STOCHASTIC LAGRANGIAN FLOWS AND THE NAVIER-STOKES EQUATION

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Abstract. We describe how some diffusion processes are obtained as critical points of variational principles, focusing in the case where the stochastic Lagrangian flows are associated to the (classical) Navier-Stokes equation. Stability of the flows is studied. Existence of the flows in various senses is discussed.

Contents

1. Introduction

In fluid dynamics it is common to distinguish between the Eulerian and the Lagrangian representation of a motion. The first one, who has been favoured in the analytic/pde tradition in mathematics, refers to the study of the velocity of the fluid. The Lagrangian approach considers the position of the individual particles and describes their evolution in time (the Lagrangian flows).

In the non-viscous case the velocity of an incompressible fluid is described by the Euler equations,

$$\frac{\partial}{\partial t}u = -(u.\nabla)u - \nabla p, \ \operatorname{div} u = 0 \tag{1.1}$$

where p is the pressure. An initial condition u_0 as well as adequate boundary conditions for the underlying space domain are data of the problem, but the pressure is not: it is part of the solution.

The Lagrangian description consists in looking at the corresponding integral flows $g(t)(x)$, satisfying

$$\frac{\partial}{\partial t}g(t)(x) = u(t, g(t)(x)), \ g(0) = x \tag{1.2}$$

The two approaches look at a first glance fairly equivalent but actually they are not. It is not always possible to derive one from the other since regularity of the velocity field is often very low. Also the behaviour of the velocity is in many respects totally different from the behaviour of the position: just as an example, the flow $u(t,\cdot)$ is believed to be ergodic whereas $g(t)(.)$ is not (c.f. [17]). The stability properties of the velocity fields are also quite independend of those of the Lagrangian flows (c.f. some examples in [32]).

Assuming everything is smooth enough and taking second time derivatives, we have,

$$\frac{\partial^2}{\partial t^2} g = (\frac{\partial}{\partial t} u + (u.\nabla)u)(t,g)$$

and therefore, by the Euler equations satisfied by the velocity u, the acceleration of the motion is a gradient. As such, it is, at every instant, orthogonal in the L^2 sense to all vector fields with zero divergence. The space of such vector fields, if all objects are correctly defined, should be the tangent space of a "manifold" which consists of volume measure preserving maps, like the Lagrangian flows g. The statement that the acceleration is orthogonal to such manifold should mean that the motion is a geodesic.

To be more precise the Lagrangian flows $g(t)(\cdot)$ are geodesics with respect to the (right-invariant) induced metric on the group of volume preserving diffeomorphisms of the underlying configuration manifold. This view was suggested by V. Arnold ([6]) and gave rise to many interesting developments, in particular to the study of the stability of the motion (i.e. the evolution in time of the distance between particles) through the geometry of the group ([7]). The study of the geometry of the (infinite-dimensional) group of diffeomorphisms and the existence of the geodesic was carried out in [20].

Geodesics are minima of lenght and there is, indeed, a variational principle associated to Euler equations. Let us consider the configuration space to be flat and without boundary, for simplicity. The Lagrangian flows $g(t)$, with $t \in [0,T]$, can be characterized as critical paths for the action functional

$$S[g] = \frac{1}{2}\int_0^T \int |\dot{g}(t)(x)|^2 dx dt = \frac{1}{2}\int_0^T ||\dot{g}(t)||^2_{L^2} dt \tag{1.3}$$

where $\dot{g}$ denotes the derivative in time of g. Namely, they are critical points with respect to the L^2 metric on the underlying space.

Writing $L(g,\dot{g},t) = \frac{1}{2}||\dot{g}(t)||^2_{L^2(dx)}$ for the Lagrangian, it is indeed easy to check that the Euler-Lagrange equations

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{g}}) - \frac{\partial L}{\partial g} = 0$$

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$$\frac{d}{dt}[\dot{g}(t)] = 0$$

which is, formally, Euler equation (in the weak L^2 sense). In the incompressible case we have to add in the right of both equations $-\nabla p(t)$, where $p(t)$ is the pressure at time t, specified a posteriori.

Euler's fluid equations are a special case of Lagrangian systems treated in Geometric Mechanics via variational principles in general Lie groups ([31], [25]).

Replacing the deterministic paths $g(t)$ by suitable semimartingales and defining the Lagrangian on the drift of those semimartingales, this drift playing the rôle of the time derivative of this processes which is no longer well defined without a conditional expectation, one can construct a similar framework for Navier-Stokes equations. The critical paths for the action will be diffusions whose drift satisfies these equations. The approach is non perturbative, since the equations describing the velocity of the fluid are still the deterministic ones, as expected, but the position, described by the Lagrangian paths, becomes random. This is an alternative way to describe the motion of particles, which can be justified by a stochastic least action principle.

This stochastic variational principle for Navier-Stokes equations was proved in the case where the configuration space is the two-dimensional flat torus in [14] and later generalized to compact Riemannian manifolds in [3]. The origin of the ideas behind such stochastic variational principles can be found in early works such as [33] and [35]. An analogous stochastic least action principle was derived in [21]. Here the author uses backward instead of forward semimartingales and the variations, unlike ours, are assumed to be of bounded variation.

Actually stochastic Lagrangian variational principles have been initially motivated by quantum mechanics and its Feynman's path integral approach as well as stochastic optimal control problems. We refer to [13] for the first perspective and to [24] as well as to the work of J. M. Bismut [9] for the second.

Our stochastic variational principles in fluid dynamics generalizes to the viscous case Arnold's characterization of Euler's equation for ideal fluids as geodesics on the group of volume-preserving diffeomorphisms. The same kind of stochastic variational principles can be derived on general Lie groups: this is the content of reference [1].

Let us also mention that, originated in Bismut's "mécanique aléatoire" ([9]) a different kind of stochastic generalization or, more precisely, a random perturbation of geometric mechanics has been developed in [11].

Many examples of deterministic Euler-Arnold geodesics, namely geodesics on a Lie group equipped with an invariant metric, have been studied. They include not only Euler but also many interesting equations such as Camassa-Holm or Korteweg-de Vries, formulated in infinite-dimensional Lie groups with suitable metrics. For a survey of this subject we refer to [26]. In principle one can expect to have random counterparts of all such geodesics.

We observe that stochastic Lagrangian flows associated with Navier-Stokes equations such as the ones we study here also appear in representation formulae in the work of Constantin and collaborators ([15], [16]).

It is a natural question to ask how to prove existence of the stochastic Lagrangian flows. One possible way is via their characterization through forward-backward stochastic systems, a subject we are currently investigating. Another is to consider a weaker formulation of the variational problem, in the line of Monge-Kantorovich problems.

