Introduction to Feichtinger's Theory of Modulation Spaces

Abstract. I will present in these two lectures the basics of the theory of Feichtinger's modulation spaces, which play a central role in time frequency analysis and signal theory, and whose importance in quantum mechanics is only beginning to be realized. The simplest example of a modulation space is the so-called Feichtinger algebra $S_0(\mathbb{R}^n)$ whose elements are characterized by the property that their Wigner transform is in $L^1(\mathbb{R}^n \oplus \mathbb{R}^n)$. It is of course not obvious at all that with *this* definition $S_0(\mathbb{R}^n)$ is a vector space, even less an algebra! We will show that it is, and that $S_0(\mathbb{R}^n)$ is in fact a Banach algebra (both for pointwise product and convolution), characterized by the following interesting property: it is the smallest Banach space containing the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and being invariant under the action of the inhomogeneous metaplectic group. This makes of course $S_0(\mathbb{R}^n)$ an interesting substitute for $\mathcal{S}(\mathbb{R}^n)$ as space of wavepackets, especially since $(S_0(\mathbb{R}^n), L^2(\mathbb{R}^n), S_0'(\mathbb{R}^n))$ is a Gelfand triple. We will give a straightforward application to the theory of Schrödinger's equation with quadratic (inhomogeneous) Hamiltonian function. We thereafter discuss the more general modulation spaces $M_s^q(\mathbb{R}^n)$ which are defined in a way similar to the Feichtinger algebra but in terms of weighted L_s^q spaces with polynomially bounded weight $v_s(z) = (1 + |z|^2)^{s/2}$. We finally consider the modulation spaces $M_s^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$ (in the case $s = 0$ we obtain the "Sjöstrand class" $M_0^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$), and show that these spaces are particularly well-adapted to the study of deformation quantization, because of the following two properties. (i) Let $\widehat{A}$ be the Weyl operator with symbol a, etc. If we have $\widehat{C} = \widehat{A}\widehat{B}$ with $a, b \in M^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$ then $c \in M^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$; (ii) $\widehat{A}$ with $a \in M^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$ is invertible with inverse $\widehat{B} \overset{\text{Weyl}}{\longleftrightarrow} b$ then $b \in M^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$ (the "Wiener property").