STOCHASTIC GEODESICS AND STOCHASTIC BACKWARD EQUATIONS ON LIE GROUPS

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Abstract. We describe how to generalize to the stochastic case the notion of geodesic on a Lie group equipped with an invariant metric. We give a characterization of the stochastic geodesic equations in terms of a backward stochastic differential equation.

When the group is the diffeomorphisms group this corresponds to a probabilistic description of the Navier-Stokes equations.

Contents

1. Introduction

The study of stochastic Lagrangian variational principles has been motivated initially by quantum mechanics, especially Feynman's path integral approach of this theory and also by optimal control problems. We refer to [16] and [7] for the first perspective. The second one can be found in [12] but mainly in the early and groundbreaking work of J. M. Bismut, who introduced also the concept of backward stochastic differential equations (c.f. for example [4]).

More recently these stochastic methods and ideas have been re-introduced in the context of fluid dynamics: in [8] the critical stochastic flows for the kinetic energy on the volume-preserving diffeomorphisms group of the torus were described as those whose drift satisfies the Navier-Stokes equations. The torus can be replaced by any (reasonable) Riemannian manifold, as was shown in [2]. This result generalizes to the viscous case of Arnold's characterization of Euler's equation for ideal fluids as geodesics on the group of volume-preserving diffeomorphisms ([3]).

Actually, the same kind of stochastic variational principles can be derived on general Lie groups: this is the content of reference [1]. The result is a stochastic version of the classical approach in the theory of Geometric Mechanics ([15]).

Originated in Bismut's approach to mechanics ([5] a different type of stochastic generalization or, more precisely, a random perturbation of geometric mechanics has been developed in [6].

Many examples of deterministic Euler-Arnold geodesics, namely geodesics on a Lie group equipped with an invariant metric, have been studied with this point of view. They include not only Euler but also many interesting equations such as Camassa-Holm or Korteweg-de Vries, formulated in infinite-dimensional Lie groups with suitable metrics. For a survey of this subject we refer to [13].

In this work we first recall the formulation of the stochastic variational principle on Lie groups derived in [1]. Then we give a relation between stochastic geodesic equations and a backward stochastic differential equation. This relation turns out to be very simple for the case of the volume preserving diffeomorphisms group. For the subject of backward and forward-backward stochastic equations we refer to [14].

2. The variational principles

Let G denote a Lie group, e its identity element and $\mathscr{G}$ the corresponding Lie algebra which can be identified with T_eG. Assume that G is endowed with a right-invariant Riemannian metric.

The Lagrangian corresponding to the kinetic energy is

$$\mathscr{L}(u_g) = \frac{1}{2} < T_g R_{g^{-1}} u_g, T_g R_{g^{-1}} u_g >, \quad \forall u_g \in T_g G,$$

for $g \in G$, where R_g is the right translation by the element g on G and $T_a R_g : T_a G \to T_{ag} G$ is its differential. This Lagrangian is right-invariant, but one can work as well with left-invariant Lagrangians.

For all the Lagrangian paths $g(.) \in C^1([0,T];G)$, using Euler-Poincaré reduction (c.f.[15]), the critical flows $g(t)$ for the action functional $\int_0^T \mathscr{L}(\frac{d}{dt}g(t))dt$ (the geodesics for the corresponding metric) are solutions of the following equations:

$$\begin{cases} \frac{d}{dt}g(t) = T_e R_{g(t)} u(t) \\ \frac{d}{dt}u(t) = -ad^*_{u(t)} u(t) \end{cases}$$

where ad^* is the dual of the ad operator with respect to the metric, i.e. $< ad^*_u v, w >=< v, ad_u w >$.

We describe the extension of the notion of geodesics to the stochastic case.

For this we fix a right-invariant Riemannian connection (not necessarily the Levi-Civita connection) on the Lie group, that we denote by ∇ and take a sequence of vectors H_k in T_eG, not necessarily a basis of the Lie algebra.

The paths $g(t)$ will now be stochastic processes, more precisely semi-martingales of the form

$$dg^u(t) = T_e R_{g^u(t)} (\sum_k H_k o dW_t^k - \frac{1}{2}\nabla_{H_k} H_k dt + u(t)dt), \ g^u(0) = e, \ t \in [0,T] \tag{2.1}$$

Here W_t^k are real valued independent Brownian motions, o denotes Stratonovich integration, and $u(.) \in C^1([0,T];T_eG)$ a non random map.

