BERNSTEIN PROCESSES ASSOCIATED WITH A MARKOV PROCESS

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Abstract. A general description of Bernstein processes, a class of diffusion processes, relevant to the probabilistic counterpart of quantum theory known as Euclidean Quantum Mechanics, is given. It is compatible with finite or infinite dimensional state spaces and singular interactions. Although the relations with statistical physics concepts (Gibbs measure, entropy,...) is stressed here, recent developments requiring Feynman's quantum mechanical tools (action functional, path integrals, Noether's Theorem,...) are also mentioned and suggest new research directions, especially in the geometrical structure of our approach.

0. Introduction

This is a review of various recent developments regarding the construction and properties of Bernstein processes, a class of diffusions originally introduced for the purpose of Euclidean Quantum Mechanics (EQM), a probabilistic analogue of Quantum Theory [1, 2].

The first section describes their construction, in a rather general setting, compatible with singular interactions. Most of Bernstein processes are not Markovian. The original characterization of the Markovian ones in terms of a maximal entropy principle goes back to E. Schrödinger [3], the originator of EQM, and has been mathematically substantiated by H. Föllmer [4]. An adaptation in the present setting is given in section 2.

For the relations with quantum dynamics, however, the above characterization is not directly relevant. It is more natural to introduce a concept of action functional on a class of processes, along the line of Feynman's path integral, and to look for the minimal point of this action. This is done in section 3.

The next section considers the relations between a crucial factorization, which is the probabilistic counterpart of Born's interpretation of the (complex) wave function ψ_t solving Schrödinger's equation ($\bar{\psi}_t(x)\psi_t(x)\,dx$ should be a probability), and a martingale problem associated with the probability measure of the Bernstein processes.

Section 5 describes the regularity of the (positive) solutions of the pair of adjoint PDEs which are the basis of the construction.

Section 6 is devoted to the dynamical characterization of the Bernstein processes, with some applications to the case where the state space E is finite dimensional, Euclidean or Riemannian, then to the case where E is the Wiener space $C([0,1];\mathbb{R}^d)$.

Finally, section 7 formulates in the simplest situation (E finite dimensional and Euclidean) the Noether Theorem associated with the action functional of section 3, together with some interesting open problems suggested by it. This Theorem relates the presence of symmetries of the action functional under some space-time transformations to the existence of some martingales of the Bernstein processes.

The whole framework has been designed to be the closest possible analogue of quantum theory using (Kolmogorovian) probabilistic concepts. It has recently partially justified this claim in showing that, after the proper analytic continuation in the time parameter, the abovementioned stochastic Noether Theorem turns into a new Theorem of regular quantum theory, providing more symmetries than the usual results of this framework.

It is therefore the aim of EQM to build up progressively a complete stochastic counterpart of quantum theory, allowing to transfer as many concepts and structures as possible from stochastic analysis to quantum theory. And, doing so, to convince theoretical physicists that probability theory may provide new conceptual insights in this area. Reciprocally, one may hope to alleviate the traditional frustration of probabilists in relation with the regular presentations of quantum physics.

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1. Bernstein processes: the definition

1.1. Notations regarding the free Markov process. Let

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$$

be the canonical realization of a continuous homogeneous Hunt-Markov process with values in a Polish space E, where $\Omega = C(\mathbb{R}^+, E)$, $(X_t)_{t\geq 0}$ is the family of coordinates on Ω, $(\mathcal{F}_t)_{t\geq 0}$ is the natural increasing filtration, $\mathbb{P}_x$ is the law of our Markov process starting from $x \in E$, $(\theta_t)_{t\geq 0}$ is the semigroup of shifts on Ω defined by $(\theta_t\omega)(s) = \omega(t+s)$. We denote by $(P_t(x,dy))_{t\geq 0}$ its semigroup of transition kernels on $(E, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-field. Throughout this paper we assume that α is a σ-finite measure on E such that

(1) α is (P_t)-invariant, i.e., $\int_E P_t f\, d\alpha = \int_E f\, d\alpha, \forall t \geq 0$ and $f \geq 0$;
(2) the dual Markov semigroup (P_t^*), acting on $L^\infty(\alpha)$, of (P_t) on $L^1(\alpha)$ can be realized as the semigroup of transition kernels $(\hat{P}_t(x,dy))$ of a continuous Hunt-Markov process $(\hat{\mathbb{P}}_x)_{x\in E}$ on $\Omega = C(\mathbb{R}^+, E)$ (the dual process).

This process is used to modelize the evolution of a special realization of the free quantum system (i.e., without interaction potential). In the simplest quantum mechanical case of a system of particles in a potential, it will be the Brownian Motion on $E = \mathbb{R}^d$ (in the flat case) or a Riemannian manifold. And it will be an Ornstein-Uhlenbeck process (ground state of an harmonic oscillator) in the quantum field case.

For every initial measure $\nu \in M_1(E)$ ($M_1(\cdot)$ denotes the space of probability measures on a measurable space $\cdot$), we write $\mathbb{P}_\nu = \int_E \mathbb{P}_x \nu(dx)$ and $\hat{\mathbb{P}}_\nu = \int_E \hat{\mathbb{P}}_x \nu(dx)$. We denote by $\mathbb{E}^\nu(\cdot)$ (respectively, $\hat{\mathbb{E}}^\nu(\cdot)$) the expectation with respect to $\mathbb{P}_\nu$ (respectively, $\hat{\mathbb{P}}_\nu$).

1.2. Definition of Bernstein processes. Let $V : E \to \mathbb{R}$ be a Borel measurable potential of interaction, which is singular in general. We assume always that, for all $t > 0$,

$$\mathbb{E}^x \exp \int_0^t V^-(X_s)\, ds < +\infty \quad \text{and} \quad \mathbb{P}_x\left(\int_0^t V^+(X_s)\, ds < +\infty\right) = 1, \quad \alpha - \text{a.e.}, \tag{1.1}$$

where $V^- = \max(-V, 0)$ and $V^+ = \max(V, 0)$.

According to Ruelle [6], the *Gibbs measure* (or *specification*) associated with V knowing $(X_0, X_1) = (x, y)$ is a probability measure on $\mathcal{F}_1^0 = \sigma(X_t; 0 \le t \le 1)$, given by

$$\mathbb{P}^V(d\omega \mid 0, x; 1, y) := \frac{\exp(-\int_0^t V(X_s)\, ds)}{Z(0, x; 1, y)} \mathbb{P}(d\omega \mid 0, x; 1, y), \tag{1.2a}$$

where $\mathbb{P}(d\omega \mid 0, x; 1, y)$ is the regular conditional distribution on $\mathcal{F}_1^0$ of $\mathbb{P}_x$ knowing $X_1 = y$, and

$$Z(0, x; 1, y) := \int_\Omega \mathbb{P}(d\omega \mid 0, x; 1, y) \exp\left(-\int_0^t V(X_s)\, ds\right) \tag{1.2b}$$

denotes the normalization constant for $\mathbb{P}^V$. (1.2a), (1.2b) are well defined $\alpha \otimes P_1(x, dy)$ – a.e.

Given two marginal laws $\mu_0, \mu_1 \in M_1(E)$ such that

$$\mu_0 \ll \alpha, \quad \mu_1(\cdot) \ll \alpha P_1(\cdot) := \int_E \alpha(dx) P_1(x, \cdot), \tag{1.3}$$

consider the space of measures

$$\mathcal{M}(\mu_0, \mu_1) := \{\mu \in M_1(E \times E) \mid \mu(X_t \in \cdot) = \mu_t(\cdot) \text{ for } t = 0, 1 \text{ and } \mu \ll \mu_0 \otimes P_1\}. \tag{1.4}$$

According to Zambrini [1] (see also Jamison [7] and Cruzeiro-Zambrini [8]), let us introduce the

Definition 1.1. For $\mu \in \mathcal{M}(\mu_0, \mu_1)$, the probability measure on $\mathcal{F}_1^0$ given by

$$\mathbb{P}_\mu^V(d\omega) := \int_{E \times E} \mu(dx, dy)\, \mathbb{P}^V(d\omega \mid 0, x; 1, y) \tag{1.5}$$

is called the measure of a **Bernstein process** associated with V and the boundary condition μ.

