connected spacetime and a 5-dimensional isometry group.

Conclusion 1 Recall that above we showed that there are closed time-like curves in the Gödel universe and that for certain points P the light rays through P rejoin (in the 3-dimensional reduced setting in $\mathcal{M}'$.)

From Observation 5 it follows that there are closed time-like curves through every point in the Gödel universe and that the other result mentioned above is true for every point P.

Conclusion 2 Observation 6 can be used to give an abstract definition of the Gödel solutions.

Conclusion 3 The homogeneous 3-dimensional Riemannian manifolds have been completely classified by Bianchi. A similar classification is possible for arbitrary 3-dimensional pseudo-Riemannian manifolds This puts the Gödel solutions in a systematic framework.

5 Causality notions: overview

There is the following "hierarchy" of notions of causality for a given space time M:

non-totally vicious	(= not through every point there is a closed time-like curve)
chronological	(= there are no closed time-like curve)
causal	(= there are no non-trivial closed causal curves)
strongly causal	(see below)
stably causal	(see below)
globally hyperbolic	(see below)
Condition 2	(see above)

Remark 5 We emphasize that the notion "globally hyperbolic" has little or nothing to do with the notion of a "hyperbolic manifold" of Lecture 2.

Let (M, g) be a fixed space-time.

Definition 14 A **time orientation** on M is a smooth vector field X on M which is time-like

Assume that there is a time-orientation X on M.

Definition 15 For $x, y \in M$ we set

i) $x \ll y$ iff there is a "future-directed" time-like curve from x to y

ii) $x < y$ iff there is a "future-directed" causal curve from x to y

Definition 16 For each $x \in M$ set

$$
\begin{aligned}
I^+(x) &:= \{y \in M \mid x \ll y\} && (\text{"chronological future"}) \\
I^-(x) &:= \quad .. \qquad \gg .. && (\text{"chronological past"}) \\
J^+(x) &:= \quad .. \qquad < .. && (\text{"causal future"}) \\
J^-(x) &:= \quad .. \qquad > .. && (\text{"causal past"})
\end{aligned}
$$

Observe that $x \ll y$ implies $x < y$ and therefore $I^\pm(x) \subset J^\pm(x)$

Reformulation/Formal definitions:

Definition 17

i) M is non-totally vicious iff $x \not\ll x$ for some x

ii) M is chronological iff $x \not\ll x$ for all x

iii) M is causal iff $x < y$ and $y < x$ imply $x = y$

iv) M is strongly causal iff for every $x \in M$ and every neighborhood U of x there is a neighborhood $V \subset U$ which is hit by all time-like curves at most once

v) M is stably causal: see Wikipedia or Wald

vi) M is globally hyperbolic iff M is strongly causal and for all x the set $J^+(x) \cap J^-(x)$ is compact.

Digression 3 It can be shown that the following statements are equivalent

- M is globally hyperbolic
- M has a "Cauchy-surface" Σ (i.e. Σ is an "achronal" 3-dimensional submanifold of M and every "inextendible" time-like curve in M hits Σ exactly once)
- $M \cong \mathbb{R} \times \Sigma$ and each $\Sigma_t \cong \{t\} \times \Sigma$ is a Cauchy surface.

In view of the last characterization Condition 2 above implies global hyperbolicity.