After briefly recalling in section 2 some aspects of the geometry of the diffeomorphisms group and the description of (deterministic) Lagrangian Euler flows as geodesics according to V. Arnold, we introduce the stochastic variational principles and the corresponding stochastic Lagrangian flows in section 3. Their stability properties are presented in the next paragraph. We consider the case where the configuration space is the two-dimensional torus in order to simplify the exposition and to concentrate on the main ideas rather than in the more technical geometrical aspects. In section 5 we give an alternative description of the flows using forward-backward stochastic differential equations. Finally, in the last section, we consider a generalization of this notion, in the spirit of Brenier's generalized flows, that allows to tackle the problem of existence in a weaker sense.

2. Euler equations in Arnold's approach

Let M be a compact Riemannian manifold without boundary (starting from the next section we shall specifically consider the case of the two-dimensional flat torus) and denote by dx its volume measure. Consider the space

$$G^s(M) = \{g \in H^s(M;M) : g \text{ bijective},\ g^{-1} \in H^s(M;M)\} \tag{2.1}$$

where H^s is the Sobolev space of order s. If d is the dimension of the manifold M and $s > \frac{d}{2} + 1$, then by Sobolev imbedding theorems the maps in G^s are diffeomorphisms and G^s is a (infinite-dimensional) Hilbert manifold, which is locally diffeomorphic to

$$H^s_g(M) = \{X \in H^s(M;TM) : \pi \circ X = g\}$$

where TM stands for the tangent space of M and $\pi : TM \to M$ for the canonical projection. A chart at g, $\phi : H^s_g(M) \to G^s(M)$ is defined by $\phi(X)(\cdot) = \exp \circ X(.)$, where exp is the exponential map in the manifold M. Also $G^s(M)$ is a topological group for the composition of maps (not quite a Lie group because left composition is not a smooth operation).

The tangent space at the identity of the group ($e(x) = x$) can be identified with the space $\mathscr{G}(M) = H^s_e(M)$, consisting of the vector fields on M which are H^s regular. On $G^s(M)$ we consider the L^2 Riemannian metric defined as

$$< X_g, Y_g >_{L^2} = \int_M (X_g(x).Y_g(x))_{g(x)} dx \tag{2.2}$$

with $g \in G^s(M)$, $X, Y \in T_g(G^s(M)) \equiv H^s_g(TM)$. Here dx denotes integration with respect to the volume measure. Note that the metric does not coincide with the one that defines the topology (this is called a *weak Riemannian structure* in [20]).

The abovementioned spaces are suitable to formulate Burger's equation, for example, but not good enough to consider imcompressible equations such as Euler or Navier-Stokes. For these we need to restrict ourselves to volume-preserving maps. The volume-preserving counterparts of the abovementioned spaces are

$$G^s_V(M) = \{g \in G^s(M) : \int f(g(x))dx = \int f(x)dx\ \forall f\} \tag{2.3}$$

$$\mathscr{G}^s_V(M) = \{X \in G^s(M) : \operatorname{div} X = 0\} \tag{2.4}$$

$$(H_V^s)_g(M) = \{X \in H_g^s(M) : \operatorname{div} X = 0\}$$

One can study these structures either directly or by regarding G_V^s as a submanifold of G^s.

A right invariant Levi-Civita connection ∇^0 with respect to the L^2 metric, such that

$$\nabla_X^0 Y = P_e(\nabla_X Y)\ \forall X, Y \in \mathscr{G}_V$$

can be defined, where ∇ is the Levi-Civita connection on M and P_e is the orthogonal projection in the Hodge decomposition

$$H^s(TM) = \operatorname{div}^{-1}(\{0\}) \oplus grad\ H^{s+1}(M)$$

In the case where M is the two-dimensional torus $\mathbb{T}^2 \simeq [0, 2\pi] \times [0, 2\pi]$, we shall endow the tangent spaces $\mathscr{G}_V^s(\mathbb{T}^2)$ with orthogonal basis (c.f. [17] for higher dimensional torus). For this, let $\tilde{\mathbb{Z}}^2 = \{k = (k_1, k_2) \in \mathbb{Z}^2 : k_1 > 0 \text{ or } k_1 = 0, k_2 > 0\}$. We consider the vector fields $\{A_0, B_0, A_k, B_k,\ k \in \tilde{\mathbb{T}}^2\}$ defined as

$$A_0 \equiv (1, 0),\ B_0 \equiv (0, 1)$$

$$A_k(\theta) = (k_2, -k_1)\cos(k.\theta),\ B_k(\theta) = (k_2, -k_1)\sin(k.\theta) \tag{2.5}$$

where $k.\theta = k_1\theta_1 + k_2\theta_2$, $\theta \in \mathbb{T}^2$.

Multiplying the vector fields by suitable renormalization constants we can obtain orthonormal basis of $\mathscr{G}_V^s(\mathbb{T}^2)$.

In [20] a detailed study of the geometry of diffeomorphisms groups was presented.

Ebin and Marsden proved that geodesics are locally well defined in G_V^s (up to a time that does not depend on the value of s), provided the initial velocities are regular enough (and satisfy some extra topological condition).

Such geodesics are the *Lagrangian Euler flows;*more precisely, $g(t)$ is a geodesic in G_V^s, with $s > \frac{d}{2}$ if and only if $u(t) = \dot{g}(t) \circ g^{-1}(t)$ satisfies (in the classical sense) the equation,

$$\frac{\partial}{\partial t} u(t) = -(u.\nabla)u(t) - \nabla p(t)$$

for some function p, together with the conditions $\operatorname{div} u = 0$ and given initial condition $u_0 \in H^s$.

3. Stochastic variational principles and the Navier-Stokes equations

As mentioned before, from now on we restrict ourselves to the case of a flat compact manifold, namely the two-dimensional torus $\mathbb{T}^2$.

3.1. *Stochastic variational principles.*

Let us start with a formal computation. Consider the simplest possible vector fields in $\mathscr{G}_V^s(\mathbb{T}^2)$, namely the constant ones A_0, B_0. Let $W(t) = W^1(t)A_0 + W^2(t)B_0$ be the canonical Brownian motion (W^1, W^2 are real-valued i.i.d. Brownian motions).

For a time dependent vector field on the torus $u(t,\cdot)$ such that $\operatorname{div} u(t,\cdot) = 0\ \forall t \in [0,T]$ and for a constant $\nu > 0$, let g^u be the solution of the stochastic differential equation

$$dg_t^u(\theta) = \sqrt{2\nu}dW_t + u(t, g_t^u(\theta))dt \tag{3.1}$$

with $g_0^u(\theta) = \theta$, $t \in [0,T]$.