Most of Bernstein processes are not Markovian [7]. Let us consider one of the possible characterizations of the Markovian ones, inspired by Schrödinger [3] and substantiated mathematically by Föllmer [4].

2. The maximal entropy principle

2.1. **Recalls on relative entropy.** Let $(E, \mathcal{B})$ be a countably generated measurable space and $\mu, \nu \in M_1(E)$. The relative entropy or Kullback information of ν with respect to μ is defined as

$$h_{\mathcal{B}}(\nu; \mu) := \int_E \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu}\, d\mu, \text{ if } \nu \ll \mu, \text{ and } = +\infty, \text{ otherwise.} \tag{2.1}$$

For a fixed pair $\mu, \nu \to h_{\mathcal{B}}(\nu; \mu)$ is nonnegative, convex and

$$h(\nu; \mu) = 0 \Leftrightarrow \nu = \mu.$$

Let $\mathcal{G} \subset \mathcal{B}$ be a *sub-σ-algebra*. By desintegration, we have

$$h_{\mathcal{B}}(\nu; \mu) = h_{\mathcal{G}}(\nu; \mu) + \int_E \nu(dz) h_{\mathcal{B}}(\nu_z; \mu_z), \tag{2.2a}$$

where $\{\nu_z(\cdot) = \nu(\cdot \mid \mathcal{G})(z)\}$ (resp. $\{\mu_z\}$) is the regular conditional distribution of ν (resp. μ) knowing $\mathcal{G}$ (see [19]). In particular,

$$h_{\mathcal{B}}(\nu; \mu) \ge h_{\mathcal{G}}(\nu; \mu) \quad \text{and} \quad (h_{\mathcal{B}}(\nu; \mu) = h_{\mathcal{G}}(\nu; \mu)) \Leftrightarrow (\nu_z = \mu_z, \nu - \text{a.s.}). \tag{2.2b}$$

We will also need that ([9])

$$h(\nu; \alpha) - h(\nu; \mu) = \int_E \nu(dx) \log \frac{d\mu}{d\alpha} \tag{2.2c}$$

for $\mu \ll \alpha$ and $h(\nu; \alpha) \wedge h(\nu; \mu) < +\infty$.

2.2. **Maximal entropy principle.** Consider the Feynman-Kac semigroup, for A a Borelian

$$P_t^V(x, A) := \mathbb{E}^x 1_A(X_t) \exp\left(-\int_0^t V(X_s)\, ds\right), \tag{2.3a}$$

and the dual Feynman-Kac semigroup

$$\hat{P}_t^V(x, A) := \hat{\mathbb{E}}^x 1_A(X_t) \exp\left(-\int_0^t V(X_s)\, ds\right). \tag{2.3b}$$

Let

$$\begin{aligned} \widetilde{\mathbb{P}}_x^V|_{\mathcal{F}_1^0} &:= (P_1^V 1)^{-1}(x) \exp\left(-\int_0^1 V(X_s)\, ds\right) \cdot \mathbb{P}_x, \quad \forall x \in E \\ \widetilde{\mathbb{P}}_{\mu_0}^V &:= \int_E \mu_0(dx) \widetilde{\mathbb{P}}_x^V \end{aligned} \tag{2.4a}$$

be the normalized Feynman-Kac measures and

$$\widetilde{P}_1^V(x, A) := \widetilde{\mathbb{P}}_1^V(X_1 \in A) \tag{2.4b}$$

its transition kernel. The following is a simple application of Csiszär theorem [9]:

Proposition 2.1. *Assume that there is some $\mu \in \mathcal{M}(\mu_0, \mu_1)$ such that*

$$h(\mu; \mu_0 \otimes \widetilde{P}_1^V) < +\infty. \tag{2.5}$$

Then there is a unique $\mathbb{Q}$ attaining the

$$\inf\{h_{\mathcal{F}_1^0}(\mathbb{Q}; \widetilde{\mathbb{P}}_{\mu_0}^V); \mathbb{Q} \in M_1(\Omega, \mathcal{F}_1^0) \text{ and } \mathbb{Q}((X_0, X_1) \in \cdot) \in \mathcal{M}(\mu_0, \mu_1)\}, \tag{2.6}$$

which is the probability measure $\mathbb{Q} = \mathbb{P}_{\mu^B}^V$ of a Bernstein process with $\mu^B \in \mathcal{M}(\mu_0, \mu_1)$ minimizing

$$\inf\{h(\mu; \mu_0 \otimes \widetilde{P}_1^V); \mu \in \mathcal{M}(\mu_0, \mu_1)\}. \tag{2.7}$$

Moreover $((X_t), \mathbb{P}_{\mu^B}^V)$ is Markov.

If there is some $\mu \in \mathcal{M}(\mu_0, \mu_1)$ such that

$$h(\mu; \mu_0 \otimes \widetilde{P}_1^V) < +\infty \text{ and } \mu \sim \mu_0 \otimes P_1, \tag{2.8}$$

then μ^B is characterized by

$$\mu^B(dx, dy) = p(x) q(y) \mu_0(dx) \widetilde{P}_1^V(x, dy) \tag{2.9}$$

for some nonnegative measurable functions p, q on E.

Proof. For any $\mathbb{Q} \in M_1(\Omega, \mathcal{F}_1^0)$, let $\mu = \mathbb{Q}((X_0, X_1) \in \cdot)$. By (2.2),

$$\begin{aligned} h_{\mathcal{F}_1^0}(\mathbb{Q}; \widetilde{\mathbb{P}}_{\mu_0}^V) &= h(\mu; \mu_0 \otimes \widetilde{P}_1^V) \\ &\quad + \int_{E \times E} \mu(dx, dy) \cdot h_{\mathcal{F}_1^0}(\mathbb{Q}(\cdot \mid 0, x; 1, y); \mathbb{P}^V(\cdot \mid 0, x; 1, y)). \end{aligned}$$

Then $\mathbb{Q}$ attains the infimum in (2.6) if and only if its boundary law μ minimizes (2.7) and

$$\mathbb{Q}(\cdot \mid 0, x; 1, y) = \mathbb{P}^V(\cdot \mid 0, x; 1, y), \quad \mu - \text{a.e.}$$

Under the condition (2.5), the variational problem (2.7) admits a unique solution μ^B by the well known entropy projection theorem of Csiszär [9, Th. 2.1]. Hence $\mathbb{Q} = \mathbb{P}_{\mu^B}^V$ is the unique solution of (2.6). Since $((X_t), \widetilde{\mathbb{P}}_{\mu_0}^V)$ is Markov, then $((X_t), \mathbb{P}_{\mu^B}^V)$ is Markov too.

Moreover, μ^B is characterized by (2.9) under (2.8), by [9, Corollary 3.1]. □

Remark 2.1. The idea of using Csiszär entropy projection for this problem is due to Föllmer [4]. See Nagasawa [10], Brunaud [11], Cattiaux and Léonard [12] and the references therein for further developments. The above argument is simply an adaptation of that in [4] (in the Brownian Motion case) to the present setting.

Remark 2.2. If $\mu^B \in \mathcal{M}(\mu_0, \mu_1)$ is given by (2.9), then it minimizes (2.7), without the condition (2.8).

Remark 2.3. The usual maximal entropy principle is expressed here as the minimal relative entropy principle (2.6). The reason why $\widetilde{\mathbb{P}}^V_{\mu_0}$ is chosen as the reference measure will be clear afterwards. Since the concept of entropy is, in fact, irrelevant to quantum mechanics, section 3 will provide another interpretation of the Bernstein-Markov process along the line suggested by Feynman's path integrals [13].