In terms of Itô integration the process $g(t)$ can be written as

$$dg^u(t) = T_e R_{g^u(t)} (\sum_k H_k dW_t^k + u(t)dt) \tag{2.2}$$

When H_k is an orthonormal basis of T_eG, ∇ is the Levi-Civita connection, $u(t) = 0$ for all t and $\nabla_{H_k} H_k = 0$ for all k then $g(.) \equiv g^0(.)$ is the Brownian motion associated to the Laplace-Beltrami operator.

Our results hold for the finite dimensional Lie group, as well as some infinite dimensional groups, for example, the group of diffeomorphism on Riemannian manifolds.

Remark that if $H_k = 0$ for each k, we are back to deterministic paths with velocity given by the vector field $u(t)$.

Since the paths are now not differentiable with the time parameter, we must replace their derivative in time by a "mean" generalized derivative. For a continuous G-valued semi-martingale $\xi(.)$ with $\xi(0) = x$ we consider the T_xG-valued semi-martingale $\eta(t) = \int_0^t \mathscr{T}_{0\leftarrow s} od\xi(s)$, where $\mathscr{T}_{t\leftarrow s} : T_{\xi(s)}G \rightarrow T_{\xi(t)}G$ is the (stochastic) parallel transport along $\xi(.)$ associated to the connection ∇ and o denotes Stratonovich integration. Although the path $\xi(.)$ is not differentiable, the parallel transport is well defined, as was shown by Itô. Then we consider the generalized derivative for the T_xM valued semi-martingale $\eta(.)$

$$D_t\eta(t) = lim_{\epsilon\rightarrow 0}\frac{1}{\epsilon}E\Big[(\eta(t+\epsilon) - \eta(t)\Big|\mathscr{F}_t\Big] \tag{2.3}$$

where $\mathscr{F}_t$ denotes the natural filtration generated by $\xi(.)$, i.e. $\mathscr{F}_t = \sigma\Big\{\xi(s), \quad s \in [0,t]\Big\}$. Finally we define,

$$D_t^{\nabla}\xi(t) := \mathscr{T}_{t\leftarrow 0} D_t\eta(t) \tag{2.4}$$

This notion depends, of course, on the chosen connection ∇.

Define the action functional as

$$J^{\nabla,< >} := E\int_0^T \mathscr{L}(D_t^{\nabla}\xi(t))dt, \tag{2.5}$$

in fact, it is associated with the following (stochastic) kinetic energy for the semi-martingale $\xi(.)$

$$\mathscr{L}(D^{\nabla}\xi(t)) = \frac{1}{2} < T_{\xi(t)}R_{\xi(t)^{-1}}D_t^{\nabla}\xi(t), T_{\xi(t)}R_{\xi(t)^{-1}}D_t^{\nabla}\xi(t) > \tag{2.6}$$

Remark that the action functional $J^{\nabla,< >}$ depends on the choice of the metric as well as the connection, which can be chosen independently.

We say that a G-valued semimartingale $\xi(.)$ is critical for the action functional above if for every $v(.) \in C^1([0,T]; T_eG)$ with $v(0) = v(T) = 0$ we have

$$\frac{d}{d\epsilon}|_{\epsilon=0} J^{\nabla,< >}(\xi(.)e_{\epsilon,v}(.)) = 0$$

where $e_{\epsilon,v}(.)$ is the flow in G generated by $\epsilon v(.)$,

$$\begin{cases} \frac{d}{dt}e_{\epsilon,v}(t) = \epsilon T_e R_{e_{\epsilon,v}(t)}\frac{d}{dt}v(t) \\ e_{\epsilon,v}(0) = e \end{cases} \tag{2.7}$$

By generalizing Euler-Poincaré reduction methods to the stochastic case we proved the following

Theorem ([1]). *A G-valued semi-martingale of the form (2.1) is critical for $J^{\nabla,< >}$ if and only if the vector field $u(.)$ satisfies the following (reduced) equation:*

$$\frac{d}{dt}u(t) = -ad^*_{\tilde{u}(t)}u(t) - K(u(t)) \tag{2.8}$$

where

$$\tilde{u}(t) = u(t) - \frac{1}{2}\sum_k \nabla_{H_k} H_k \tag{2.9}$$

and $K : T_eG \to T_eG$ *is defined by*

$$< K(u), v >= - < u, \frac{1}{2}\sum_k (\nabla_{ad_v H_k} H_k + \nabla_{H_k}(ad_v H_k)) >, \ \forall u, v \in T_eG \tag{2.10}$$

Analogous results (modulo changes of signs in the equations) hold for left-invariant metrics and left-invariant connections.

