ORNSTEIN-UHLENBECK SEMIGROUPS ON RIEMANNIAN PATH SPACES

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1. Introduction

The Ornstein-Uhlenbeck semigroup on the classical Wiener space X (and actually, on much more general Gaussian spaces) can be defined by the following formula:

$$(T_t f)(x) = \int_X f(e^{-t}s + \sqrt{1-e^{-2t}}y)d\mu(y)$$

where μ denotes the Wiener measure. This corresponds to an extension to finite dimensions of the Mehler's formula. There are other ways to introduce this semigroup, notably through its action on the finite dimensional Wiener chaos or by associating the semigroup to the generator, the so-called Ornstein-Uhlenbeck operator, and constructing the correspondent diffusion. For this last approach one can proceed at least in two different ways: using Dirichlet form theory ([?][?][?]) or defining a two-parameter diffusion (i.e., a stochastic process with values on the Wiener space) as a perturbation of a two-parameter Brownian motion ([?]). The Wiener measure is invariant for the Ornstein-Uhlenbeck semigroup, which is a positive self-adjoint contraction operator on the spaces $L^p(X,\mu)$ for any $p \geq 1$. Nelson's hypercontractivity also holds true, namely

$$||T_t f||_{L^{q_t}} \leq ||f||_{L^p}$$

with $q_t = 2^{2t}(p-1)+1$, $p>1$. This semigroup plays an important rôle in Malliavin calculus ([19]). And it corresponds to the number operator, a fundamental object in Quantum Mechanics and in Quantum Field Theory.

How can one define such an object on the path space of a Riemannian manifold? The Mehler's formula or the chaos decomposition approaches are not available in the nonlinear setting. The first construction of the Ornstein-Uhlenbeck semigroup on the curved path space was done via Dirichlet form theory by Driver and Röckner ([11]). The corresponding Ornstein-Uhlenbeck process was defined by Kazumi by solving the associated martingale problem in [15], where an expression for the generator was also derived. The Norris "twisted sheet" ([20]) correspond to the two-parameter stochastic process approach. In fact this last construction gives exactly the Driver-Röckner process only when the Ricci curvature of the underlying manifold is zero.

In [7] a systematic approximation of the geometrical objects on the path space by finite dimensional ones was defined and studied. The (Driver-Röckner) Ornstein-Uhlenbeck semigroup, in particular, was approximated by semigroups defined on finite dimensional manifolds and convergence in the weak sense was proved. In this work we show the strong (L^2) convergence of these objects.

2. The Riemannian path space

Let M be a d-dimensional compact Riemannian manifold where we consider Levi-Civita connection and $O(M)$ be the corresponding bundle of orthogonal frames, namely

$$O(M) := \{(m,r) : m \in M \text{ and } r : \mathbb{R}^d \to T_m(M) \text{ is a Euclidean isometry}\}$$

The horizontal Laplacian on $O(M)$ is defined by $\Delta_{O(M)} = \sum_{k=1}^d A_k^2$, where A_k are the canonical horizontal vector fields. It satisfies the relation $\Delta_{O(M)}(f \circ \pi) = (\Delta_M f) \circ \pi$, where Δ_M denotes the Laplace-Beltrami operator and $\pi : O(M) \to M$ denotes the canonical projection. The stochastic (Stratonovich) differential equation

$$dr_x = \sum_{k=1}^{d} A_k(r_x) \circ dx$$

with initial condition $r_x(0) = r_0$ defines a flow of diffeomorphisms on $O(M)$, the lift of the Brownian motion associated with Δ_M (cf.[19]).

We consider the path space

$$P_{m_0}(M) = \{\text{continuous } p : [0,1] \to M \text{ with } p(0) = m_0\}$$

for a fixed $m_0 \in M$. This space is endowed with the Wiener measure μ (the law of the Brownian motion on M) and with its natural past filtration. The path space of $\mathbb{R}^d$, the classical Wiener space, will be simply denoted by X. The Itô map $I : X \to P_{m_0}(M)$, namely

$$I(x)(\tau) = \pi(r_x(\tau))$$

was defined in [19] as a map which is a.s. bijective and provides an isomorphism between the corresponding Wiener measures.

The path space geometry constructed in [4] is a Cartan-type moving frame geometry based on the parallel transport along Brownian paths, which was constructed by Itô as an extension of the parallel transport over smooth trajectories. The Itô parallel transport along a path $p \in P_{m_0}(M)$ is defined by

$$t^p_{\tau \leftarrow \tau_0} = r_p(\tau) r_p(\tau_0)^{-1}$$

where r_p is the horizontal lift of p.

For a cylindrical functional on the path space $F(p) = f(p(\tau_1), ...p(\tau_m))$, with $0 < \tau_1 < \ ... \ \tau_m \leq 1$ and f a smooth function on M^m, the derivation operators in the sense of Malliavin calculus are defined by

$$D_\tau F = \sum_{k=1}^{m} 1_{\tau < \tau_k} t^p_{\tau \leftarrow \tau_k}(\partial_k f)$$

These operators are closable in L^q, $q > 1$, with respect to the norm

$$\|DF\|(p) = \Big(\sum_{\alpha=1}^{d} \int_0^1 (D_{\tau,\alpha} F)^2 d\tau \Big)^{\frac{1}{2}}$$

where $D_{\tau,\alpha} F = (t^p_{0 \leftarrow \tau} D_\tau F | \varepsilon_\alpha)$ and $\{\varepsilon_\alpha\}$ denotes the canonical basis in $\mathbb{R}^d$.

