Solutions of the Bethe Ansatz Equations as Spectral Determinants

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Excursions in Integrability

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References

The talk is based on three recent papers with R. Conti and A. Raimondo:

- R. Conti and D.M., *On solutions of the Bethe Ansatz for the Quantum KdV model*. arXiv 2022
- R. Conti and D.M., *Counting Monster Potentials*. JHEP 2021
- D.M. and Andrea Raimondo, *Opers for higher states of quantum KdV models*, Commm. Math. Phys, 2020.

A family of anharmonic oscillators

$$-\Psi''(x) + \left(x^{2\alpha} + \frac{\ell(\ell+1)}{x^2} - E\right)\Psi(x) = 0, \alpha > 1, \ell \geq 0, E \in \mathbb{C}.$$

E is said an eigenvalue if $\exists \Psi \neq 0$ such that

$$\lim_{x \to 0^+} \Psi(x) = \lim_{x \to +\infty} \Psi(x) = 0.$$

The **spectrum is discrete, simple and positive**, $E_n(\ell), n \in \mathbb{N}$:

$$E_n(\ell) \sim \left(\frac{2\Gamma\left(\frac{2\alpha+1}{2\alpha}\right)}{\sqrt{\pi}\Gamma\left(\frac{3\alpha+1}{2\alpha}\right)}\right)^{\frac{2\alpha}{\alpha+1}} (4n + 2\ell + 3)^{\frac{2\alpha}{\alpha+1}}, \; n \to +\infty.$$

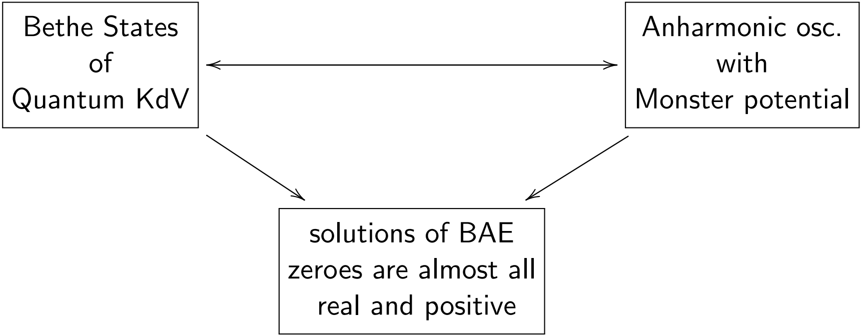
Spectral determinant $D_\ell(E)$ is an entire function of order $\frac{1+\alpha}{2\alpha}$.

The Dorey-Tateo discovery

- Dorey and Tateo, J.Phys A, (1998) noticed that $D_\ell(E)$ satisfies the following countable collection of identities:

$$e^{-i\pi\frac{4\ell+2}{\alpha+1}}\frac{D_\ell\left(e^{-\frac{2\pi i}{\alpha+1}}E_n\right)}{D_\ell\left(e^{\frac{2\pi i}{\alpha+1}}E_n\right)}=-1,\ \forall n\geq 0$$

- These are the Bethe Ansatz Equations (BAE) of an Integrable Quantum Field Theory known Quantum KdV model!
(CFT with $c<1 \approx 6$ Vertex model with $-1<\Delta<1$)
- The spectral determinant $D_\ell(E)$ should correspond to the ground state of the model.


Bethe States
of
Quantum KdV
Anharmonic osc.
with
Monster potential
solutions of BAE
zeroes are almost all
real and positive

Topological classification of solutions

- Problem: Classify solutions of the BAE, $Q(E)$, whose zeros are **all** real, positive and are asymptotics to $E_n(\ell)$ as $n \to +\infty$.
- Use as "topological index" the sequence of root numbers.

Roots and Root-Numbers

Let $Q(E)$ be a solution and $\{x_k\}$ be the increasing sequence of those positive real numbers such that

$$e^{-i\pi\frac{4l+2}{\alpha+1}}\frac{Q\left(e^{-i\frac{2\pi}{\alpha+1}}x_k\right)}{Q\left(e^{i\frac{2\pi}{\alpha+1}}x_k\right)} = -1.$$

We say that $k \in \mathbb{Z}$ is a root-number if $Q(x_k) = 0$.
Root-numbers $\{k_n\}_{n\in\mathbb{N}}$ form an increasing sequence of integers.