An important observation is that, in the right-invariant case, if ∇ is the Levi-Civita connection with respect to the metric and we assume that $\nabla_{H_k} H_k = 0$ for every k, then the operator K reduces to

$$K(u) = -\frac{1}{2}\sum_k (\nabla_{H_k}\nabla_{H_k} u + R(u, H_k)H_k), \tag{2.11}$$

where R is the Riemannian curvature tensor. In particular, if H_k is an orthonormal basis of T_eG, K coincides with the minus of de Rham-Hodge Laplacian. This is important for applications, especially to derive Navier-Stokes equations (cf. [8] and [1]).

We shall call a *stochastic geodesic* on a Lie group endowed with a right-invariant metric and connection a G-valued semi-martingale $g^u(.)$ of the form

$$dg^u(t) = T_e R_{g^u(t)}(\sum_k H_k o dW_t^k - \frac{1}{2}\nabla_{H_k} H_k dt + u(t)dt), \ g^u(0) = e$$

which is critical for an action functional $J^{\nabla, < >}$ as defined above.

3. Relation with backward differential equations

In this section we give a characterization of the stochastic geodesics as solutions of a stochastic backward differential equation.

We shall consider the case referred above, namely ∇ is the Levi-Civita connection with respect to the metric, H_k is an orthonormal basis of T_eG, $\nabla_{H_k} H_k = 0$ for every k. Then the operator K is the de Rham-Hodge Laplacian. In particular K coincides with its dual.

Denote by $\bar{u}(t, .) = -u(T - t, .)$. We have,

$$\frac{d}{dt}\bar{u}(t) = -ad^*_{\bar{u}(t)}\bar{u}(t) + K(\bar{u}(t)) \tag{3.1}$$

Consider then the process $g(.)$ defined for $t \in [0, T]$ by

$$dg(t) = T_e R_{g(t)}(\sum_k H_k o dW_t^k), \ g(0) = e \tag{3.2}$$

where, since we are assuming $\nabla_{H_k} H_k = 0$ for every k, the Stratonovich integral coincides with the Itô one.

Let $X(t) := T_eR_{g(t)}\bar{u}(t)$, note that $X(t) \in T_{g(t)}G$. We denote the stochastic covariant derivative along $g(.)$ associated with the connection ∇ by $\mathscr{D}_t$, then we have,

(3.3)

$$\mathscr{D}_t X(t) = T_eR_{g(t)}\Big(\sum_k \nabla_{H_k}\bar{u}(t)\circ dW_t^k + \frac{d}{dt}\bar{u}(t)dt\Big)$$
$$= T_eR_{g(t)}\Big(\sum_k \big(\nabla_{H_k}\bar{u}(t)dW_t^k + \frac{1}{2}\nabla_{H_k}\nabla_{H_k}\bar{u}(t)dt\big) - \big(ad^*_{\bar{u}(t)}\bar{u}(t) - K(\bar{u}(t))\big)dt\Big),$$

where in the second step we use the equation (3.1) for $\bar{u}(t)$ and decompose the Stratonovich integral $\nabla_{H_k}\bar{u}(t)\circ dW_t^k$ into the associated Itô integral the contraction terms by the connection ∇.

Let $Z_k(t) := T_eR_{g(t)}\nabla_{H_k}\bar{u}(t)$, and a linear map Γ be defined by,

$$\Gamma(u,v) := T_eR_g\Big(\nabla_{T_gR_{g^{-1}}u}T_gR_{g^{-1}}v\Big), \quad \forall u,v \in T_gG,$$

We can write

$$T_eR_{g(t)}\big(\nabla_{H_k}\nabla_{H_k}\bar{u}(t)\big) = \Gamma(\tilde{H}_k(g(t)), Z_k(t)),$$

where $\tilde{H}_k$ denote the right invariant vector fields associated with H_k.

$$T_eR_{g(t)}(ad^*_{\bar{u}(t)}\bar{u}(t)) = \sum_k \langle ad^*_{\bar{u}(t)}\bar{u}(t), H_k\rangle \tilde{H}_k(g(t))$$
$$= \sum_k \langle \bar{u}(t), ad_{\bar{u}(t)}H_k\rangle \tilde{H}_k(g(t)) = \sum_k \langle \bar{u}(t), \nabla_{H_k}\bar{u}(t) - \nabla_{\bar{u}(t)}H_k\rangle \tilde{H}_k(g(t))$$
$$= \sum_k \Big(\langle X(t), Z_k(t) - \Gamma(X(t), \tilde{H}_k(g(t)))\rangle\Big)\tilde{H}_k(g(t)).$$