If we consider maps $Z_p(\tau) \in T_{p(\tau)}(M)$ such that $z(\tau) = t^p_{0 \leftarrow \tau}(Z(\tau))$ belongs to the Cameron-Martin subspace H of the Wiener space, then we can also define derivation along the "vector field" Z by

$$D_Z F = \int_0^1 D_{\tau,\alpha} F \dot{z}^\alpha(\tau) d\tau$$

Differential calculus on the path space of a Riemannian manifold can be "transported" to differential calculus on the Wiener space through the Itô map. The price to pay is that Cameron-Martin tangent space is not preserved. This phenomena leads to a necessary extension of the tangent space and the definition of the so-called tangent processes (cf. [10],[4]). The corresponding result is the following:

Theorem 2.1. *(Driver [10], Fang-Malliavin [12] and Cruzeiro-Malliavin [4]) A scalar valued functional F defined on the path space $P_{m_0}(M)$ is differentiable along an adapted vector field Z if and only if $F \circ I$ is differentiable on the Wiener space along a semimartingale ξ given by*

$$\begin{cases} d\xi(\tau) = \dot{z} d\tau + \rho \circ dx(\tau) \\ d\rho(\tau) = \Omega(\circ dx(\tau), z) \end{cases}$$

where Ω denotes the curvature tensor of the manifold read on the frame bundle ($\Omega_r(u,v) = r^{-1}\Omega^M(ru, rv)$) , $z(\tau) = t^p_{0\leftarrow\tau}(Z(\tau))$; furthermore we have the intertwinning formula

$$(D_Z F) \circ I = D_\xi (F \circ I).$$

One consequence of the intertwinning formula is Bismut's integration by parts formula (cf. [4]), namely

$$E^\nu(D_Z F) = E^\nu\Big((F \circ I) \int_0^1 \Big[\dot{z} + \frac{1}{2}\mathrm{Ricc}(z)\Big] dx\Big)$$

which holds for adapted Cameron-Martin vector fields Z and functionals $F \in L^2$ whose derivative is also in L^2. We shall write

$$\delta(z) = \int_0^1 \Big[\dot{z} + \frac{1}{2}\mathrm{Ricc}(z)\Big] dx.$$

In [11] the Ornstein-Uhlenbeck Dirichlet form

$$\mathcal{E}(F,G) = E\Big(\sum_\alpha \int_0^1 D_{\tau,\alpha} F D_{\tau,\alpha} G d\tau\Big)$$

was defined and studied. In particular a process and a corresponding semigroup can be associated to it. The generator, computed on cylindrical functions of the form $F(p) = f(p(\tau_1), ..., p(\tau_m)), f \in C^\infty(M^m)$ has the form (cf.[15] and also [5]):

$$LF = \sum_\alpha \sum_{i,j=1}^m s_i \wedge s_j D_{z_{i,\alpha}} D_{z_{j,\alpha}} f - \sum_{\alpha,i,j} s_i \wedge s_j \delta(z_{i,\alpha}) D_{z_{j,\alpha}} f$$

where

$$\dot{z}_{i,\alpha} := \Big(\frac{1}{\triangle_{i-1} s} 1_{[s_{i-1}, s_i)} - \frac{1}{\triangle_i s} 1_{[s_i, s_{i+1})}\Big)\varepsilon^\alpha, i = 1, \cdots, n-1$$

$$\dot{z}_{n,\alpha} := \frac{\varepsilon^\alpha}{1 - s_{n-1}} 1_{[s_{n-1}, 1]}$$

This operator coincides with the Norris Ornstein-Uhlenbeck operator when the Ricci curvature of the manifold M is zero. We shall denote by T_t the semigroup associated with L.

3. Finite dimensional approximations

Following [7] we consider, for a finite partition of the time interval $\mathcal{P} = \{0 = s_0 < s_1 < ... < s_n = 1\}$, the space of piecewise geodesics paths which change directions only at the partition points, namely:

$$H^n(M) = \{\sigma \in P_{m_0}(M) \cap C^2(I \setminus \mathcal{P}) : \nabla \dot{\sigma}(s)/ds = 0 \text{ for } s \notin \mathcal{P}\}.$$

The development map I_n is a diffeomorphism between the spaces $H^n(\mathbb{R}^d)$(simplified as H^n) and $H^n(M)$ that associates to a path $x \in H^n$ the unique $\sigma = I_n(x) \in H^n(M)$ verifying

$$\dot{\sigma}(s) = t^{\sigma}_{s\leftarrow 0}\dot{x}(s), \quad \sigma(0) = m_0,$$

where $t^{\sigma}_{\cdot\leftarrow 0}$ denotes the parallel transport along σ.