The measure $d\theta$ is invariant for these processes, namely we have, for every function f,

$$\int f(g_t^u(\theta))d\theta = \int f(\theta)d\theta \tag{3.2}$$

For a general semimartingale ξ with values on $\mathbb{T}^2$,

$$d\xi_t(\theta) = \sigma_t(\theta)dW_t + y_t(\theta)$$

with the same invariance property (3.2), define the functional

$$S[\xi] = \frac{1}{2}E\int_0^T \int |y_t(\theta)|^2 d\theta dt$$

and compute S on the diffusions g^u. We are interested in derivating $S[g^u]$, in the sense that we want to consider variations of the paths g^u for which the functional above is still well defined. Take the exponential type functions

$$e_t(\epsilon v)(\theta) = \theta + \epsilon \int_0^t \dot{v}(s, e_s(\epsilon v)(\theta))ds$$

with $\epsilon > 0$ and where $v(t,\cdot)$ is a time dependent vector field such that $v(0) = v(T) = 0$ and $\operatorname{div} v(t,\cdot) = 0$ for every $t \in [0,T]$.

Notice that, up to the first order in ϵ we have $e_t(\epsilon v)(\theta) \simeq \theta + \epsilon v(t,\theta)$.

The variations of the paths $g^u(t)$ will be defined by left composition:

$$g_t^{u,\epsilon} = e_t(\epsilon v) \circ g^u(t)$$

We have,

$$dg_t^{u,\epsilon} = \nabla e_t(\epsilon v)(g_t^{u,\epsilon})\sqrt{2\nu}dW_t + [\dot{e}_t(\epsilon v) + (u.\nabla)e_t(\epsilon v) + \nu\Delta e_t(\epsilon v)](g_t^{u,\epsilon})]dt$$

and therefore,

$$\frac{d}{d\epsilon}|_{\epsilon=0} S[g_t^{u,\epsilon}] = E\int_0^T \int \left(u(t,\theta).[\dot{v} + (u.\nabla)v + \nu\Delta v]\right)(t, g_t^u)dtd\theta$$

Because of the invariance of the volume measure on the torus we also have

$$\frac{d}{d\epsilon}|_{\epsilon=0} S[g_t^{u,\epsilon}] = \int_0^T \int \left(u.[\dot{v} + (u.\nabla)v + \nu\Delta v]\right)(t,\theta)dtd\theta$$

$$= -\int_0^T \int \left([\dot{u} + (u.\nabla)u - \nu\Delta u].v\right) dtd\theta$$

where we have used integration by parts and the assumptions $\operatorname{div} u = 0$ as well as $v(0) = v(T) = 0$.

We conclude that the derivative of the action functional S when we consider the variations above is zero (the process g_t^u is critical for S) if and only if the vector field $u(t,\cdot)$ solves Navier-Stokes equations in the L^2 weak sense.

3.2. *Brownian motions on the group of measure-preserving diffeomorphisms.*

We are going to formulate the last statement in a more rigorous way and considering critical processes driven by more general diffusion coefficients.

We consider a Brownian motion with values in $\mathscr{G}_V^s(\mathbb{T}^2)$ of the form,

$$dx(t) = \sum_{\tilde{\mathbb{Z}}^2 \cup \{0\}} \lambda_k (A_k(\theta) dW_k^1(t) + B_k(\theta) dW_k^2(t)) \tag{3.3}$$

We can choose for example $\lambda_k = \frac{1}{|k|^{s+1}}$ where $|k| = \sqrt{k_1^2 + k_2^2}$, but for our purposes, we can simply take a finite number of λ_k.

This Brownian motion induce on the group G_V^s the processes g satisfying the following Stratonovich stochastic differential equation,

$$dg(t) = (\circ dx(t)) g(t), \ g(0) = e \tag{3.4}$$

More explicitly, for $i = 1, 2$,

$$dg^i(t)(\theta) = \sum_{k \in \mathbb{Z}^2 \cup \{0\}} \lambda_k (A_k^i(g(t)(\theta)) \circ dW_k^1(t) + B_k^i(g(t)(\theta)) \circ dW_k^2(t))$$

with $g(t)(\theta) = \theta$.

The regularity of the process g as a function of the space variable θ depends on the chosen coefficients λ_k (c.f. [30] and [14]). If we chose a finite number of such coefficients, from the classical theory of stochastic flows ([28]), $g(t)$ will be well defined as a stochastic flow of diffeomorphisms.

Two important properties of the process g are the following

Proposition 3.1. *The Stratonovich differentiation in (3.4) coincides with the Itô one.*

Proof.

Since, for each k,

$$d(A_k^1(g(t)) = (\partial_1 A_k^1(g(t)) \circ dg^1(t) + (\partial_2 A_k^1(g(t)) \circ dg^2(t)$$

$$= \sum_m \lambda_m (\partial_1 A_k^1(g(t)) [A_m^1(g(t)) \circ dW_m^1(t) + B_m^1(g(t)) \circ dW_m^2(t)]$$

$$+ \sum_m \lambda_m (\partial_2 A_k^2(g(t)) [A_m^2(g(t)) \circ dW_m^1(t) + B_m^2(g(t)) \circ dW_m^2(t)],$$

we have

$$d(A_k^1(g(t)).dW_k^1(t) = \lambda_k [(\partial_1 A_k^1(g(t)) A_k^1(g(t)) + \partial_2 A_k^1(g(t)) A_k^2(g(t))] dt$$

$$= \lambda_k [-(k_2)^2 k_1 \sin(k.\theta) \cos(k.\theta) + (k_2)^2 k_1 \sin(k.\theta) \cos(k.\theta)] dt = 0$$

All other Itô contraction terms can be shown to vanish in a similar way.

Proposition 3.2. When computed on functionals of the form $F(g)(\theta) = f(g(\theta)),\ f \in C^2(\mathbb{T}^2)$, the generator $\mathscr{L}$ of the process g coincides with the Laplacian multiplied by a constant. More precisely,

$$\mathscr{L}(F)(g)(\theta) = c\Delta f(\theta)$$

with $c = \frac{1}{2}\sum_k \lambda_k((k_1)^2 + 1)$

Proof.
Explicit computation (c.f. [14], Theorem 2.2).

From now on we shall consider the coefficients of the Brownian motion to be such that $c = 1$.