2.3. **Factorial equation of Schrödinger.** Let us make a remark on the relation between Csiszär factorial form (2.9) and a factorial equation due to Schrödinger [3]:

Corollary 2.2. *If the condition* (2.8) *is satisfied, then Schrödinger's factorial equation for* ϕ *and* ψ*,*

$$\psi P_1^V \phi = \frac{d\mu_0}{d\alpha}, \quad \phi \hat{P}_1^V \psi = \frac{d\mu_1}{d\alpha}, \quad \alpha - a.e. \text{ on } E, \tag{2.10}$$

admits a solution with ϕ, ψ *nonnegative measurable functions on* E *satisfying, in addition,*

$$\log \phi \in L^1(\mu_1), \quad \log P_1^V 1 - \log P_1^V \phi \in L^1(\mu_0). \tag{2.11}$$

Moreover, μ^B*, determined in Proposition 2.1, is given by*

$$\mu^B(dx, dy) = \psi(x)\,\phi(y)\,\alpha(dx)\,P_1^V(x, dy). \tag{2.12}$$

Proof. By Proposition 2.1, (2.9) is valid. As $P_1^V(x, dy) = P_1^V 1(x) \cdot \widetilde{P}_1^V(x, dy)$, (2.9) can be rewritten as

$$\begin{aligned}\mu^B(dx, dy) &= p(x)\frac{d\mu_0}{d\alpha}(x)[P_1^V 1(x)]^{-1} q(y) \cdot \alpha(dx) P_1^V(x, dy) \\ &= \psi(x)\,\phi(y)\,\alpha(dx)\,P_1^V(x, dy),\end{aligned}$$

where

$$\phi(y) := q(y), \quad \psi(x) := p(x)\frac{d\mu_0}{d\alpha}(x)[P_1^V 1(x)]^{-1} \text{ (Borel version).}$$

Since $\mu^B \in \mathcal{M}(\mu_0, \mu_1)$ and $\alpha(dx)\,P_1^V(x, dy) = \alpha(dy)\hat{P}_1^V(y, dx)$, we have

$$\begin{aligned}\mu_0(dx) &= \psi(x)\,P_1^V \phi(x)\,\alpha(dx) \\ \mu_1(dy) &= \phi(y)\,\hat{P}_1^V \psi(y)\,\alpha(dy)\end{aligned} \tag{2.13}$$

and then (2.10) follows. (2.11) is a translation of the second claim in (2.9). □

Corollary 2.3. *Assume moreover that* (P_t) *is symmetric and ergodic with respect to* α*. Then the solution* (ϕ, ψ) *of* (2.10) *satisfying* (2.11)*, when it exists, is unique up to a constant factor, i.e., if* (ϕ', ψ') *is another such couple, then there is a constant* $C > 0$ *such that*

$$\phi = C\phi', \quad \psi = \frac{1}{C}\psi', \ \alpha - a.e.$$

Proof. Notice that (P_t) is strictly positive improving (well known, see [35, p. 255]), as well as (P_t^V) by our assumption (1.1). For two solutions (ϕ, ψ) and (ϕ', ψ') of (2.10) satisfying (2.11), define

$$q(y) := \phi(y), \quad p(x) := \frac{P_1^V 1(x)}{P_1^V \phi(x)}, \tag{2.14}$$

and similarly (q', p') corresponding to (ϕ', ψ'). Then both (p, q) and (p', q') satisfy (2.9). Since the measure given by (2.9) is the entropical projection (i.e., minimizing (2.7)) by Remark 2.2, it is unique by Proposition 2.1. Thus $p(x)\,q(y) = p'(x)\,q'(y)$, $\mu_0(dx)P_1(x, dy) -$ a.e. on E^2 or equivalently for $\mu_0 -$ a.e. $x \in E$,

$$\log p(x) - \log p'(x) = \log q(y) - \log q'(y), \quad P_1(x, dy) - \text{a.e. } y \in E \tag{2.15}$$

in $[-\infty, +\infty)$, by Fubini's theorem.

If, on the contrary, $\log q - \log q'$ were not constant $\alpha -$ a.e., then there would be $-\infty \le a < b < +\infty$, such that $\alpha(\log q - \log q' \le a) \wedge \alpha(\log q - \log q' \ge b) > 0$. By the strict positive improving property, for $\alpha -$ a.e. $x \in E$,

$$P_1(x; [y; \log q(y) - \log q'(y) \le a]) \wedge P_1(x; [y; \log q(y) - \log q'(y) \ge b]) > 0,$$

which is obviously in contradiction with (2.15).

Consequently $\log q - \log q'$ is constant, $\alpha -$ a.e.. Returning to ϕ by the first formula in (2.14), $\phi = C\phi'$, $\alpha -$ a.e. for some constant $C > 0$. By the strict positivity of $P_1^V \phi$ and the first equation in (2.10), $\psi = \frac{1}{C}\psi'$, $\alpha -$ a.e. too. □

Remark 2.4. The desintegration formula (2.12) is much more convenient that (2.9), especially for the time reversal. See §4, 5.

Remark 2.5. If there is $\mu \in \mathcal{M}(\mu_0; \mu_1)$ such that

$$h(\mu; \mu_0 \otimes P_1) < +\infty \text{ and } \mu \sim \mu_0 \otimes P_1, \tag{2.16a}$$

$$\log P_1^V 1 \in L^1(\mu_0) \text{ and } \log Z(0, x; 1, y) \in L^1(\mu), \tag{2.16b}$$

then the condition (2.8) is verified.

In fact, by (2.2c), we have under (2.12)

$$\begin{aligned}
+\infty &> h(\mu; \mu_0 \otimes P_1) - h(\mu; \mu_0 \otimes \widetilde{P}_1^V) \\
&= \int \mu \log \frac{\widetilde{P}_1^V(x, dy)}{P_1(x, dy)} \\
&= \int \mu(dx, dy) \Big[-\log P_1^V 1(x) + \log \mathbb{E}^x \Big(\exp\Big(-\int_0^1 V(X_s)\, ds \Big) \;\Big|\; X_1 = y \Big) \Big] \\
&= \int \log Z(0, x; 1, y)\, \mu(dx, dy) - \int \log P_1^V 1(x)\, \mu_0(dx) \\
&> -\infty.
\end{aligned}$$

Then $h(\mu; \mu \otimes \widetilde{P}_1^V) < +\infty$, as desired.

A direct proof of existence and uniqueness of positive solutions of (2.10) for strictly positive μ_0 and μ_1 has been given by Beurling [14]. See also [7].

3. The least action principle

We extend the least action principle for Bernstein processes [1, 8] to the general setting of section 1.

3.1. **Forward and backward kinetic energies.**

The presentation of this paragraph is inspired by [4] and especially [12].