The tangent space inherited from the tangent space of the Gaussian vector space H^n through the map I_n consists of maps of the form $Z(s) := t^{\sigma}_{s\leftarrow 0}(z(s))$ such that

$$\ddot{z}(s) = \Omega_{r(s)}(\dot{x}(s), z(s))\dot{x}(s) \text{ on } I \setminus P$$

with $\sigma \in H^{(n)}(M)$, $x = I_n^{-1}(\sigma)$, r the horizontal lift of σ and $z \in H^n$ (cf. [1]).

We endow $H^n(M)$ with a Gaussian measure ν_n such that $\nu_n \circ I_n = \mu_n$, where $\mu_n = \mu \circ (\pi_n^X)^{-1}$ is the finite dimensional Gaussian measure on H^n.

For $\varepsilon \in [0, 1]$, we consider the following spaces:

$$M^n_\varepsilon := \{v \in M^n : d(v_i, v_{i+1}) < \zeta_\varepsilon, \text{ for } i = 0, 1, \cdots, n-1\},$$

$$H^n_\varepsilon(M) := \{\sigma \in H^n(M) : \int_{s_i}^{s_{i+1}} |\dot{\sigma}(s)| ds < \zeta_\varepsilon, \text{ for } i = 0, 1, \cdots, n-1\},$$

$$H^n_\varepsilon := \{z \in H^n : \|z(s_{i+1} - z(s_i)\| < \zeta_\varepsilon, \text{ for } i = 0, 1, \cdots, n-1\},$$

where $\zeta_\varepsilon := \varepsilon(\rho \wedge 4/K_\Omega)$, ρ is the radius of injectivity of M, $K_\Omega = \sup_{r\in O(M)} \|\Omega_r\| < \infty$.

M^n_ε is an open subset of M^n and therefore is a differentiable manifold. We associate to $v \in M^n_\varepsilon$ the piecewise geodesic curve σ_v defined by linking the points v_i, v_{i+1} by the minimizing geodesic. For $v \in M^n_\varepsilon$, we consider the map

$$[\Theta^n_v]^{-1} : H^n \mapsto T_v(M^n_\varepsilon)$$

given by

$$Z(s_i) = t^{\sigma_v}_{s_i\leftarrow 0}(z(s_i)) \in T_{v_i}(M), i = 1, \cdots, n$$

where $z \in H^n$. Then Θ^n determines a parallelism on M^n_ε.

A Riemannian metric is defined on M^n_ε by the condition that Θ^n_v is an isometry of $T_v(M^n_\varepsilon)$ onto H^n.

Under the maps π^W_n, I_n, where π^W_n denotes the projection from $P_{m_0}(M)$ to H^n, namely

$$\pi^W_n(p) := (p(s_1), \cdots, p(s_n))$$

we can identify the spaces M^n_ε, $H^n_\varepsilon(M)$ and H^n_ε and we have

$$d(v_i, v_{i+1}) = \int_{s_i}^{s_{i+1}} |\dot{\sigma}_v(s)| ds = \|x_v(s_{i+1}) - x_v(s_i)\|$$

where $\dot{x}_v(s) = t^{\sigma_v}_{0\leftarrow s}\dot{\sigma}_v(s)$.

For a function $f \in C^\infty(M^n_\varepsilon)$ we define the derivatives

$$(D^n_{s,\lambda} f)(v) := \sum_{k=1}^{n} 1_{s<s_k} \langle t^{\sigma_v}_{0\leftarrow s_k} \partial_k f, \varepsilon_\lambda \rangle_{m_0},$$

and, if Y is a smooth vector field in $T(M_\varepsilon^n)$,

$$D_Y^n f := \int_0^1 D_{s,\lambda}^n f \cdot \dot{y}^\lambda(s) ds = \sum_{k=1}^n \langle \partial_k f, Y(s_k)\rangle_{v_k} = Y f.$$

On M_ε^n we consider the measure $\nu_{n,\varepsilon} := \hat{\varphi}_n^2 d\nu_n$ and on H_ε^n the measure $\mu_{n,\varepsilon} := \varphi_n^2 d\mu_n$, where $\hat{\varphi}_n(v) = \varphi_n(I_n^{-1}(\sigma_v))$ and $\varphi_n \geq 0$ is a cutoff function on H_ε^n such that

$$\begin{cases} \varphi_n(b) = 1, b \in H_{\varepsilon'}^n, \\ \varphi_n(b) = 0, b \notin H_\varepsilon^n, \\ \sup_k \|1 - \varphi_n \circ \pi_n^X\|_{D_2^p(X)} \leq c \exp\{-cn\}, p > 1, \end{cases}$$

$\varepsilon' < \varepsilon$ being fixed, c a positive constant.