3.3. *Stochastic differential equations on the diffeomorphisms group.*

Let $u(t)(\cdot), t \in [0,T]$ be a time-dependent vector field on the torus $\mathbb{T}^2$ such $\operatorname{div} u(t) = 0$ for all t. We want to consider the following stochastic differential equations

$$dg_u(t)(\theta) = (\sqrt{2\nu}\ \circ dx(t) + u(t)dt)(g_u(\theta)),\ g_u(0)(\theta) = \theta \tag{3.5}$$

The generator of the diffusion g_u satisfies

$$\mathscr{L}_u(F)(g)(\theta) = \nu\Delta f(\theta) + (u.\nabla f)(\theta)$$

for $F(g)(\theta) = f(g(\theta)),\ f \in C^2)$.

The existence and regularity properties of the flows g_u depend on the regularity of the drift u. In the recent years much attention has been given to the construction of solutions of stochastic differential equations with non-regular drifts, both in the weak and in the strong sense. References [27], [23] or [22] are just a few examples of works on this subject.

In [14] we proved the existence of weak solution g_u with values in $G_V^0(\mathbb{T}^2)$ for the s.d.e. above when $u \in L^2([0,T];\mathscr{G}_V^0(\mathbb{T}^2))$. We will come back to regularity questions in section 6, but for now we assume that the vector fields u are smooth.

3.3. *Stochastic variational principle on the diffeomorphisms group.*

Denote by $\mathscr{P}$ the set of $G_V^0(\mathbb{T}^2)$-valued semimartingales $g(t)$ such that $g(0) = e$ and consider, for a functional defined on G_V^0 the mean time derivative

$$D_t F(g(t)) = \lim_{\epsilon\to 0} \frac{1}{\epsilon} E_t(F(g(t+\epsilon)) - F(g(t)) \tag{3.6}$$

where E_t denotes conditional expectation with respect to the filtration generated by $\{g(\tau), \tau \in [0,t]\}$.

Let

$$\mathscr{H} = \{v \in C^1([0,T];\mathscr{G}_V^0), v \text{ smooth in the space variable, } v(0)(\cdot) = v(T)(\cdot) = 0\}$$

As explained before, we consider the solutions of the ordinary differential equation driven by $\dot{v}(t)(.)$, namely

$$\frac{d}{dt} e_t(v) = \dot{v}(t, e_t(v)),\ e_0(v) = e \tag{3.7}$$

For all $v \in \mathscr{H}$, if $g \in \mathscr{P}$ we have $e_t(v)og\ (t) \in \mathscr{P}$.

Let S is a functional defined on $\mathscr{P}$ with values in $\mathbb{R}$. We define its left and right derivatives in the direction $h(\cdot) = e.(v)$ at a process $g \in \mathscr{P}$ respectively, by

$$(D_L)_h J[g] = \frac{d}{d\epsilon}|_{\epsilon=0} S[\ (e.(\epsilon v)) \circ g(\cdot)]$$
$$(D_R)_h J[g] = \frac{d}{d\epsilon}|_{\epsilon=0} S[g(\cdot) \circ e.(\epsilon v)]$$

A process $g \in \mathscr{P}$ wil be called a *critical point* of the functional S if

$$(D_L)_h S[g] = (D_R)_h S[g] = 0,\ \forall h = e.(v)$$

The action functional defined by the stochastic kinetic energy is the following:

$$S[g] = \frac{1}{2} E \int_0^T \Big(\int_{\mathbb{T}^2} |D_t g_u(t)(\theta)|^2 d\theta\Big) dt \tag{3.8}$$

Note that if $\nu = 0$ the paths g are deterministic and the stochastic kinetic energy functional reduces to the classical one.

Then we have the following

Theorem 3.3. ([14], [3]) *Let $(t, \theta) \mapsto u(t, \theta)$ be a smooth time dependent divergence-free vector field on $\mathbb{T}^2$, defined on $[0, T] \times \mathbb{T}^2$. Let $g_u(t)$ be a stochastic Brownian flow with diffusion coefficient $\sqrt{2\nu}$ and drift u (as in (3.1)). The stochastic process $g_u(t)$ is a critical point of the energy functional S if and only if there exists a function p such that the vector field $u(t)$ verifies the Navier-Stokes equation*

$$\frac{\partial u}{\partial t} = -(u.\nabla)u + \nu \Delta u - \nabla p$$

in the weak L^2 sense.

We shall write a proof of this result which is a little different from the argument given in subsection 3.1. and, in some sense, closer to stochastic calculus.

Proof of Theorem 3.1.

Since the energy functional S is right invariant, we only need to consider its left derivative. Recall that

$$e_t(\epsilon v) = e + \int_0^t \dot{v}(s, e_s(\epsilon v)) ds$$

Hence,

$$\frac{d}{d\epsilon}|_{\epsilon=0} e_t(\epsilon v) = \int_0^t \dot{v}(s, e) ds = v(t, \cdot)$$

Therefore we have,

$$\frac{d}{d\epsilon}|_{\epsilon=0} S[e.(\epsilon v) \circ g_u(\cdot)] = E \int_0^T \left(\int_{\mathbb{T}^2} D_t g_u(t)(\theta).D_t v(g_u(t)(\theta)\right) d\theta) dt$$

By Itô's formula

$$d \int D_t g_u(t)(\theta).v(g_u(t)(\theta)) d\theta = \int dD g_u(t)(\theta).v(g_u(t)(\theta)) d\theta + \int D g_u(t)(\theta).dv(g_u(t)(\theta)) d\theta$$

$$+\int dD_t g_u(t)(\theta).dv(g_u(t)(\theta))d\theta$$

The last Itô's contraction term is equal to

$$2\nu(\int(\nabla v\otimes\nabla u)(g_u(t)(\theta))d\theta)dt$$

where $\nabla v\otimes\nabla u=\sum_{i,j=1}^2\partial_j v^i\partial_j u^i$. Since $v(0)=v(T)=0$ this implies,

$$\frac{d}{d\epsilon}|_{\epsilon=0}S[e.(\epsilon v)\circ g_u(\cdot)]=-E\int_0^T(\int(D_tD_tg_u(t)(\theta)d\theta)dt-2\nu E\int_0^T(\int(\nabla v\otimes\nabla u)(g_u(t)(\theta))d\theta)dt$$

On the other hand

$$D_tD_tg_u(t)=(\frac{\partial}{\partial t}u+(u.\nabla)u+\nu\Delta u)(g_u(t))$$

and therefore, using the invariance of the measure $d\theta$ with respect to the process g_u, we obtain

$$\frac{d}{d\epsilon}|_{\epsilon=0}S[e.(\epsilon v)\circ g_u(\cdot)]=-E\int_0^T\int\left(\frac{\partial}{\partial t}u+(u.\nabla)u-\nu\Delta u(t,g_u(t)(\theta).v(t,g_u(t)(\theta)d\theta\right)dt$$

$$=-\int_0^T\left(\int[\frac{\partial}{\partial t}u+(u.\nabla)u-\nu\Delta u].v(t,\theta))d\theta\right)dt$$

from which the result follows.