Fixing Ambiguities

- Numbering ambiguity: $x_k \to x_{k+m_1}$ with $m_1 \in \mathbb{Z}$
  Fix the numbering by imposing: $k_n = n$ for n large enough .

- Phase/Momentum ambiguity

$$e^{-i\pi\frac{4l+2}{\alpha+1}} = e^{-4ip},\ p \to p + \frac{m_2}{2}$$

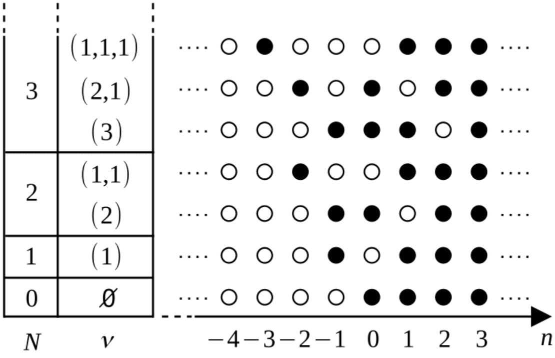
  Fix the momentum by imposing: $2p - \frac{1}{2} \leq k_{min} < 2p + \frac{1}{2}$,
  $k_{min} = -\min_k\{x_k \geq 0\}$

Roots and integer partitions

Root-numbers are sequences that stabilizes: $k_n = n, \text{ if } n \gg 0$.

$$\Downarrow$$

Root-numbers sequences are classified by integer partitions $\{k_n^\lambda\}$.



The ODE/IM Conjecture for Quantum KdV

Bazhanov-Lukyanov-Zamolodchikov, Adv. Theor. Math. Phys., (2003) made the following conjecture:

1. Let $N \in \mathbb{N}$ and $2p \geq N + \frac{1}{2}$. For every $\lambda \vdash N$, the BAE admit a unique (normalised) solution $Q_p^\lambda(E)$ whose sequence of root-numbers coincide with $\{k_n^\lambda\}_{n\in\mathbb{N}}$.
2. Any solution of the BAE coincides with the spectral determinant of a certain anharmonic oscillator.

Our results. 1. Well-posedness of BAE

(1) Theorem, M. - Conti 2022

Fix $\alpha > 1$, $(N, \lambda \vdash N)$. If p is sufficiently large:
The BAE admit a unique solution $Q_p^\lambda(E)$ whose sequence of root-numbers coincide with $\{k_n^\lambda\}_{n\in\mathbb{N}}$.

$+$ Uniform asymptotics of roots/holes positions.

Earlier results:

- Well-posedness for $\alpha > 1$, $p = \frac{1}{2\alpha+2}$ and $\lambda = \emptyset$ by A. Avila in Comm. Math. Phys. (2004) - after Voros.
- Well-posedness for 2α integer and $\lambda = \emptyset$ by Hilfiker and Runke, Ann. Henri Poincaré (2020), using TBA.

Idea beyond the proof

- Introducing the counting function,

$$z(x) = -2p + \frac{1}{2\pi i} \log \frac{Q\left(e^{-i\frac{2\pi}{\alpha+1}} x\right)}{Q\left(e^{i\frac{2\pi}{\alpha+1}} x\right)}, x \geq 0,$$

- The BAE becomes (cfr. Spohn's talk)

$$z(x_{k_n}) = k_n + \frac{1}{2}, \ n \in \mathbb{N}$$

- Transform the logarithmic BAE into a Free-Boundary Nonlinear Integral Equation (known as Destri-De Vega).
- Do mathematics!

Destri-De Vega Integral Equation

Given $\lambda \vdash N$, call $H = -k_0$ (k_0 is the lowest root number).
The unknown is a tuple $(\omega, h_1, \dots, h_H, z)$

- $[\omega, +\infty[$, $\omega > 0$, is the integration interval.
- $h_1 < \dots < h_H$ are the holes greater than the lowest root.
- $z : C^1([\omega, \infty[)$, strictly monotone, $z(x) \sim x^{\frac{1+\alpha}{2\alpha}}, x \to +\infty$.