For a function f on M_ε^n we define its lift to path space as follows:

$$\tilde{f}(p) = \varphi_n(\pi_n^X \circ I^{-1}(p)) \cdot f(I_n \circ \pi_n^X \circ I^{-1}(p)).$$

Finely, for f defined on path space, its projection to M_ε^n is given by

$$f_n(\sigma) := E^\mu(f \circ I(x) | \pi_n^X(x) = I_n^{-1}(\sigma)).$$

Let

$$\mathcal{E}^n(f, g) := \int_{M_\varepsilon^n} \Big(\sum_\alpha \int_0^1 D_{\tau,\alpha}^n f D_{\tau,\alpha}^n g d\tau \Big) d\nu_{n,\varepsilon}$$

defined for $f, g \in C_r^\infty(M_\varepsilon^n)$, be a Dirichlet form on the Hilbert space $L^2(M_\varepsilon^n, d\nu_{n,\varepsilon})$. It is a regular Dirichlet form with local property in the sense of Fukushima. Its generator is given by:

$$L^n f = \sum_{\alpha,i,j} s_i \wedge s_j D_{z_{i,\alpha}}^n D_{z_{j,\alpha}} f - \sum_{\alpha,i,j} s_i \wedge s_j \delta^{(n)}(z_{i,\alpha}) D_{z_{j,\alpha}}^n f$$

where

$$\dot{z}_{i,\alpha} := \Big(\frac{1}{\triangle_{i-1} s} 1_{[s_{i-1}, s_i)} - \frac{1}{\triangle_i s} 1_{[s_i, s_{i+1})} \Big) \varepsilon^\alpha, i = 1, \cdots, n-1$$

$$\dot{z}_{n,\alpha} := \frac{\varepsilon^\alpha}{1 - s_{n-1}} 1_{[s_{n-1}, 1]}$$

$\delta^{(n)}$ denoting the divergence (i.e., the L^2 dual of the derivative) with respect to the measure $d\nu_{n,\varepsilon}$.

In [7] we have proved the following results:

Theorem 3.1. *If f is a cylindrical function on the path space, then*

$$\tilde{L}^n f \to L f \ \text{in} \ L^2(P_{m_0}(M)).$$

Now we denote the resolvents associated to the Dirichlet forms $(\mathcal{E}, D(\mathcal{E}))$ and $(\mathcal{E}^n, D(\mathcal{E}^n))$, respectively, by $(G_\alpha)_{\alpha>0}$ and $(G_\alpha)_{\alpha>0}^n$.

Theorem 3.2. *Let $g_n \in L^2(M_\varepsilon^n, d\nu_{n,\varepsilon})$ be a sequence of functions such that $\tilde{g}_n$ converge weakly to $g \in L^2(P_{m_0}(M))$. Then, for any $\alpha > 0$, we have*

$$\tilde{G}_\alpha^n g_n \to G_\alpha g \ \text{weakly in} \ L^2(P_{m_0}(M)).$$

Concerning the convergence of the semigroups, we have:

Theorem 3.3. *For any $g \in C_b(P_{m_0}(M))$ and any $t > 0$ the following convergence holds*

$$\tilde{T}_t^n g_n \to T_t g \ \text{weakly in} \ L^2(P_{m_0}(M))$$

g_n denoting the projection of g.

4. Strong convergence results

A classical Trotter-Kato theorem ([22]) states that convergence of the resolvents is equivalent to convergence of the semigroups. Moreover, for Feller semigroups, this convergence is equivalent to the one of the generators (corresponding to Feller process) (cf.[14, p.331, Theorem 17.25]). However, in these results, all the objects are defined on the same space. Sometimes, and this is the case in our finite dimensional approximation scheme for the path space, the operators are defined on different spaces. We shall follow the method used by Röckner and Zhang to prove the convergence of semigroups starting from the generators' convergence. Our result will be stated in a general frame.

Let $\{H_n, \|\cdot\|_n, n = 1, \cdots, +\infty\}$ be a sequence of separable Hilbert spaces, $\{(X_n, \mu_n), n = 1, \cdots, +\infty\}$ a sequence of measure spaces. We now consider the separable Hilbert spaces of $L^2(X_n, \mu_n; H_n) =: L^2_n$. Let $(L^{(n)}, \mathcal{D}(L^{(n)}))$ be positive self-adjoint operators on L^2_n. When $n = +\infty$, we shall omit it for the simplicity of notation. We make the following assumptions:

(i) $L^2_n \hookrightarrow L^2$, the linear embedding map is given by i_n, which satisfies that for each $f \in L^2(X_n, \mu_n; H_n)$

$$\|i_n f\|_{L^2} = \|f\|_{L^2_n};$$

(ii) there is a projection $j_n : L^2 \mapsto L^2_n$ such that

$$\lim_{n\to\infty} \|i_n(j_n f) - f\|_{L^2} = 0;$$

(iii) there is a dense subset $\mathcal{C}$(called core of L) of $\mathcal{D}(L)$ such that $j_n\mathcal{C} \subset \mathcal{D}(L^{(n)})$, and for every $f \in \mathcal{C}$

$$\lim_{n\to\infty} \|i_n L^{(n)}(j_n f) - Lf\|_{L^2} = 0,$$

Let $\{G^{(n)}_\lambda\}_{\lambda>0}$ be the resolvent associated with $L^{(n)}$, $\{T^{(n)}_t\}_{t>0}$ the semigroup. That is

$$G^{(n)}_\lambda := (\lambda - L^{(n)})^{-1}, \quad T^{(n)}_t = e^{tL^{(n)}}; n = 1, \cdots, +\infty.$$