Let $\mathbb{Q} \in M_1(\Omega_1 := C([0, 1], E), \mathcal{F}_1^0)$ such that $\mathbb{Q} \ll \mathbb{P}_\alpha$. Denote by $\mathbb{Q}_t := \mathbb{Q}(X_t \in \cdot)$ the (marginal) law of X_t under $\mathbb{Q}$. Assume $\mathbb{Q}_0 = \mu_0$, $\mathbb{Q}_1 = \mu_1$. Then $\mathbb{Q} \ll \mathbb{P}_{\mu_0}$. Consider the density martingale

$$M_t = \frac{d\mathbb{Q}}{d\mathbb{P}_{\mu_0}}\bigg|_{\mathcal{F}_t^0}, \forall t \in [0, 1]$$

which can be chosen continuous $\mathbb{P}_{\mu_0}-$a.s. (this will be assumed afterwards) because of our continuous path assumption for $((X_t),\mathbb{P})$. Define

$$\tau^{\mathbb{Q}} := \inf\{t \in [0,1]; M_t = 0\} \quad (\text{convention: } \inf\emptyset = +\infty), \tag{3.1}$$

which is stopping time with respect to $(\mathcal{F}_t^{\mu_0})$, the completion of $(\mathcal{F}_t^0)$ by $\mathbb{P}_{\mu_0}$. It is well known that

$$\mathbb{P}_{\mu_0}(M_t = 0, \forall t \in [\tau^{\mathbb{Q}}, 1]) = 1,$$

and then

$$\mathbb{Q}(\tau^{\mathbb{Q}} > 1) = \mathbb{E}^{\mu_0} M_1 1_{[t^{\mathbb{Q}}>1]} = \mathbb{E}^{\mu_0} M_1 1_{[M_1>0]} = 1. \tag{3.2}$$

We define the stochastic integral, $\forall t < \tau^{\mathbb{Q}}$,

$$L_t^{\mathbb{Q}} = \int_0^t \frac{1}{M_s}\, dM_s. \tag{3.3}$$

It is a local martingale in $t \in [0, \tau^{\mathbb{Q}})$. By Itô's formula and the fact that $M_0 = 1$, we have $\mathbb{P}_{\mu_0}$ − a.s.,

$$\log M_t = \int_0^t \frac{1}{M_s}\, dM_s - \frac{1}{2}\int_0^t \frac{1}{M_s^2} d\langle M\rangle_s = L_t^{\mathbb{Q}} - \frac{1}{2}\langle L^{\mathbb{Q}}\rangle_t, \quad \forall t < \tau^{\mathbb{Q}},$$

where $\langle\cdot\rangle$ denotes the usual quadratic variational previsible process. Then

$$M_t = 1_{[t<\tau^{\mathbb{Q}}]} \exp\left(L_t^{\mathbb{Q}} - \frac{1}{2}\langle L^{\mathbb{Q}}\rangle_t\right) := \mathcal{E}(L^{\mathbb{Q}})_t, \quad \forall t \in [0,1]. \tag{3.4}$$

On the other hand by [16, p. 149, (3.25)], up to $\mathbb{P}_{\mu_0}$-equivalence,

$$[\langle L^{\mathbb{Q}}\rangle_{1\wedge\tau^{\mathbb{Q}}} = \infty] = \left[\lim_{t\nearrow 1\wedge\tau^{\mathbb{Q}}} \mathcal{E}(L^{\mathbb{Q}})_t = 0\right] = [\tau^{\mathbb{Q}} \le 1]. \tag{3.5}$$

From the condition $\mathbb{Q} \ll \mathbb{P}_{\mu_0}$ on $\mathcal{F}_1^0$, we get therefore

$$\mathbb{Q}[\langle L^{\mathbb{Q}}\rangle_{1\wedge\tau^{\mathbb{Q}}} < \infty] = \mathbb{Q}(\tau^{\mathbb{Q}} > 1) = 1. \tag{3.6}$$

Definition 3.1. For $\mathbb{Q} \in M_1(\Omega_1, \mathcal{F}_1^0)$,

$$K^+(\mathbb{Q}) := K^+(\mathbb{Q};\mathbb{P}) := \frac{1}{2}\mathbb{E}^{\mathbb{Q}}\langle L^{\mathbb{Q}}\rangle_1 1_{[\tau^{\mathbb{Q}}>1]}, \text{ if } \mathbb{Q} \ll \mathbb{P}_\alpha, \text{ and } +\infty \text{ otherwise} \tag{3.7a}$$

is called the **forward kinetic energy** of $\mathbb{Q}$ with respect to the free process $\mathbb{P}$. And the **backward kinetic energy** is defined as

$$K^-(\mathbb{Q}) := K^+(\hat{\mathbb{Q}};\hat{\mathbb{P}}), \tag{3.7b}$$

where $(\gamma\omega)(t) = \omega(1-t)$ is the time reversal on Ω_1 and $\hat{\mathbb{Q}} := \gamma\mathbb{Q}$ is the law of the time reversed process $(\hat{X}_t := X_{1-t})_{t\in[0,1]}$ under $\mathbb{Q}$.

Lemma 3.2. *It holds that for any $\mathbb{Q} \in M_1(\Omega_1, \mathcal{F}_1^0)$ with $\mathbb{Q}_0 = \mu_0$, $\mathbb{Q}_1 = \mu_1$, if $\mathbb{Q} \ll \mathbb{P}_\alpha$, the kinetic energies coincide with the Kullback entropies*

$$K^+(\mathbb{Q}) = h_{\mathcal{F}_1^0}(\mathbb{Q};\mathbb{P}_{\mu_0}); \tag{3.8a}$$

if $\hat{\mathbb{Q}} \ll \hat{\mathbb{P}}_\alpha$,

$$K^-(\mathbb{Q}) = h_{\mathcal{F}_1^0}(\hat{\mathbb{Q}};\hat{\mathbb{P}}_{\mu_1}). \tag{3.8b}$$

Proof. (Following [12].) Let $\tau_n = \inf\{0 \le t \le \tau^{\mathbb{Q}}; \langle L^{\mathbb{Q}}\rangle_t \ge n\} \wedge \tau^{\mathbb{Q}}$ ($\inf\emptyset := \infty$). By (3.5) and (3.6), $1\wedge\tau_n \uparrow 1\wedge\tau^{\mathbb{Q}}$, $\mathbb{P}_{\mu_0} + \mathbb{Q}$ − a.e.. Thus

$$h_{\mathcal{F}_{1\wedge\tau_n}^{\mu_0}}(\mathbb{Q},\mathbb{P}_{\mu_0}) \longrightarrow h_{\mathcal{F}_{1\wedge\tau^{\mathbb{Q}}}^{\mu_0}}(\mathbb{Q},\mathbb{P}_{\mu_0}) = \mathbb{E}^{\mathbb{Q}} \log M_{1\wedge\tau^{\mathbb{Q}}} = h_{\mathcal{F}_1^0}(\mathbb{Q};\mathbb{P}_{\mu_0}),$$

as n tends to infinity (where the last equality follows from (3.2)). On the other hand,

$$
\begin{aligned}
h_{\mathcal{F}^0_{1\wedge\tau_n}}(\mathbb{Q},\mathbb{P}_{\mu_0}) &= \mathbb{E}^{\mathbb{Q}}\Big(L^{\mathbb{Q}}_{1\wedge\tau_n} - \frac{1}{2}\langle L^{\mathbb{Q}}\rangle_{1\wedge\tau_n}\Big) \\
&= \frac{1}{2}\mathbb{E}^{\mathbb{Q}}(\langle L^{\mathbb{Q}}\rangle_{1\wedge\tau_n}),
\end{aligned}
$$

where the last equality follows from Girsanov's formula: in fact $(L^{\mathbb{Q}}_{t\wedge\tau_n} - \langle L^{\mathbb{Q}}\rangle_{t\wedge\tau_n})$, being a $\mathbb{Q}$-local martingale with bounded quadratic variational process, is a $\mathbb{Q}$-martingale.