The Destri-De Vega (DDV) equation is

$$\begin{cases} 1.\ z(x) = -2p + \int_\omega^\infty K_\alpha(x/y) \left\lceil z(y) - \frac{1}{2} \right\rceil \frac{dy}{y} + H\, F_\alpha\left(\frac{x}{\omega}\right) - \sum_{k=1}^{H} F_\alpha\left(\frac{x}{h_k}\right), \\ \qquad K_\alpha(x) := \frac{\sin\left(\frac{2\pi}{1+\alpha}\right)}{\pi} \frac{x}{1 + x^2 - 2x\cos\left(\frac{2\pi}{1+\alpha}\right)} = x F'_\alpha(x) \\ 2.\ \left\lceil z(\omega) - \frac{1}{2} \right\rceil = -H \\ 3.\ z(h_k) = \sigma(k) + \frac{1}{2},\ k = 1 \dots N,\ \sigma(k) = \text{hole number of } h_k \end{cases}$$

$$l_{\omega,p}(x) = -2p + \int_{\omega}^{\infty} K_\alpha(x/y) l_{\omega,p}(y) \frac{dy}{y}, \ l_{\omega,p}(x) \sim x^{\frac{\alpha+1}{2\alpha}}, x \to \infty.$$

It is a Wiener-Hopf equation, solutions can be expressed via

$$\tau(\xi)=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{\alpha^{\frac{\alpha s}{1+\alpha}}}{2\sqrt{\pi}(1+\alpha)^{s-1}} \frac{\Gamma\left(-\frac{1}{2}-\frac{\alpha s}{1+\alpha}\right)\Gamma\left(1-\frac{s}{1+\alpha}\right)}{s^2\,\Gamma(-s)}\,\xi^{-s} ds, \quad \xi=x/\omega.$$

We discovered a (much more useful) formula in terms of a WKB integral

$$\tau(\xi) = \frac{1}{\pi}\int_{u_-}^{u_+} \sqrt{u^2\xi - u^{2\alpha+2} - \ell(\ell+1)}\frac{du}{u}, \ \sqrt{\cdots}_{|u=u_\pm} = 0.$$

This is a first hint of the ODE/IM correspondence.

We need to analyse integrals like

$$A_p[f,\varepsilon] = \int_1^\infty K_\alpha\left(\frac{x}{y}\right)\langle pf(y)+\varepsilon(y)\rangle \frac{dy}{y},\ \langle z\rangle = z - \left\lceil z - \frac{1}{2}\right\rceil$$

$$B_p[f,\varepsilon] = \int_1^\infty K_\alpha\left(\frac{x}{y}\right)\left\lceil pf(y)+\varepsilon(y) - \frac{1}{2}\right\rceil \frac{dy}{y}$$

As an example, we showed that if $f \sim x^{\frac{\alpha+1}{2\alpha}}$ and $\varepsilon, \tilde{\varepsilon}$ are bounded (+ some further hypotheses), then

$$\left| \| B_p[f,\varepsilon] - B_p[f,\tilde{\varepsilon}] \|_\infty - \frac{\alpha+1}{2\alpha}\|\varepsilon-\tilde{\varepsilon}\|_\infty \right| \lesssim_f \frac{\|\varepsilon-\tilde{\varepsilon}\|_\infty}{p}$$

$\Longrightarrow$ contractiveness of the perturbation operator $B_p[I,\cdot]$ when p is large.

Monster potentials

Monster potentials, BLZ (2003)

1. Let P be a monic polynomial of degree N. The spectral determinant $D_\ell^P(E)$ w.r.t the potential

$$V^P = x^{2\alpha} + \frac{\ell(\ell+1)}{x^2} - 2\frac{d^2}{dx^2}\log P(x^{2\alpha+2})$$

satisfies the BAE if the monodromy about the additional poles is trivial for every E.
2. Assuming that the roots of P are distinct, the trivial monodromy is equivalent to the BLZ system

$$\sum_{j\neq k} \frac{z_k\left(z_k^2+(3+\alpha)(1+2\alpha)z_k z_j+\alpha(1+2\alpha)z_j^2\right)}{(z_k-z_j)^3} - \frac{\alpha z_k}{4(1+\alpha)} + \Delta(\ell,\alpha) = 0\ , \quad k=1,\dots,N.$$

Wronskian of Hermite polynomials

Rational extensions of the harmonic oscillator

- A rational extension of degree N is a potential

$$V^U(t) = t^2 - 2\frac{d^2}{dt^2}\ln U(t),$$

where U a polynomial of degree N such that all monodromies of $\psi''(t) = (V^U(t) - E)\psi$ are trivial for every E.