Proposition 4.1. *Let $g_n \in L^2_n$ be such that $i_n g_n \to g$ in L^2. Then for any $\lambda > 0$ and $m \in \mathbb{N}$, we have*

$$i_n(G^{(n)}_\lambda)^m g_n \to (G_\lambda)^m g \text{ in } L^2.$$

Proof. It suffices to prove this for $m = 1$. We take a family of functions $f_m \in \mathcal{C}$ such that

$$\|(\lambda - L)f_m - g\|_{L^2} \to 0.$$

Set $g'_m := (\lambda - L)f_m$ and $g_{n,m} := (\lambda - L^{(n)})(j_n f_m)$. For any $\varepsilon > 0$, let m be large enough such that

$$\|g'_m - g\|_{L^2} \le \lambda\varepsilon.$$

Then we have

$$\begin{aligned}
& \|i_n(G^{(n)}_\lambda g_n) - G_\lambda g\|_{L^2} \\
\le\ & \|i_n(G^{(n)}_\lambda g_n) - i_n(G^{(n)}_\lambda g_{n,m})\|_{L^2} + \|i_n(G^{(n)}_\lambda g_{n,m}) - G_\lambda g'_m\|_{L^2} + \|G_\lambda g'_m - G_\lambda g\|_{L^2} \\
=\ & \|G^{(n)}_\lambda(g_n - g_{n,m})\|_{L^2_n} + \|i_n(j_n f_m) - f_m\|_{L^2} + \|G_\lambda(g'_m - g)\|_{L^2} \\
\le\ & \frac{1}{\lambda}\|g_n - g_{n,m}\|_{L^2_n} + \|i_n(j_n f_m) - f_m\|_{L^2} + \frac{1}{\lambda}\|g'_m - g\|_{L^2} \\
\le\ & \frac{1}{\lambda}\Big(\|i_n g_n - g\|_{L^2} + \|g - g'_m\|_{L^2} + \|g'_m - i_n g_{n,m}\|_{L^2}\Big) + \|i_n(j_n f_m) - f_m\|_{L^2} + \varepsilon \\
\le\ & \frac{1}{\lambda}\Big(\|i_n g_n - g\|_{L^2} + \|i_n L^{(n)}(j_n f_m) - Lf_m\|_{L^2}\Big) + 2\|i_n(j_n f_m) - f_m\|_{L^2} + 2\varepsilon
\end{aligned}$$

Let n tend to infinity, by the assumption (iii), we obtain

$$\|i_n(G_\lambda^{(n)}g_n) - G_\lambda g\|_{L^2} \leq 2\varepsilon,$$

which gives the convergence in the lemma. $\square$

Define the bounded operators

$$L^{(n,\lambda)} := \lambda(\lambda G_\lambda^{(n)} - I)$$

and the associated semigroup $T_t^{(n,\lambda)} := e^{tL^{(n,\lambda)}}; n = 1\cdots, +\infty$.

Lemma 4.2. *For $g_n \in \mathcal{D}((L^{(n)})^2)$, assume that*

$$\sup_n \|(L^{(n)})^2 g_n\|_{L_n^2} < \infty,$$

then for fixed $T > 0$, $T_t^{(n,\lambda)}g_n$ converges uniformly (with respect to n and $t \in (0,T]$) to $T_t^{(n)}g_n$ in L_n^2 as $\beta \to \infty$.

Proof. Note that

$$\frac{\partial}{\partial s}(T_{t-s}^{(n,\lambda)}T_s^{(n)}g_n) = T_{t-s}^{(n,\lambda)}T_s^{(n)}(L^{(n)} - L^{(n,\lambda)})g_n,$$

Since $T_t^{(n,\lambda)}$ and $T_t^{(n)}$ are contractive, we have

$$\|T_t^{(n)}g_n - T_t^{(n,\lambda)}g_n\|_{L_n^2} \leq t\|(L^{(n)} - L^{(n,\lambda)})g_n\|_{L_n^2}.$$

On the other hand,

$$\begin{aligned}
L^{(n,\lambda)}g_n &= \lambda(\lambda G_\lambda^{(n)}g_n - g_n) \\
&= \lambda^2 \int_0^\infty e^{-\lambda t}(T_t^{(n)}g_n - g_n)dt \\
&= \lambda^2 \int_0^\infty e^{-\lambda t}dt\Big(\int_0^t T_s^{(n)}L^{(n)}g_n\ ds\Big) \\
&= \lambda \int_0^\infty T_s^{(n)}L^{(n)}g_n\ ds\Big(\int_s^\infty e^{-\lambda t}dt\Big) \\
&= \lambda \int_0^\infty e^{-\lambda s}T_s^{(n)}L^{(n)}g_n\ ds \\
&= \int_0^\infty e^{-s}T_{s/\lambda}^{(n)}L^{(n)}g_n\ ds,
\end{aligned}$$