Combining these two facts, we obtain (3.8a) by Fatou's lemma. Applying (3.8a) to $\hat{\mathbb{Q}}$ with respect to $\hat{\mathbb{P}}$, we get (3.8b). □

Fundamental example 3.3. [17, 4, 5] Let $E = \mathbb{R}^d$ and $((X_t),(\mathbb{P}_x))$ be the standard Brownian Motion such that $\mathbb{P}_x(X_0 = x) = 1$. Assume $K^+(\mathbb{Q}) < +\infty$. Then the coordinates process (X_t) satisfies Itô's stochastic differential equation

$$dX_t = dB_t + v_t^+\,dt,$$

where (B_t) is a $\mathbb{Q}$-Brownian Motion and

$$v_t^+ = \lim_{\epsilon\to 0+}\frac{1}{\epsilon}\mathbb{E}^{\mathbb{Q}}(X_{t+\epsilon} - X_t \mid \mathcal{F}^0_t) := D^+_t X_t \tag{3.9}$$

is the forward velocity of $((X_t),\mathbb{Q})$ (see [4] for a precise description of (3.9)). Fix a Borel version of v^+. By the Girsanov formula,

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}_{\mu_0}}\right|_{\mathcal{F}^0_t} = M_t = \mathcal{E}(L^{\mathbb{Q}})_t,$$

where $L^{\mathbb{Q}}_t = \int_0^t v_s^+\,dX_s$, $\forall t < \tau := \inf\{t; \int_0^t |v_s^+|^2\,ds = +\infty\} = \tau^{\mathbb{Q}}$ (by (3.5) under $\mathbb{P}_{\mu_0}$. Then, by definition,

$$K^+(\mathbb{Q}) = \frac{1}{2}\mathbb{E}^{\mathbb{Q}}\langle L^{\mathbb{Q}}\rangle_1 1_{[1<\tau]} = \frac{1}{2}\mathbb{E}^{\mathbb{Q}}\int_0^1 |v_t^+|^2\,dt, \tag{3.10a}$$

where the last expression is precisely the forward kinetic energy, justifying our general definition (3.7a).

Similarly, assume $K^-(\mathbb{Q}) < +\infty$ and let

$$v_t^- = \lim_{\epsilon\to 0+}\frac{1}{\epsilon}\mathbb{E}^{\mathbb{Q}}(X_{t-\epsilon} - X_t \mid \mathcal{F}^t_1) := D^-_t X_t$$

be the backward velocity of $((X_t),\mathbb{Q})$; then

$$K^-(\mathbb{Q}) = \frac{1}{2}\mathbb{E}^{\mathbb{Q}}\int_0^1 |v_t^-|^2\,dt, \tag{3.10b}$$

justifying (3.7b).

Remark 3.1. In the expressions (3.10a) and (3.10b), the mass is assumed to be one. If there is a nontrivial mass tensor, or $((X_t),\mathbb{P}_x)$ is the Brownian Motion on a Riemannian manifold whose metric is determined by the mass tensor, we can still justify that (3.7a) and (3.7b) define, respectively, natural forward and backward kinetic energies (cf. section 6).

Lemma 3.4. *Let $\mu_t(x) = \frac{d\mu_t}{d\alpha}(x)$ for $t = 0,1$. If $K^+(\mathbb{Q}) < +\infty$, and*

$$\mu_0(x)\log\mu_0(x) - \mu_1(x)\log\mu_1(x) \in L^1(\alpha), \tag{3.11a}$$

then the relation between the two kinetic energies can be written as

$$K^-(\mathbb{Q}) = K^+(\mathbb{Q}) + \int_E [\mu_0(x)\log\mu_0(x) - \mu_1(x)\log\mu_1(x)]\,\alpha(dx). \tag{3.11b}$$

Proof. By (3.8b) and (2.2),

$$\begin{aligned}
K^-(\mathbb{Q}) = h(\hat{\mathbb{Q}}; \hat{\mathbb{P}}_{\mu_1}) &= \mathbb{E}^{\hat{\mathbb{Q}}} \log \frac{d\hat{\mathbb{Q}}}{d\hat{\mathbb{P}}_\alpha} \cdot \frac{1}{\mu_1(X_0)} \\
&= \mathbb{E}^{\mathbb{Q}} \log \frac{d\mathbb{Q}}{d\mathbb{P}_\alpha} \cdot \frac{1}{\mu_1(X_1)} \quad \left(\text{since } \frac{d\hat{\mathbb{Q}}}{d\hat{\mathbb{P}}_\alpha} = \frac{d\mathbb{Q}}{d\mathbb{P}_\alpha}(\gamma)\right) \\
&= \mathbb{E}^{\mathbb{Q}} \log \frac{d\mathbb{Q}}{d\mathbb{P}_{\mu_0}} \cdot \frac{\mu_0(X_0)}{\mu_1(X_1)} \\
&= K^+(\mathbb{Q}) - \mathbb{E}^{\mathbb{Q}}[\log \mu_0(X_0) - \log \mu_1(X_1)] \\
&= K^+(\mathbb{Q}) - \int_E [\mu_0(x) \log \mu_0(x) - \mu_1(x) \log \mu_1(x)] \, d\alpha(x).
\end{aligned}$$

□

3.2. **The least action principle.** From now on we always assume

$$\int_E \mu_0(dx) P_t^V 1(x) < +\infty, \qquad \int_E \mu_1(dx) \hat{P}_t^V 1(x) < +\infty. \tag{3.12}$$

Lemma 3.5. *Under* (3.12)*, if one of* $K^\pm(\mathbb{Q})$ *is finite, then*

$$\mathbb{E}^{\mathbb{Q}} \int_0^1 V^-(X_s)\, ds < +\infty, \quad \textit{or, equivalently,} \quad \mathbb{E}^{\mathbb{Q}} \int_0^1 V(X_s)\, ds \in (-\infty, +\infty]. \tag{3.13}$$

Proof. We treat only the case where $K^+(\mathbb{Q}) < +\infty$. By Lemma 3.2, $h(\mathbb{Q}; \mathbb{P}_{\mu_0}) = K^+(\mathbb{Q}) < +\infty$. By (2.2c),

$$\begin{aligned}
+\infty > h(\mathbb{Q}; \mathbb{P}_{\mu_0}) &- h(\mathbb{Q}; \widetilde{\mathbb{P}}^V_{\mu_0}) \\
&= \mathbb{E}^{\mathbb{Q}} \log \frac{d\widetilde{\mathbb{P}}^V_{\mu_0}}{d\mathbb{P}_{\mu_0}} \\
&= \mathbb{E}^{\mathbb{Q}} \log \frac{\exp(-\int_0^1 V(X_s)\, ds)}{P_1^V 1(X_0)} \\
&= -\mathbb{E}^{\mathbb{Q}} \int_0^1 V(X_s)\, ds - \int_E d\mu_0(x) \, \log P_1^V 1(x).
\end{aligned}$$

Thus (3.13) follows by the condition (3.12). □

It follows from this lemma that the potential energy $\mathbb{E}^{\mathbb{Q}} \int_0^1 V(X_s)\, ds$ of the Bernstein process X_s is well defined.

This allows us to introduce action functionals:

Definition 3.6. For $\mathbb{Q} \in M_1(\Omega_1, \mathcal{F}_1^0)$ with $\mathbb{Q}_t = \mu_t, t = 0, 1,$

$$\mathcal{A}^\pm(\mathbb{Q}) = K^\pm(\mathbb{Q}) + \mathbb{E}^{\mathbb{Q}} \int_0^1 V(X_s)\, ds, \quad \text{when } K^\pm(\mathbb{Q}) < +\infty \quad (= +\infty \text{ otherwise}) \tag{3.14}$$

is called respectively the forward and backward action functional of $\mathbb{Q}$. We shall use occasionally

$$\mathcal{A}(\mathbb{Q}) = \frac{1}{2}[\mathcal{A}^+(\mathbb{Q}) + \mathcal{A}^-(\mathbb{Q})], \tag{3.15}$$

called the symmetrized action functional of the process.