- Oblomkov's theorem (1999)

$$U \propto U^\lambda := Wr[H_{\lambda_1+j-1}, \dots, H_{\lambda_j}], \text{ for a } \lambda := (\lambda_1, \dots, \lambda_j) \vdash N.$$

(2) (Conditional) Theorem, M. - Conti 2021/2022

- Assume there exists a sequence P_ℓ of monster potentials with $\ell \to \infty$, then – up to subsequences –

$$z_k = \frac{\ell^2}{\alpha} + \frac{(2\alpha+2)^{\frac{3}{4}}}{\alpha} v_k^\lambda \ell^{\frac{3}{2}} + O\big(\ell\big),\ k = 1, \dots, N$$

where v_k^λ are the roots of U^λ.

- (If a monster potential with a such an asymptotics exists and) $D_\ell^\lambda(E)$ is the corresponding spectral determinant, then

$$D_\ell^\lambda(E) = Q_p^\lambda\big(E/\eta\big),\ p = \frac{2\ell+1}{\alpha+1} \text{ and } \eta = \left(\frac{2\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+\frac{1}{2\alpha}\right)}{\Gamma\left(1+\frac{1}{2\alpha}\right)}\right)^{\frac{2\alpha}{1+\alpha}}.$$

An unproven identity

Let $\lambda \vdash N$, assume U^λ has N distinct zeroes (see conjecture by Felder-Hemery-Veselov 2010). Consider the Jacobian

$$J^\lambda_{ij}(\underline{t}) = \delta_{ij}\left(1 + \sum_{l\neq j}\frac{6}{(v_i^\lambda - v_j^\lambda)^4}\right) - (1-\delta_{ij})\frac{6}{(v_i^\lambda - v_j^\lambda)^4}, \, {\scriptstyle i,j=1,\dots,N.}$$

The eigenvalues of J^λ are the square numbers $\mu_k = \left(\rho_k^\lambda\right)^2$ computed from the Tableau as follows:

Example: $\lambda = (3,2,2,1,1)$ yields $\underline{\rho}^\lambda = {\color{red}\{1,1,1,2,2,4,4,5,7\}}$.

$$\begin{array}{|c|c|c|c|c|}\hline 5 & 4 & 3 & {\color{red}2} & {\color{red}1} \\ \hline 6 & {\color{red}5} & {\color{red}4} \\ \cline{1-3} {\color{red}7} \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 3 & {\color{red}2} & {\color{red}1} \\ \hline {\color{red}4} \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|}\hline {\color{red}1} \\ \hline\end{array} \longrightarrow \emptyset.$$

$\lambda=(N)$ stated/proven in Ahmed, Bruschi, Calogero, Olshanetsky, and Perelomov ('79).

This is just the tip of an iceberg!

The Big ODE/IM Conjecture, M. - Raimondo (2020)

Every solution of the BAE of every integrable quantum field theory is the **spectral determinant of a linear differential operator**.
$\to$ Bethe Roots are eigenvalues of a (possibly self-adjoint) differential operator (cf. Hilbert-Pólya Conjecture).

Ongoing work: M - Raimondo after Feigin-Frenkel and M -R- Valeri

$\widehat{\mathfrak{g}}$ an affine Kac-Moody Lie-algebra and ${}^L\widehat{\mathfrak{g}}$ the Langlands dual,

$$\left\{\text{Bethe states of } \widehat{\mathfrak{g}} - \text{quantum KdV}\right\} \dashleftarrow\dashrightarrow \left\{{}^L\widehat{\mathfrak{g}} - \text{opers on } \mathbb{C}^*\right\}.$$

MANY THANKS FOR YOUR ATTENTION!