Thus

$$\begin{aligned}
L^{(n,\lambda)}g_n - L^{(n)}g_n &= \int_0^\infty e^{-s}ds\Big(T_{s/\lambda}^{(n)}L^{(n)}g_n - L^{(n)}g_n\Big) \\
&= \int_0^\infty e^{-s}ds\Big(\int_0^{s/\lambda} T_t^{(n)}(L^{(n)})^2 g_n dt\Big).
\end{aligned}$$

Therefore

$$\|L^{(n,\lambda)}g_n - L^{(n)}g_n\|_{L_n^2} \leq \Big(\int_0^\infty e^{-s}s\ ds\Big)\|(L^{(n)})^2 g_n\|_{L_n^2}/\lambda \leq C/\lambda,$$

which yields the result. $\square$

Theorem 4.3. *Let $g_n \in L^2_n$ be such that $i_n g_n \to g$ in L^2. Then for any $T > 0$, we have*

$$\lim_{n\to\infty} \sup_{t\in(0,T]} \|i_n T_t^{(n)} g_n - T_t g\|_{L^2} = 0.$$

Proof. We first prove this for special g and g_n. For $h \in L^2$, set

$$g = (I-L)^{-2}h = (G_1)^2 h, \quad g_n = (I-L^{(n)})^{-2}(j_n h) = (G_1^{(n)})^2(j_n h).$$

Then from Proposition 4.1, we know

$$\lim_{n\to\infty} \|i_n g_n - g\|_{L^2} = 0.$$

Hence

$$\sup_n \|(L^{(n)})^2 g_n\|_{L^2_n} = \sup_n \|j_n h + g_n - 2G_1^{(n)}(j_n h)\|_{L^2_n} < \infty.$$

Since $T_t^{(n,\beta)}$ can be written as

$$T_t^{(n,\beta)} := e^{-t\lambda} \sum_{m=0}^{\infty} \frac{(t\lambda)^m}{m!} (\lambda G_\lambda^{(n)})^m \quad ; t, \lambda > 0, n \in \mathbb{N}.$$

by Lemma 4.2 and Hille-Yoshida approximation, changing the order of the limits we obtain

$$\begin{aligned}
\lim_{n\to\infty} i_n T_t^{(n)} g_n &= \lim_{n\to\infty} \lim_{\lambda\to\infty} i_n T_t^{(n,\lambda)} g_n \\
&= \lim_{\lambda\to\infty} \lim_{n\to\infty} e^{-t\lambda} \sum_{m=0}^{\infty} \frac{(t\lambda)^m}{m!} i_n (\lambda G_\lambda^{(n)})^m g_n \\
&= \lim_{\lambda\to\infty} e^{-t\lambda} \sum_{m=0}^{\infty} \frac{(t\lambda)^m}{m!} \lim_{n\to\infty} i_n (\lambda G_\lambda^{(n)})^m g_n \\
&= \lim_{\lambda\to\infty} e^{-t\lambda} \sum_{m=0}^{\infty} \frac{(t\lambda)^m}{m!} (\lambda G_\lambda)^m g \\
&= T_t g, \text{ in } L^2
\end{aligned}$$

where we used Lebesgue's dominated convergence theorem in the third step.

Next we prove this for arbitrary $g \in L^2$. Since $\mathcal{D}((I-L)^2)$ is dense in L^2, there exist $h_k \in \mathcal{D}((I-L)^2)$ such that

$$\lim_{k\to\infty} \|h_k - g\|_{L^2} = 0.$$

Set

$$g'_n = j_n g, \quad h_{n,k} = j_n h_k.$$

Then

$$\begin{aligned}
&\|i_n T_t^{(n)} g'_n - T_t g\|_{L^2} \\
\le\ & \|i_n T_t^{(n)} g'_n - i_n T_t^{(n)} h_{n,k}\|_{L^2} + \|i_n T_t^{(n)} h_{n,k} - T_t h_k\|_{L^2} + \|T_t h_k - T_t g\|_{L^2} \\
\le\ & \|h_{n,k} - g'_n\|_{L^2_n} + \|h_k - g\|_{L^2} + \|i_n T_t^{(n)} h_{n,k} - T_t h_k\|_{L^2}.
\end{aligned}$$

First letting $n \to \infty$, and then $k \to \infty$, we get

$$\lim_{n\to\infty} \|i_n T_t^{(n)} g'_n - T_t g\|_{L^2} = 0.$$

Lastly, for any $g_n \in L^2_n$, if

$$\lim_{n\to\infty} \|i_n g_n - g\|_{L^2} = 0.$$

Then

$$\begin{aligned}\lim_{n\to\infty}\|i_nT_t^{(n)}g_n - T_tg\|_{L^2} &= \lim_{n\to\infty}\|i_nT_t^{(n)}g_n - i_nT_t^{(n)}g_n'\|_{L^2}\\ &\leq \lim_{n\to\infty}\|g_n - g_n'\|_{L^2_n}\\ &= 0,\end{aligned}$$

we complete the proof. □

5. Some properties of the O.U. semigroup

In [8] we have shown some properties that hold in the general theory of Dirichlet forms for self-adjoint Markovian semigroups. They hold in particular for the Ornstein-Uhlenbeck semigroup on the path space we have been studying.