The following result is along the line of variational principles in [1, 2, 8]:

Proposition 3.7. *Assume* (3.12)*. The condition* (2.5) *of Proposition 2.1 is equivalent to*

$$\inf\{\mathcal{A}^+(\mathbb{Q}) \mid \mathbb{Q} \in M_1(\Omega, \mathcal{F}_1^0), \mathbb{Q}_0 = \mu_0, \mathbb{Q}_1 = \mu_1\} < +\infty. \tag{3.16}$$

In that case, there is a unique probability measure $\mathbb{Q}$ *attaining the infimum in* (3.16)*, which is given by the Bernstein-Markov measure* $\mathbb{P}^V_{\mu^B}$ *of Proposition 2.1*

Proof. By Lemma 3.2 and the proof of Lemma 3.5, we have

$$\mathcal{A}^+(\mathbb{Q}) = h(\mathbb{Q}; \widetilde{\mathbb{P}}^V_{\mu_0}) - \int_E d\,\mu_0(x) \log P_1^V 1(x), \quad \text{if } K^+(\mathbb{Q}) < +\infty. \tag{3.17}$$

Since the last constant in (3.17) is finite by condition (3.12), the conclusion follows directly from Proposition 2.1. □

Another extension of the variational principles of EQM involves the symmetrized action functional:

Theorem 3.8. *Assume* (2.5), (3.11a) *and* (3.12)*. Then*

$$\inf\{\mathcal{A}(\mathbb{Q}) \mid \mathbb{Q} \in M_1(\mathcal{F}_1^0), \mathbb{Q}_0 = \mu_0, \mathbb{Q}_1 = \mu_1\} < +\infty \tag{3.18}$$

and it is attained by a unique $\mathbb{Q}$*, which is given by the Bernstein-Markov measure* $\mathbb{P}^V_{\mu^B}$ *determined in Proposition 2.1.*

Proof. By Proposition 3.7, there exists $\mathbb{Q}$ satisfying $\mathbb{Q}_t = \mu_t$ for $t = 0, 1$ and $\mathcal{A}^+(\mathbb{Q}) < +\infty$. By Lemma 3.4 (3.11b), we have, for any such $\mathbb{Q}$,

$$\mathcal{A}(\mathbb{Q}) = \mathcal{A}^+(\mathbb{Q}) + \frac{1}{2} \int_E [\mu_0(x) \log \mu_0(x) - \mu_1(x) \log \mu_1(x)]\, d\alpha(x), \tag{3.19}$$

which is finite. Since the last term in (3.19) is independent on $\mathbb{Q}$ (i.e., depends only on (μ_0, μ_1)), this result follows from Proposition 3.7. □

4. Schrödinger's factorization and node estimate

In this section we assume that (2.8) holds.

According to (2.12), there are two nonnegative Borel measurable functions ϕ and ψ such that $\mu^B(dx, dy) = \psi(x)\phi(y)\alpha(dx)P_1^V(x, dy)$. They will be fixed from now on. Consequently, the Bernstein-Markov measure determined in Proposition 2.1 is given by

$$\mathbb{P}^V_B := \mathbb{P}^V_{\mu^B} = \psi(X_0)\, \phi(X_1)\, \exp\Big(-\int_0^1 V(X_t)\, dt\Big) \cdot \mathbb{P}_\alpha, \quad \text{on } \mathcal{F}_1^0. \tag{4.1}$$

Recalling the definition of the Feynman-Kac semigroups (P_t^V) and $(\hat{P}_t^V)$ given in (2.3), we define for all $(t, x) \in [0, 1] \times E$,

$$\phi(t, x) := P_{1-t}^V \phi(x); \quad \psi(t, x) := \hat{P}_t^V \psi(x). \tag{4.2}$$

They are Borel-measurable on $[0, 1] \times E$ with values in $[0, +\infty]$. The following lemma is the key of this section.

Lemma 4.1.

a) *Let* (M_t^B) *be the* $\mathbb{P}_\nu$*-continuous martingale version of*

$$\mathbb{E}^\nu\Big[\psi(X_0)\, \phi(X_1)\, \exp\Big(-\int_0^1 V(X_t)\, dt\Big) \,\Big|\, \mathcal{F}_t^\nu\Big], \tag{4.3}$$

where ν *is a probability measure equivalent to* α *with* $d\nu/d\alpha$ *bounded, and* $(\mathcal{F}_t^\nu)$ *is the completion of* $(\mathcal{F}_t^0)$ *by* $\mathbb{P}_\nu$*. Then, with* $\mathbb{P}_\nu$*-probability one,*

$$M_t^B = \psi(X_0)\, \phi(t, X_t)\, \exp\Big(-\int_0^t V(X_s)\, ds\Big), \quad \forall t \in [0, 1]; \tag{4.4}$$

b) *the map* $t \to \psi(X_0)\phi(t, X_t)$ *is* $\mathbb{P}_\alpha$*-a.e. continuous and finite on* $[0, 1]$*;*
c) *the map* $t \to \phi(X_1)\psi(t, X_t)$ *is* $\mathbb{P}_\alpha$*-a.e. continuous and finite on* $[0, 1]$.

Proof. **Step 1.** We prove at first that $t \to \phi(t, X_t)$ is optional w.r.t. $(\mathcal{F}^\nu_t)$, and for any stopping time $0 \le \tau \le 1$ w.r.t. $(\mathcal{F}^\nu_t)$,

$$\mathbb{E}^\nu \left[\phi(X_1) \exp\Big(- \int_\tau^1 V(X_s)\, ds \Big) \,\Big|\, \mathcal{F}^\nu_\tau \right] = \phi(\tau, X_\tau). \tag{4.5}$$

For this purpose, let $V^N_n = [V \vee (-n)] \wedge N$ and $\phi^n = \phi \wedge n$ for any $n, N \in \mathbb{N}$. Let

$$\phi^N_n(t, x) := P^{V^N_n}_{1-t} \phi_n(x) = \mathbb{E}^x \phi_n(X_{1-t}) \exp\Big(- \int_0^{1-t} V^N_n(X_s)\, ds \Big).$$

□

Since $t \to m_t := \exp\big(-nt - \int_0^t V^N_n(X_s) ds\big)$ is multiplicative, then $\mathbb{Q}. := m_t \mathbb{P}.$ defines a *right* Markov process by Sharpe [36, Th. (61.5), p. 287]. Applying [36, Th. (7.4).(viii), p. 31] to $\mathbb{Q}.$, we get that

$$t \to e^{-nt} \phi^N_n(t, X_t) \text{ is right continuous on } [0, 1],\ \mathbb{P}_\nu - \text{a.s.}$$

Hence $(\phi^N_n(t, X_t))_{t \in [0,1]}$ is optional w.r.t. $(\mathcal{F}^\nu_t)$.

By the same argument as in [36, Th.(7.4). (iv)⇒(vi),p. 33] (w.r.t. $\mathbb{Q}.$), we have

$$\mathbb{E}^\nu \left[\phi_n(X_1) \exp\Big(- \int_\tau^1 V^N_n(X_s)\, ds \Big) \,\Big|\, \mathcal{F}^\nu_\tau \right] = \phi^N_n(\tau, X_\tau). \tag{4.6}$$

Now by dominated convergence and Fatou's lemma, for all $(t, x) \in [0, 1] \times E$,

$$\phi(t, x) = \lim_{n \to \infty} \uparrow \lim_{N \to \infty} \downarrow \phi^N_n(t, x).$$

Thus for all $(t, \omega) \in [0, 1] \times \Omega$,

$$\phi(t, X_t(\omega)) = \lim_{n \to \infty} \uparrow \lim_{N \to \infty} \downarrow \phi^N_n(t, X_t(\omega)),$$

where the desired optionality of $t \to \phi(t, X_t)$ (w.r.t. $(\mathcal{F}^\nu_t)$) follows. Finally, taking at first $N \to \infty$ and next $n \to \infty$ (as above), (4.6) becomes (4.5).

Step2: part a). Since $M^B_1 = \psi(X_0)\phi(X_1)\exp\big(-\int_0^1 V(X_t)\, dt\big) \in L^1(\mathbb{P}_\alpha)$ by (4.1), we have $M^B_1 \in L^1(\mathbb{P}_\nu)$ as well by our assumption on ν. By Doob's stopping time theorem, for any stopping time $0 \le \tau \le 1$ w.r.t. $(\mathcal{F}^\nu_t)$,

$$\begin{aligned} M^B_\tau &= \mathbb{E}^\nu(M_1 \mid \mathcal{F}^\nu_t) \\ &= \psi(X_0) \exp\Big(- \int_0^\tau V(X_t)\, dt \Big) \cdot \mathbb{E}^\nu \left[\phi(X_1) \exp\Big(- \int_\tau^1 V(X_t)\, dt \Big) \,\Big|\, \mathcal{F}^\nu_\tau \right] \\ &= \psi(X_0) \exp\Big(- \int_0^\tau V(X_t)\, dt \Big) \cdot \phi(\tau, X_\tau), \end{aligned} \tag{4.7}$$

where the last equality follows from (4.5).