The theorem above can be regarded as a particular case of stochastic variational principles on (Lie) groups, when one considers the measure preserving diffeomorphisms group on the torus. General results in Lie groups are studied in [1].

4. Stability properties of the stochastic Lagrangian flows

4.1. *Stability of Euler Lagrangian flows.*

One of the reasons for the success of Arnold's approach in Hydrodynamics has been its application to the study of stability properties of Euler flows. In finite dimensions it is well known that the behaviour of geodesics according to the different initial conditions can be expressed in terms of the curvature of the underlying manifold via the Jacobi equation. This is still true in some sense for geodesics on the infinite-dimensional relevant space for Hydrodynamics, namely the space of volume preserving diffeomorphisms.

Arnold could show, in many cases, that the curvature of these spaces is negative and therefore that the fluid trajectories are unstable (or "chaotic"), i.e. their distance, starting from different initial conditions, grows exponentially during time evolution (c.f. [7]).

Recall from Riemannian geometry that, given a family of geodesics $\{X_\tau(\cdot)\}$ defined on a Riemannian manifold which is endowed with a connection ∇, the Jacobi field $J(t)=\frac{\partial X(t)}{\partial\tau}|_{\tau=0}$ describes the behaviour of the geodesics in a neighborhood of X_0. The geodesic equation is $\nabla_{\dot X}\dot X=0$ and the Jacobi field satisfies the Jacobi equation (its linearisation),

$$\nabla_{\dot{X}}\nabla_{\dot{X}}J = -R(J,\dot{X})\dot{X}$$

where R denotes the Riemannian curvature tensor.

In [32] existence of unbounded Jacobi fields in the diffeomorphism group G_V^s was shown:

Theorem 4.1. ([32]) *Let g ge a geodesic in G_V^s and $J(t)$ a non zero solution of the Jacobi equation $\nabla_{\dot{g}}\nabla_{\dot{g}}J = -R(J,\dot{g})\dot{g}$ with $J(0) = 0$, $\nabla_{\dot{g}}J(0) = \dot{J}_e \in T_e(G_V^s)$ and such that the two dimensional curvature of the plane spanned by $J(t)$ and $\dot{g}(t)$ is non positive for all t. Then $||J(t)||_{L^2} \geq ct$ for all t and some positive constant c depending on $\dot{J}_e$.*

The corresponding geodesics, in this case, are said to be Lagrangian unstable. In the specific case of the two dimensional torus curvatures are shown no be negative, which implies the exponential instability of the geodesics (c.f. [7]).

4.2. *Stability of stochastic Lagrangian flows.*

The stochastic Lagrangian flows are, as we have seen, critical points of a generalized stochastic energy functional, and can therefore be regarded as generalized geodesics. We shall discuss in Section 5 how we can give a meaning to the corresponding geodesic equation, but it is still not clear how to define the associated Jacobi fields. Instead, we describe stability of the stochastic Lagrangian flows by explicitly computing the formula for the distance between two such flows.

For viscous flows it is expected that the trajectories of the particles become closer and closer after some possible initial stretching. Of course these are dissipative systems and they have no standard geodesic formulation. For our model, at least in the case of the two dimensional torus, we could show that the sensitivity with respect to initial conditions of the trajectories is enhanced by their stochasticity. Their behaviour will depend of the choice of the coefficients λ_k in (3.3) or, in other words, on which scales and with what strenght the motion is excited.

We shall describe here the study of the L^2 distance between trajectories, following our study in [3]. Some particular solutions and simulations were discussed in [5] where we have considered the ponctual distance.

Let g_u and $\tilde{g}_u$ be two stochastic Lagrangian trajectories associated with the same drift and starting from two different diffeomorphisms ϕ and ψ on the torus. Namely,

$$dg_t = \sqrt{2\nu}(odx(t))(g_t) + u(t,g_t)dt, \qquad d\tilde{g}_t = \sqrt{2\nu}(odx(t))(\tilde{g}_t) + u(t,\tilde{g}_t)dt$$

with

$$g_0 = \phi, \qquad \tilde{g}_0 = \psi, \qquad \phi \neq \psi$$

We consider the L^2 distance of the particles defined by

$$\rho^2(\phi,\psi) = \int_{\mathbb{T}^2} |\phi(\theta) - \psi(\theta)|^2 d\theta$$

Denoting $\rho_t = \rho(g_t,\tilde{g}_t)$ and $\tau(g,\tilde{g}) = \inf\{t > 0 : \rho_t = 0\}$. We have the following result:

Proposition *The stopping time $\tau(g,\tilde{g})$ is infinite.*

Proof. By uniqueness of the solution of the s.d.e. for $\tilde{g}_t$, for all $t > 0$ we can write $\tilde{g}_t(\theta) = g_t((\phi^{-1} \circ \psi)(\theta))$. Since g_t, φ and ψ are diffeomorphisms, if $\varphi(\theta) \neq \psi(\theta)$ then $g_t(\theta) \neq g_t((\phi^{-1} \circ \psi)(\theta))$.

As $\phi \neq \psi$, the set $\left\{\theta \in \mathbb{T}^2,\ \tilde{g}_t(\theta) \neq g_t(\theta)\right\}$ has positive measure and this implies that $\rho_t > 0$. Therefore $\tau(g, \tilde{g})$ is infinite.

Denote by $L_t(\theta)$ the local time of the process $|g_t(\theta) - \tilde{g}_t(\theta)|$ when $(g_t(\theta), \tilde{g}_t(\theta))$ reaches the cutlocus of $\mathbb{T}^2$. By Itô calculus we have

$$d\rho_t = \frac{1}{\rho_t} \sum_k \lambda_k \sqrt{2\nu} \left\langle g_t - \tilde{g}_t, (A_k(g_t) - A_k(\tilde{g}_t))\, dW_k^1(t) + (B_k(g_t) - B_k(\tilde{g}_t))\, dW_k^2(t) \right\rangle_{\mathbb{T}}^2$$

$$+\frac{1}{\rho_t} \langle g_t - \tilde{g}_t, u(t, g_t) - u(t, \tilde{g}_t) \rangle_{\mathbb{T}}^2\, dt - \frac{1}{\rho_t} \int_{\mathbb{T}^2} |g_t - \tilde{g}_t|(\theta) dL_t(\theta)$$

$$+\frac{1}{\rho_t} \sum_k \lambda_k^2 \nu \left(\|A_k(g_t) - A_k(\tilde{g}_t)\|_{\mathbb{T}^2} + \|B_k(g_t) - B_k(\tilde{g}_t)\|_{\mathbb{T}^2}\right) dt$$

$$-\frac{1}{\rho_t^3} \sum_k \lambda_k^2 \nu \left(\langle g_t - \tilde{g}_t, A_k(g_t) - A_k(\tilde{g}_t) \rangle_{\mathbb{T}^2} + \langle g_t - \tilde{g}_t, B_k(g_t) - B_k(\tilde{g}_t) \rangle_{\mathbb{T}^2}\right) dt$$