For example, for $\alpha > 0$ and any exponent $1 < p < \infty$, we have

$$\|L^\alpha T_t f\|_p \leq \frac{c_p}{t^\alpha}\|f\|_p$$

and, for f in the domain of L and with constant sign

$$\|df\|_p \leq c_p\|f\|_p^{\frac{1}{2}}\|Lf\|_p^{\frac{1}{2}}, 1 < p \leq 2.$$

Concerning the derivatives of the semigroup, the de Rham-Hodge L^2 contractivity, namely

$$\|T_tf\|_{1,2} \leq \|f\|_{1,2}$$

where $\|.\|_{1,2}$ denotes the Sobolev norm correspondent to the first derivative in L^2, was also derived in [8]. This property had been previously proved by [13] using different methods.

The semigroup also satisfies the following Harnack inequality:

$$\|dT_tf\|_p \leq \frac{C_p}{\sqrt{t}}\|f\|_p$$

for $1 < p \leq 2$.

An Harnack theorem for the corresponding heat kernel has been announced in [6].

Finally we also refer to [9], where a Littlewood-Paley type inequality on the path space was proved.

6. The lifted semigroup

In [4] a Markovian connection on the path space was introduced in order to renormalize the Levi-Civita connection, which produces a divergent curvature. If Z_1, Z_2 are adapted vector fields, and $z_i(\cdot) = t^p_{0\leftarrow\cdot}(Z(\cdot))$, $i = 1,2$, are the corresponding Cameron-Martin vectors, the Markovian connection is defined by

$$\frac{d}{d\tau}(\nabla_{z_1}z_2) = D_{z_1}z_2 + Q_{z_1}\cdot\left(\frac{d}{d\tau}z_2\right)\ ,\ Q_z(\tau) = \int_0^\tau \Omega(z, \circ dx)$$

where Ω denotes the curvature tensor of the manifold M and $\circ dx$ stands for Stratonovich stochastic integration. Here we have identified the covariant derivative with its image through the parallel transport.

In [7] we have defined on the finite manifold M^n_ε a Markovian connection which is Riemannian : for any smooth vector fields $Y, Z \in T(M^n_\varepsilon)$, we put

$$\frac{d}{ds}(\nabla^n_Y Z)^\lambda(v, s^-) := D^n_Y\dot{z}^\lambda(s^-) + \int_0^{s^-}\Omega^\lambda_{\gamma\lambda\beta}(\sigma_v(\tau))y^\gamma(\tau)d[I_n^{-1}(\sigma_v)]^\lambda(\tau)\cdot\dot{z}^\beta(s^-).$$

where $s^- = max\{s_i \leq s\}$.

The operator L^n can be lifted to the frame bundle $O(M_\varepsilon^n)$ through the connection ∇^n, thus defining an operator $\mathcal{L}^n_{O(M_\varepsilon^n)}$ such that, for any smooth function f,

$$\mathcal{L}^n_{O(M_\varepsilon^n)}(f \circ \pi) = (L^n f) \circ \pi$$

where π denotes the canonical bundle projection. Furthermore, if Z is a vector field on M_ε^n and $F_Z(r) = r^{-1}(Z) \in H^n$ denotes its scalarizition, we have:

$$\mathcal{L}^n_{O(M_\varepsilon^n)} F_Z = F_{\mathcal{L}^n Z},$$

where

$$\mathcal{L}^n Z = \sum_{\alpha,i,j} s_i \wedge s_j \nabla^n_{z_{i,\alpha}} \nabla_{z_{j,\alpha}} f - \sum_{\alpha,i,j} s_i \wedge s_j \delta^{(n)}(z_{i,\alpha}) \nabla^n_{z_{j,\alpha}} f.$$

On the other hand, an operator $\mathcal{L}$ on vector fields on the path space associated to the Markovian connection was defined in [3]. For cylindrical vector fields $Z \in \mathcal{C}(H) = \{Z(p) = \sum_{i=1}^k F_i(p) h_i, F_i \text{ cylindrical }\}$, where $\{h_i\}$ denotes a basis in H, it can be written as:

$$\mathcal{L}Z = \sum_\alpha \int_0^1 \nabla^2_{\tau,\alpha} Z d\tau - \sum_\alpha \int_0^1 \nabla_{\tau,\alpha} Z \circ dx^\alpha(\tau)$$

In [7] we have proved the following:

Theorem 6.1. *For any $Z \in \mathcal{C}(H)$ we have*

$$(\tilde{\mathcal{L}}^n + I) Z_n \to (\mathcal{L} + I) Z$$

in $L^2(P_{m_0}(M), \nu; H)$, where

$$Z_n(.) = \int_0^{\cdot} \Big(\sum_{i=1}^n 1_{[s_i, s_{i+1})}(s) \Big(\frac{1}{s_{i+1} - s_i} \int_{s_i}^{s_{i+1}} \dot{z}(\tau) d\tau \Big) \Big) ds.$$