Now notice that $(M^B_t)_{t \in [0,1]}$ and $\big(\psi(X_0)\, \phi(t, X_t)\, \exp\big(-\int_0^t V(X_s)\, ds\big)\big)_{t \in [0,1]}$ are two optional processes satisfying (4.7). By the well known section theorem in Dellacherie and Meyer [15, Vol. 1,chap. IV] (or [36, Th. (A4.13).(ii), pp. 389,390]), these two processes are $\mathbb{P}_\nu$-indistinguishable. Then (4.4) follows. Part a) is proved.

Step 3: part b). By (4.4), $\mathbb{P}_\nu$ – a.s., we have for all $t \in [0, 1]$,

$$\psi(X_0)\, \phi(t, X_t) = M_t \exp\Big(\int_0^t V(X_s)\, ds \Big).$$

But the last process above is continuous and finite on $[0, 1]$, $\mathbb{P}_\nu \sim \mathbb{P}_\alpha$ – a.e..

Step 4: part c). By reversing the time in (4.1), we have

$$\gamma \mathbb{P}_B^V = \hat{\mathbb{P}}_B^V = \phi(X_0)\,\psi(X_1)\,\exp\Big(-\int_0^1 V(X_s)\,ds\Big)\cdot \hat{\mathbb{P}}_\alpha. \tag{4.8}$$

Let $(\hat{M}_t^B)$ be the $\hat{\mathbb{P}}_\nu$-continuous martingale version of

$$\hat{\mathbb{E}}^\nu\Big[\phi(X_0)\,\psi(X_1)\,\exp\Big(-\int_0^1 V(X_s)\,ds\Big)\,\Big|\,\mathcal{F}_t\Big].$$

By the same proof as in part a) above, we have

$$\hat{M}_t^B = \phi(X_0)\,\psi(1-t,X_t)\,\exp\Big(-\int_0^t V(X_s)\,ds\Big),\quad \forall t\in[0,1],\quad \hat{\mathbb{P}}_\nu \sim \hat{\mathbb{P}}_\alpha - \text{a.e.} \tag{4.9}$$

Thus, still by the time reversed process $\hat{X}_t := X_{1-t}$, we get

$$\begin{aligned} 0 &= \hat{\mathbb{P}}_\alpha\big(t\to\phi(X_0)\,\psi(1-t,X_t) \text{ is not continuous on } [0,1]\big)\\ &= \mathbb{P}_\alpha\big(t\to\phi(X_1)\,\psi(1-t,\hat{X}_t) \text{ is not continuous on } [0,1]\big)\\ &= \mathbb{P}_\alpha\big(s\to\phi(X_1)\,\psi(s,X_s) \text{ is not continuous on } [0,1]\big),\end{aligned}$$

the desired claim c).

Proposition 4.2. *Let $\mu_t := \mathbb{P}_B^V(X_t\in\cdot)$ be the marginal law of the Bernstein-Markov process $\mathbb{P}_B^V$ (given in (4.1)) for $t\in[0,1]$. Then it holds that*

$$\mu_t(dx) = \phi(t,x)\,\psi(t,x)\,\alpha(dx),\quad \forall t\in[0,1]; \tag{4.10}$$

$$\mathbb{P}_B^V\big(\phi(t,X_t)\,\psi(t,X_t)>0,\,\forall t\in[0,1]\big)=1. \tag{4.11}$$

Remark 4.1. The relation (4.10) is called Euclidean Born interpretation in [1, 2]. The reason of this terminology will be clear in the next section, when we will come back to the fundamental example 3.3. Let us only observe here that the multiplicative form (4.10), essential to the structure of EQM, was the original motivation of E. Schrödinger [3]. The equality (4.11) means that under $\mathbb{P}_B^V$, the process (X_t) cannot reach the nodal set $\{(t,x)\in[0,1]\times E;\phi(t,x)\psi(t,x)=0\}$. It was at first established by Zheng [21] in a particular (and different) context. The node estimate (4.11) was stated in [10, Th. 5.3, pp. 128,129] (for $\mathcal{L}=\frac{1}{2}\Delta$), but its proof (only three lines, from line -6 to -4 of p. 129 in [10]) is both far from being complete and false: it is claimed that (4.11) follows from the fact that $\mathbb{P}_B^V(\phi(t,X_t)\psi(t,X_t)>0)=1$ for each t fixed (trivial by (4.10)) and the right continuity of $t\to\phi(t,X_t)$, which was not proved.

Proof. Formula (4.10) is an immediate consequence of (4.1). To show the node estimate (4.11), observe that $\mathbb{P}_B^V(\phi(0,X_0)\psi(0,X_0)>0)=\mu_0(d\mu_0/d\alpha>0)=1$ and

$$((1/\phi(0,X_0)\psi(0,X_0))\,M_t^B)$$

is the density martingale of $\mathbb{P}_B^V$ w.r.t. $\mathbb{P}_{\mu_0}$. By (3.2) we have

$$\begin{aligned} 1 &= \mathbb{P}_B^V\Big(\frac{1}{\phi(0,X_0)\psi(0,X_0)}M_t^B>0,\forall t\in[0,1]\Big)\\ &= \mathbb{P}_B^V\Big(\phi(t,X_t)\,\exp\Big(-\int_0^t V(X_s)\,ds\Big)>0,\forall t\in[0,1]\Big)\\ &= \mathbb{P}_B^V\big(\phi(t,X_t)>0,\forall t\in[0,1]\big),\end{aligned}$$

where the second equality follows from (4.4)

Similarly from (4.9) we deduce

$$\mathbb{P}_B^V\big(\psi(t,X_t) > 0, \forall t \in [0,1]\big) = \hat{\mathbb{P}}_B^V\big(\psi(1-t,X_t) > 0, \forall t \in [0,1]\big) = \hat{\mathbb{P}}_B^V\big(\hat{M}_t^B > 0, \forall t \in [0,1]\big) = 1.$$

Combining these two estimates we get (4.11). □

5. Regularity of $\phi(t,x)$ and $\psi(t,x)$

By definition (4.2) it follows formally from Feynman-Kac formula that

$$\begin{aligned} \Big(\frac{\partial}{\partial t} + \mathcal{L}\Big)\phi(t,x) &= V(x)\,\phi(t,x) \\ \Big(-\frac{\partial}{\partial t} + \hat{\mathcal{L}}\Big)\psi(t,x) &= V(x)\,\psi(t,x) \end{aligned} \tag{5.1}$$

where $\mathcal{L}$ and $\hat{\mathcal{L}}$ are respectively the (formal) generators of (P_t) and $(\hat{P}_t)$.

At this point, it is illuminating to come back to the abovementioned **Fundamental example 3.3:** In this case we have $\mathcal{L} = \hat{\mathcal{L}} = \frac{1}{2}\Delta$. Using (5.1) we may reinterpret explicitly some of the results found in section 3 (least action principle).

Let us define the two scalar fields

$$\bar{R}(t,x) = \frac{1}{2}\log(\phi\psi)(t,x),$$
$$\bar{S}(t,x) = \frac{1}{2}\log\Big(\frac{\phi}{\psi}\Big)(t,x),$$

where ϕ and ψ are the two positive solutions of (5.1) needed for the construction. Since, formally, these equations (5.1) are time reversed of each other, together with their solutions, $\bar{R}$ is even and $\bar{S}$ odd under time reversal.