The explicit formula above allows to estimate the L^2 distance of the Lagrangian paths and, in particular, we obtained the following results

Theorem 4.2. ([3])*Let $t > 0$, $R \geq 1$ and*

$$\Omega_t = \left\{\omega \in \Omega,\ \forall s \leq t,\ \forall \theta \in \mathbb{T}^2,\ |(g_s(\theta)(\omega) - \tilde{g}_s(\theta)(\omega))| \leq \frac{\pi}{2R}\right\}.$$

If we assume the initial conditions for the L^2 distance and the L^2 norm of the initial velocity to be related as $\rho_0 - 2\|u_0\|_{\mathbb{T}^2} > 0$, then there exist a function $\sigma_t > 0$ and a constant $c(R) > 0$ such that on the set Ω_t we have,

$$\forall s \leq t, \quad \rho_s \geq e^{\int_0^t \sqrt{\nu}\sigma_s dz_s + \nu c(R)t} \left(\rho_0 - 2|u_0| \int_0^t e^{-\int_0^s \sqrt{\nu}\sigma_r dz_r - (\nu c(R) + \frac{\nu}{2})s}\, ds\right)$$

where z is a real-valued Brownian motion, the inequality holding as long as the right hand side stays positive.

Moreover both σ and $c(R)$ are explicit functions of the coefficients λ_k.

Theorem 4.3. ([3]) *If we assume that there exist constants $c_1,\ c_2 > 0$ such that for all $\theta \in \mathbb{T}^2$ and $s \in [0, t]$,*

$$|\nabla u(t, \theta)| \leq c_1 e^{-c_2 t},$$

then on Ω_t we have the more precise lower bound, holding $\forall s \leq t$,

$$\rho_s \geq \rho_0 \exp\left(\int_0^t \sigma_s\, dz_s + ct - \frac{c_1}{c_2}\left(1 - e^{-c_2 t}\right)\right).$$

For the proof of the above results we refer to [3], where we can find the explicit expressions of the functions involved in the inequalities. Analysing these expressions one can deduce that, for a fixed viscosity, the stochastic Lagrangian paths tend to get apart faster when the higher modes k (and therefore the smaller length scales) are randomly excited. In any case they spread out more than the deterministic classical Lagrangian paths.

Note that even when the velocity decays to zero at exponential rate, which is the case of many solutions of the Navier-Stokes equation, the stochastic Lagrangian flows describing the position of the fluid get apart exponentially fast, at least for short times.

We have studied stability properties of stochastic Lagrangian flows on general compact manifolds in reference [2]. Generalizing the formulae for the distance between two flows we have in particular observed that, when the Ricci curvature of the manifold is negative, the chaotic regime persists.

5. Relation with forward-backward stochastic differential systems

A geodesic curve is a solution of a second order differential equation. It is standard to solve geodesic equations with given initial position and initial velocity, although other types of related problems where the data consists of a initial and a final position may be considered (we will refer to this "shortest path problem" in section 6).

Stochastic Lagrangian flows are, in some sense, geodesic flows, as they are critical paths of (stochastic) kinetic energy functionals. And they do satisfy some generalized second order differential equations, as we shall see. We believe that the best way to describe them is to use forward-backward stochastic differential equations. Backward stochastic equations have been initially introduced by J. M. Bismut (c.f. [8]) in relation to stochastic optimal control problems. A reference for this subject is the book [29].

Let us consider a time change in the equation for the stochastic Lagrangian flow. Namely, let us consider the diffusion process on the time interval $[0, T]$,

$$X_t(\theta) = \theta + \sqrt{2\nu}\int_0^t dx(t)(X(t)(\theta)) - \int_0^t u(T-s, X(s)(\theta))ds$$

Denote by $Y(t)$ the drift of the process,

$$Y(t) = -u(T-t, X(t)) = D_t X(t)$$

By Itô's formula we have

$$Y(t)(\theta) = Y(T)(\theta) + \sqrt{2\nu}\int_t^T \nabla u(T-s, X(s)(\theta))dx(t)((X(t)(\theta))$$
$$+\int_t^T [-\frac{\partial}{\partial t}u - (u.\nabla)u + \nu\Delta u](T-s, X(s)(\theta))ds$$

If the function u satisfies the Navier-Stokes equations we have

$$Y(t)(\theta) = Y(T)(\theta)) + \sqrt{2\nu}\int_t^T \nabla u(T-s, X(s)(\theta))dx(t)((X(t)(\theta))$$
$$+\int_t^T \nabla p(T-s, X(s)(\theta))ds$$

with $Y(T)(\theta) = -u(0, X(T)(\theta))$.

The stochastic Lagrangian flows, critical paths of the kinetic energy functional, are therefore solutions of a stochastic forward-backward system with a final condition which is a function of the position. The equivalent statement is also true: solutions of such forward-backward equations are critical points of the action (c.f. [18]). More precisely, always assuming solutions of Navier-Stokes equations to be smooth, we have the following representation result,

Theorem 5.1. ([18]) *Assume that the function $p : [0, T] \to H^{s+1}$ is continuous and u_0 is a H^{s+1} divergence-free vector field on the torus. Then there exist a triple $(X_\tau(t), Y_\tau(t), Z_\tau(t))_{t\in[\tau,T]}$, with $X_\tau(t) \in G^s_V(\mathbb{T}^2)$, $Y_\tau(t) \in \mathscr{G}_V(\mathbb{T}^2)$ and $Z_\tau(t)$ defined on $G^s_V(\mathbb{T}^2)$ with values in the space of linear operators between $\mathbb{T}^2$, which is a strong solution of the forward-backward stochastic system, for $t \in [\tau, T]$,*

$$\begin{cases} dX_\tau(t) = \sqrt{2\nu}\ dx(t)(X_\tau(t)) + Y_\tau(t)dt \\ dY_\tau(t) = \sqrt{2\nu}\ Z_\tau(t)dx(t) + \nabla p(X_\tau(t))dt \\ X_\tau(\tau) = e,\ Y_\tau(T) = u_\tau(X_\tau(T)) \end{cases}$$

Reciprocally, if for every $\tau \in [0, T[$ such a solution $(X_\tau(t), Y_\tau(t), Z_\tau(t))$ exists, which is continuous in t and with values in the above spaces, then there exist a vector field $u : \mathbb{T}^2 \to \mathbb{T}^2$, $u \in H^s$, such that a.s. $Y(t) = -u(T - t, X(t)(\theta))$ and $u(t, \cdot)$ solves the Navier-Stokes equations with initial condition u_0.