In the same way, by (2.11), we get,

$$T_eR_{g(t)}K(\bar{u}(t)) = -\frac{1}{2}\sum_k \Gamma(\tilde{H}_k(g(t)), Z_k(t)) - \mathrm{Ric}(X(t)).$$

Combing all of the above equalities into (3.3), we have the following result,

Theorem 1. *If the non-random vector field $\bar{u} \in C^1([0,T];T_eG)$ solves the equation (3.1) with final condition $u(T) = u_T$, then $\Big(g(t), X(t) := T_eR_{g(t)}\bar{u}(t), \{Z_k(t) := T_eR_{g(t)}\nabla_{H_k}\bar{u}(t)\}\Big)$ solves the following forward-backward stochastic equation,*

$$\left\{\begin{array}{l} dg(t) = \sum_k T_eR_{g(t)}H_k \circ dW_t^k \\ \mathscr{D}_t X(t) = \sum_k Z_k(t)dW_t^k - \frac{1}{2}Ric(X(t))dt \\ -\sum_k \Big(\big(\langle X(t), Z_k(t) + \Gamma(X(t), \tilde{H}_k(g(t)))\rangle\Big)\tilde{H}_k(g(t))dt \\ g(0) = e, \quad X(T) = T_eR_{g(T)}u_T. \end{array}\right. \tag{3.4}$$

We notice that the appearence of the *Ricci* term is due to the choice of covariant derivative.

Hence we give an existence theorem for equation (3.4) with some special terminal condition, i.e. $T_{g(T)}R_{g^{-1}(T)}X(T)$ is non-random. But to the authors knowledge, existence and uniqueness of solutions for such backward and forward-backward equations with general random terminal condition is still unknown. In fact, due to the special terminal condition here, we do not need to use the manifold-valued martingale representation theorem to get the expression of $Z_k(t)$, and we can get $Z_k(t)$ directly by Itô formula. But for general terminal conditions, it seems that we still have to use the manifold-valued martingale representation theorem, which is also the main contribution in ([11]), where the backward SDE on some particular Lie group is studied.

Moreover, under some assumption of the solution of (3.4), by reversing the computation above, we can derive the converse of Theorem 1, namely

Theorem 2. *If $(g(t), X(t), \{Z_k(t)\})$ is a solution of (3.4) and $\bar{u}(t) := T_{g(t)} R_{g(t)^{-1}} X(t)$ is non-random and differentiable with respect to the time parameter, then $\bar{u}(t)$ solves the equation (3.1) with terminal value $\bar{u}(t) = u_T$.*

4. The Navier-Stokes equations

When G is the (right-invariant) diffeomorphisms group of volume preserving maps on a manifold, equation (2.8) is the Navier-Stokes equation (c.f. [1] and [8]). If the manifold is flat, these equations are

$$\frac{d}{dt}u(t) = -(u.\nabla)u(t) + \frac{1}{2}\Delta u(t) + \nabla p(t), \ div\ u(t) = 0 \tag{4.1}$$

considered in the weak (L^2) distributional sense.

In this case, we can get a simple expression of equation (3.4) by direct computation. For the diffeomorphism group of the three-dimensional torus, we choose the vector fields $\{H_k\}_{k=1}^3$ to be the collection of vector fields

$$H_1(\theta) = (1,0,0), \ H_2(\theta) = (0,1,0), \ H_3(\theta) = (0,0,1)$$

We have here that $g(t)$ is a standard Brownian motion, $X(t) = \bar{u}(t, g_t)$, and by Ito formula,

$$dX(t) = \sum_k \partial_k \bar{u}(t, g(t)) dW_t^k + [\partial_t \bar{u} + \frac{1}{2}\Delta \bar{u}](t, g(t))dt,$$

where ∂_k denotes the derivative with the variable θ_k. Therefore, writing $Z^k(t) = \partial_k \bar{u}(t, g(t))$, we have

$$(\bar{u}.\nabla)\bar{u}(t, g(t)) = \sum_k \big(X(t), H_k(g(t))\big)_3 Z_k(t),$$

where $(,)_3$ denotes the inner product in the three-dimensional torus. Hence by (4.1), we get the following equation,

$$\begin{cases} g(t,\theta) = W_t \\ X(t) = \sum_k Z_k dW_t^k - \sum_k \big(X(t), Z_k(t)\big)_3 dt - \nabla p(g(t))dt \\ g(0,\theta) = \theta, \quad X(T) = u_T(g(T,\theta)) \end{cases}$$

This Navier-Stokes equation on the torus has been studied from the point of view of forward-backward and backward stochastic equations in [9] and [10].

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