Dirichlet forms can be naturally associated in this framework to the operators $\mathcal{L}^n + I$ and $\mathcal{L} + I$ and in the correspondent resolvents converge weakly in L^2. Concerning the semigroups, we have constructed in [7] a process $r_t = (p_t, e_t)$ on the space $P_{m_0}(M) \times P(O(d))$ as the lift of the O.U. process p_t on the path space through the Markovian connection. The following representation formula for the semigroup associated to $\mathcal{L} + I$ holds:

Theorem 6.2. *For any $Z \in \mathcal{C}(H)$ we have*

$$(T_t^{(\mathcal{L}+I)} Z)(p) = e^{-t} E(r^{-1}(w, p, t) Z)$$

and, concerning the convergence of the semigroups, we have:

Theorem 6.3. *Let $Z \in \mathcal{C}(H)$, $Z_n \in L^2(M_\varepsilon^n, \nu_{n,\varepsilon}; H_\varepsilon^n)$. Then for any $Y \in L^2(P_{m_0}(M), \nu; H)$ we have*

$$E^\nu((\tilde{T}_t^{(\mathcal{L}^n+I)} Z_n | Y)_H) \to E^\nu((T_t^{\mathcal{L}+I} Z | Y)_H)$$

This weak convergence can be improved by the methods described in paragraph 4 that strong (L^2) convergence holds.

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References

[1] Andersson(L.) and Driver(B.), *Finite Dimensional Approximation to Wiener Measure and Path Integral Formulas on Manifolds. J. Funct. Anal*, **165** (1999), 430-498.

[2] Bismut(J. M.), *Large Deviations and the Malliavin Calculus. Birkhäuser, Basel*, 1984.

[3] Cruzeiro(A. B.) and Fang(S.), *Weak Levi-Civita Connection for the Damped Metricon the Riemannian Path Space and Vanishing of Ricci Tensor in Adapted Differential Geometry. J. Funct. Anal.*, **185** (2001), 681-698.

[4] Cruzeiro(A.B.) and Malliavin(P.), *Renormalized Differential Geometry on Path Space, Structural Equation, Curvature. J.Funct. Anal.*, 139(1996), 119-181.

[5] Cruzeiro(A.B.) and Malliavin(P.), *Frame bundle of Riemannian path space and Ricci tensor in adapted differential geometry. J.Funct. Anal.*, 177(2000), 219-253.

[6] Cruzeiro(A. B.) and Malliavin(P.), *Stochastic Calculus of Variations and Harnack inequality on Riemannian Path Space*, to appear in *C.R.Acad. Sc. Paris.*

[7] Cruzeiro(A.B.) and Zhang(X.), *Finite Dimensional Approximation of Riemannian Path Space Geometry. J.Funct. Anal.*, 2003.

[8] Cruzeiro(A.B.) and Zhang(X.), *On Harnack and Littlewood-Paley-Stein inequalities for* $1 < p \leq 2$. preprint, 2003.

[9] Cruzeiro(A.B.) and Zhang(X.), *A Littlewood-Paley type inequality on the path space*. preprint, 2003.

[10] Driver(B.) , *A Cameron-Martin type quasi-invariance theorem for Brownian motion on a compact manifold. J.Funct. Anal.*, 110 (1992), 272-376.

[11] Driver(B.) and Röckner(M.), *Construction of diffusions on path spaces and loop spaces of compact riemannian manifolds. C.R. Acad. Sciences Paris, Ser I*, 320(1995), 1249-1254.

[12] Fang(S.) and Malliavin(P.), *Stochastic analysis on the path space of a Riemannian manifold. J.Funct. Anal.*, 118 (1993), 249-274.

[13] Li(X.D.), *The de Rham-Hodge semigroup on the path space: existence and* L^2*-contractivity. C.R.Acad. Sc. Paris, t.328,Série I, n^o 15, P.161-166,1999.*

[14] Kallenberg (O): *Foundations of modern probability*, Springer-Verlag, 1997.

[15] Kazumi(T), *Le processus d'Ornstein-Uhlenbeck sur l'espace des chemins et le probleme des martingales. J.Funct. Anal.*, **144** (1997), 20–45.

[16] Kusuoka(S.), *Dirichlet forms and diffusion processes on Banach spaces. J.Fac. Sci. Univ. Tokyo Sec I A 29* (1982), 79-95.

[17] Ma (Z.M.) and Röckner (M.), *An introduction to the theory of (non-symmetric) Dirichlet forms. Berlin*, Springer-Verlag, 1992.

[18] Malliavin(P.), *Formule de la moyenne, calcul de perturbations et theorème d'annulation pour les formes harmoniques. J.Funct. Anal.*, (1974) 274-291.

[19] Malliavin(P.), *Stochastic Analysis*: *Grund. der Math. Wissen.* 313, Springer-Verlag (1997).

[20] Norris (J.R.), *Twisted Sheets. J.Funct. Anal.*, 132(1995) 273-334.

[21] Röckner(M.) and Zhang(T.S.), *Finite dimensional approximation of diffusion processes on infinite dimensional spaces. Stoch. Stoch.and Rep.*, **57**, 37-55(1996).

[22] Yosida (K): *Functional Analysis*, Sixth Edition, Springer-Verlag, 1980.

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