We have therefore a characterization of stochastic Lagrangian flows as second order differential systems. A generalization of this type of results to Lie groups is described in [12].

In [19], by considering a related backward system for the vorticity, we show existence of solutions in the two-dimensional torus case.

6. Generalized stochastic Lagrangian flows

The second order geodesic equations characterizing Lagrangian flows, both in ths deterministic and in the stochastic case, can be associated to different kinds of boundary conditions. We may prescribe an initial position $g(0)$ and initial velocity $u(0)$ and in this case we are led to solving, after a change of time, and in the stochastic (Navier-Stokes) setting, a forward-backward system, as described in last section.

One can also give an initial and final position. For the Euler equation this is called the "shortest path problem". It was solved by Ebin and Marsden ([20]) for sufficiently smooth data (together with some topological restrictions on this data) and for small time intervals. In general the problem is very difficult, maily because the action functional does not involve derivatives in space while the incompressibility condition is a condition on the determinant of the flow. It is therefore not possible to use classical methods for controlling minimizing sequences. Actually there are situations where there exist no shortest path: this was firt shown by A. I. Shnirelman in [34], where a counterexample, defined on the three dimensional cube as configuration space, was constructed.

3.3. *The deterministic case.*

In [10] Y. Brenier introduced a probabilistic concept of generalized flow for the Euler equations. This concept allows to relax the problem and consider the Lagrangian flows to be not single trajectories but measures on a space of trajectories.

Let us briefly recall the basic notions for this problem. Let M, as before, be a compact finite dimensional manifold (without boundary). Following [10] we define a *doubly stochastic probability measure* as a probability measure η on $M \times M$ such that

$$\int_{M\times M} f(x)\eta(dx,dy) = \int_{M\times M} f(y)\eta(dx,dy) = \int_M f(x)dx$$

Consider a deterministic flow of measurable maps $g(t) : M \to M$ which are invertible and volume-preserving (incompressible flows). Then $\eta_{g(t)}(dx,dy) = \delta(y - g(t)(x))dx$ defines a doubly stochastic probability measure, since we have

$$\int_{M\times M} f(x,y)\eta_{g(t)}(dx,dy) = \int_M f(x,g(t)(x))dx$$

and g is volume preserving.

The general idea is to solve the shortest path problem in the space of doubly stochastic measures rather then in some space G^s_V.

With an incompressible flow $g(t)$ we can associate a measure Q on the path space $\Omega = C([0,T];M)$ such that, for any cylindrical functional $F(\gamma) = f(\gamma(t_1),...,\gamma(t_n))$, we have,

$$\int_\Omega F(\gamma)dQ(\gamma) = \int_M f(g(t_1)(x),...,g(t_n)(x))dx$$

Then Q satisfies the following properties,

$$\forall t \in [0,T] \quad \int_\Omega f(\gamma(t))dQ(\gamma) = \int_M f(x)dx \text{ (incompressibility)}$$

and

$$\int_\Omega f(\gamma(0),\gamma(T))dQ(\gamma) = \int_{M\times M} f(x,y)\delta(y - g_T(x))dx$$

The second property tells us that the marginals of the flow define a doubly stochastic probability measure. We say that the flow reaches the final configuration $\eta(dx,dy) = \delta(y - g_T(x))dx$.

Such a measure Q is called a *generalized incompressible flow* reaching the final configuration η. We associate with it the action functional associated with the kinetic energy,

$$A(Q) = \frac{1}{2}\int_M (\int_0^T |\dot\gamma(t)|^2 dt)dQ(\gamma)$$

Theorem 6.1.([10]) *For any final configuration η, if there exists one incompressible generalized flow Q that reaches η at time T with a finite kinetic energy, then there exists such a flow that minimizes the action.*

Brenier also proved that the set of classical, deterministic Lagrangian flows for the Euler equation is contained in the set of generalized flows; therefore this is a natural framework to extend Euler Lagrangian flows. Furthermore he proved that the inclusion is strict, giving examples of generalized flows that do not come from classical ones.

3.3. *The stochastic case.*

We can generalize Brenier's notion of generalized flow for stochastic processes and give a meaning of Navier-Stokes Lagrangian flows in this weaker sense: this is the content of reference [4]. Again, for simplicity, we describe here this approach in the case where M is the two dimensional torus.

Let η be a probability measure on $\mathbb{T}^2 \times \mathbb{T}^2$ with marginals equal to $d\theta$ which, in particular, desintegrates as $\eta(d\theta, d\sigma) = d\theta\eta_\theta(d\sigma)$. Consider semimartingale flows on $\mathbb{T}^2$ defined on the time interval $[0, T]$ with the properties:

(1) $g(0)(\theta) = \theta$ and for all $x \in M$, $g(T)(\theta)$ has law η_θ;

(2) $g(\cdot)(\theta)$ satisfies the Itô equation

$$dg_u(t)(\theta) = \sqrt{2\nu}dx(t) + u(t, g_u(t)(\theta), \omega)dt;$$

where $(t, \theta, \omega) \mapsto u(t, \theta, \omega) \in T_\theta(\mathbb{T}^2) = \mathbb{T}^2$ is a time-dependent adapted drift with locally bounded variation in θ (in the sense of distributions);

(3) the kinetic energy of g

$$\begin{aligned}\mathscr{E}(g) &:= \frac{1}{2}\mathbb{E}[\int(\int_0^T |D_t g(\theta, \cdot)dt|^2)d\theta] \\ &= \frac{1}{2}\mathbb{E}[\int(\int_0^T |u(t, \theta, \cdot)dt|^2)\ d\theta]\end{aligned}$$

is finite;

(4) almost surely for all $t \in [0, T]$, $\operatorname{div} u(t, \cdot, \omega) = 0$. This together with the definition of the Brownian motion x implies that the flow is incompressible, i.e. for all t, ω a.s. for all $f \in C(\mathbb{T}^2)$,

$$\int_{\mathbb{T}^2} f(g(t)(\theta)(\omega))d\theta = \int_{\mathbb{T}^2} f(\theta)d\theta$$

Notice that when the viscosity parameter is zero we can consider $\eta_\theta = \delta_{h(\theta)}$ and these semimartingale flows coincide with Brenier's generalized flows.

It is not clear how to obtain the existence of critical points for our variational principles among semimartingale flows satisfying $g(0)(\theta) = \theta$. For this reason we have considered their corresponding transports instead.

To a semimartingale flow $g(t)(\cdot)$ we can associate a transport Θ^g defined as a map which, to functions φ, ψ that we shall consider to belong to $C^\infty(\mathbb{T}^2)$, associates the process

$$\Theta_t^g(\varphi, \psi) = \int_{\mathbb{T}^2} \phi(\theta)\psi(g(t)(\theta))d\theta$$

This process is therefore a real valued semimartingale satisfying the equation,

$$\begin{aligned}\Theta_t^g(\varphi, \psi) &= \int_{\mathbb{T}^2} \varphi\psi d\theta + \nu \sum_k \lambda_k \int_0^t \Theta_s^g(\varphi, \operatorname{div}(\psi H_k))dW_k(t) \\ &+ \int_0^t \Theta_s^g(\varphi, \operatorname{div}(\psi u(s, \cdot, \omega))\ ds + \frac{1}{2}\int_0^t \Theta_s^g(\varphi, \Delta\psi)ds\end{aligned}$$

where H_k is a generic notation for the vector fields A_k and B_k. We have,

$$\Theta_t^g(\varphi,\psi) = \int_{\mathbb{T}^2} \varphi(g(t)(\cdot)(\omega))^{-1}(\theta))\psi(\theta)d\theta$$

The time derivative of the drift of the semimartingale $\Theta_t^g(\varphi,\psi)$ is given by

$$D_t\Theta^g(\varphi,\psi) = \Theta_t^g\left(\varphi, \mathrm{div}(\psi u)(t,\cdot,\omega) + \frac{1}{2}\Delta\psi\right).$$

The semimartingales Θ_t^g posess properties (i)-(vii) below, which leads us to the following

Definition ([4]) A generalized flow with diffusion coefficient determined by the vector fields H_k and final configuration η is a bilinear map Θ, which to $\varphi,\psi \in C^\infty(\mathbb{T}^2)$ associates a continuous semimartingale $t \mapsto \Theta_t(\varphi,\psi)$, $t \in [0,T]$, with the following properties:

(i) for all $\varphi,\psi \in C^\infty(\mathbb{T}^2)$,

$$E[\Theta_T(\varphi,\psi)] = \int_{\mathbb{T}^2\times\mathbb{T}^2} \varphi(\theta)\psi(\sigma)\eta(d\theta,d\sigma);$$

(ii) for all $\varphi,\psi \in C^\infty(\mathbb{T}^2)$,

$$\Theta_t(\varphi,1) = \int_{\mathbb{T}^2}\varphi(\theta)d\theta \quad \text{and} \quad \Theta_t(1,\psi) = \int_M \phi(\theta)d\theta \quad \text{a.s. for all } t$$

(iii) for all $\varphi_1,\psi_1,\varphi_2,\psi_2 \in C^\infty(\mathbb{T}^2)$ the covariance function satisfies

$$d[\Theta(\varphi_1,\psi_1),\Theta(\varphi_2,\psi_2)]_t = \nu^2\sum_k \lambda_k^2(\Theta_t(\varphi_1,\mathrm{div}(\phi_1 H_k))(\Theta_t(\varphi_2,\mathrm{div}(\phi_2 H_k))dt.$$

(iv) for all $\varphi,\psi \in C^\infty(\mathbb{T}^2)$, the semimartingale

$$\tilde{\Theta}_t(\varphi,\psi) := \Theta_t(\varphi,\psi) - \frac{1}{2}\int_0^t \Theta_s(\varphi,\Delta\psi)\, ds$$

has absolute continuous drift with time derivative $D\tilde{\Theta}(\varphi,\psi)$. In particular

$$E[\int_0^T (D_t\tilde{\Theta}(\varphi,\psi))^2]dt \le 2\mathscr{E}'(\Theta)||\varphi||^2_{L^2(M)}||\nabla\psi||^2_{L^\infty(\mathbb{T}^2)}.$$

(v) for all $\varphi,\psi \in C^\infty(\mathbb{T}^2)$,

$$\Theta_0(\varphi,\psi) = (\varphi,\psi)_{L^2(\mathbb{T}^2)}$$

(vi) Θ is nonnegative, that is for all nonnegative $\varphi,\psi \in C^\infty(\mathbb{T}^2)$, $\Theta(\varphi,\psi)$ is a nonnegative process

(vii) for all $\varphi,\psi \in C^\infty(\mathbb{T}^2)$, a.s. for all $t \in [0,T]$,

$$||\Theta_t(\varphi,\psi)|| \le ||\varphi||_{L^2(\mathbb{T}^2)}||\psi||_{L^2(\mathbb{T}^2)}.$$

Define the kinetic energy of Θ as

$$\mathscr{E}'(\Theta) = \frac{1}{2}\sup\left\{\sum_{j=1}^m\sum_{k=1}^\ell E\left[\int_0^T \frac{\left(D\tilde{\Theta}_t(\varphi^j,\psi^k)\right)^2}{\Theta_t(\varphi^j,1)}\right] dt,\ m,\ell \ge 1,\right.$$

$$\varphi^j, \psi^k \in C^\infty(\mathbb{T}^2), \varphi^j \geq 0, \sum_{j=1}^m \varphi^j = 1, \ \psi^k \text{ s.t. } \forall v \in \mathbb{T}^2, \ \sum_{k=1}^\ell \langle \nabla \psi^k, v \rangle^2 \leq ||v||^2 \Bigg\},$$

where $D_t\tilde{\Theta}(\varphi^j, \psi^k)$ denotes the time derivative of the drift of $\tilde{\Theta}_t(\varphi^j, \psi^k)$. Notice that $\Theta_t(\varphi^j, 1) = \int_{\mathbb{T}^2} \varphi^j(\theta) d\theta$ by the incompressibility condition.

This kinetic energy is an extension of the one defined in (3), i.e., for a semimartingale flow the two definitions coincide:

Proposition 6.2. ([4]) *For a semimartingale flow g we have*

$$\mathscr{E}'(\Theta^g) = \mathscr{E}(g)$$

We have proved the following extension of Brenier's result to generalized stochastic flows:

Theorem 6.3. ([4]) *If there exists a stochastic generalized flow with fixed diffusion coefficient and final configuration η having a finite kinetic energy, then there exists one such flow that minimizes the energy.*

Under which conditions the generalized minimizer is unique or corresponds to a semimartingale flow remain, among other questions, open problems.

What we did prove (c.f. [4]) is that there exist stochastic generalized flows which do not correspond to Navier-Stokes semimartingale flows. They can be built upon weak solutions of some transport equations.

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