Let us define the differential form

$$\omega_e = d\bar{R}(t,x).$$

Using (5.1) and defining the Hamiltonian H by

$$H = -\mathcal{L} + V$$

when ϕ and ψ are regular enough we have

$$\omega_e = \frac{1}{2}\Big[\frac{H\phi}{\phi} - \frac{H\psi}{\psi}\Big]dt + \frac{1}{2}\Big[\frac{\nabla\phi}{\phi} + \frac{\nabla\psi}{\psi}\Big]dx.$$

After integration on the time interval $[0,1]$ and interpreting the space differential as a Stratonovich one along the process X_t, with probability density $\mu_t(dx) = (\phi\psi)(t,x)dx$ (cf. (4.10)) we obtain

$$\begin{aligned} &\int_{\mathbb{R}^d} dx\,[\mu_1(x)\log\mu_1(x) - \mu_0\log\mu_0(x)] \\ &= \frac{1}{2}\Big\{\mathbb{E}^a\int_0^1\Big[\frac{1}{2}|D_t^+X_t|^2 + V(X_t)\Big]dt - \mathbb{E}^a\int_0^1\Big[\frac{1}{2}|D_t^-X_t|^2 + V(X_t)\Big]dt\Big\} \\ &\equiv \mathcal{A}^+(\mathbb{Q}) - \mathcal{A}^-(\mathbb{Q}) \\ &\equiv K^+(\mathbb{Q}) - K^-(\mathbb{Q}), \end{aligned}$$

where the notations of (3.10a) and (3.10b) have been introduced for the forward and backward kinetic energies, as well as the ones of (3.14) for the associated action functionals. The last relation coincides with (3.11b).

Starting from the scalar field $\bar{S}$, which is odd under time reversal, and defining

$$\omega_0 = d\bar{S}(t,x)$$

we find as well, after a similar computation,

$$\int_{\mathbb{R}^d} dx \left[\mu_1(x) \log \frac{\phi}{\psi}(1,x) - \mu_0(x) \log \frac{\phi}{\psi}(0,x)\right]$$
$$= \frac{1}{2}\left\{\mathbb{E}^{\mathbb{Q}} \int_0^1 \left[\frac{1}{2}|D_t^+ X_t|^2 + V(X_t)\right] dt + \mathbb{E}^{\mathbb{Q}} \int_0^1 \left[\frac{1}{2}|D_t^- X_t|^2 + V(X_t)\right] dt\right\}. \quad (5.2)$$

In particular, we obtain in this way another interpretation of the action functionals, namely

$$\mathbb{E}^{\mathbb{Q}} \int_0^1 d(\bar{S} + \bar{R})(t, X_t) = \mathcal{A}^+(\mathbb{Q})$$

and

$$\mathbb{E}^{\mathbb{Q}} \int_0^1 d(\bar{S} - \bar{R})(t, X_t) = \mathcal{A}^-(\mathbb{Q}),$$

the symmetrized action functional (3.15) corresponding simply to

$$\mathbb{E}^{\mathbb{Q}} \int_0^1 d\bar{S}(t, X_t) = \mathcal{A}(Q).$$

It is natural to call Lagrangians the integrands of these action functionals. The reason why those actions are fundamental will be shown in the last section.

Also notice that the definitions of the scalar fields $\bar{R}$ and $\bar{S}$ provide us with a nontrivial decomposition of the two positive solutions ϕ and ψ of the equations (5.1) according to their behaviour under time reversal, namely

$$\phi(t,x) = e^{\bar{R}+\bar{S}}(t,x),$$
$$\psi(t,x) = e^{\bar{R}-\bar{S}}(t,x).$$

It is in this sense that $\mu_t(dx) = (\phi\psi)(t,x)\, dx = e^{2\bar{R}}(t,x)dx$ is the probabilistic (or "Euclidean") counterpart of Born's interpretation of the quantum wave function solving Schrödinger equation. The pair of equations (5.1) is the counterpart of Schrödinger's equation and its complex conjugate.

A concise interpretation of (5.1) is accessible through the

Definition 5.1. A real measurable function $u(t,x)$ on $[0,1] \times E$ belongs to the α-extended domain $\mathbb{D}_\alpha(\mathcal{A})$ (respectively $\mathbb{D}_\alpha(\hat{\mathcal{A}})$) of $\mathcal{A} := \partial/\partial t + \mathcal{L}$ (respectively $\hat{\mathcal{A}} := -\partial/\partial t + \hat{\mathcal{L}}$), if there is a measurable function $v(t,x)$ on $[0,1] \times E$ such that $\int_0^1 |v(t, X_t)|\, dt < +\infty$, $\mathbb{P}_\alpha$ – a.e. and

$$M_t(u) := u(t, X_t) - u(0, X_0) - \int_0^t v(s, X_s)\, ds, \quad t \in [0,1]$$
$$(\text{resp. } \hat{M}_t(u) := u(1-t, X_t) - u(1, X_t) - \int_0^t v(1-s, X_s)\, ds)$$

is a continuous local martingale on $[0,1]$ w.r.t. $\mathbb{P}_\nu$ (resp. $\hat{\mathbb{P}}_\nu$), where $\nu \sim \alpha$ is a probability measure. In that case, $\mathcal{A}u(t,x) := v(t,x)$ (resp. $\hat{\mathcal{A}}u(t,x) := v(t,x)$).

The above definition does not depend on the probability measure $\nu \sim \alpha$.

Proposition 5.2. a) *The process*

$$\phi(t, X_t) - \phi(X_0) - \int_0^t V(X_s)\, \phi(X_s)\, ds, \quad t \in [0,1]$$

is a $\mathbb{P}_{\mu_0}$-local continuous martingale on $[0,1]$. In particular, if $\mu_0 \sim \alpha$, then $\phi(t,x) \in \mathbb{D}_\alpha(\mathcal{A})$ and $\mathcal{A}\phi(t,x) = V(x)\phi(t,x)$, i.e., the first formula in (5.1) holds in the sense of Definition 5.1.

b) *The process*

$$\psi(1-t,X_t)-\psi(X_0)-\int_0^t V(X_s)\,\psi(1-s,X_s)\,ds,\quad t\in[0,1]$$

is a $\hat{\mathbb{P}}_{\mu_1}$-local continuous martingale on $[0,1]$. *In particular, if $\mu_1\sim\alpha$, then $\psi(t,x)\in\mathbb{D}_\alpha(\hat{\mathcal{A}})$ and $\hat{\mathcal{A}}\psi(t,x)=V(x)\psi(t,x)$, i.e., the second formula in* (5.1) *holds in the sense of Definition 5.1.*

Notice that in a) and b) $\phi(x),\psi(x)$ are the nonnegative functions used in (4.1).

Proof. a) By Lemma 4.1 (4.4), $\mathbb{P}_{\mu_0}$ − a.s., for all $t\in[0,1]$,

$$\phi(t,X_t)=\frac{1}{\psi(X_0)}M_t^B\exp\Big(\int_0^t V(X_s)\,ds\Big).$$

Writing $e_t=\exp\big(\int_0^t V(X_s)\,ds\big)$, we get by Itô's formula,

$$\phi(t,X_t)-\phi(0,X_0)=\frac{1}{\psi(x_0)}\int_0^t e_s\,dM_s^B+\frac{1}{\psi(X_0)}\int_0^t V(X_s)\,e_s\,M_s^B\,ds.$$

The first term at the right hand side above is a $\mathbb{P}_{\mu_0}$-local continuous martingale, the second term above coincides with

$$\int_0^t V(X_s)\,\phi(s,X_s)\,ds.$$

Thus part a) is shown.

Part b) can be derived from (4.9) instead of (4.4), in the same way. □

Remark 5.1. Since one of the two PDEs (5.1) does not define a well posed Cauchy problem, Proposition 4.2 shows that, in general, the life time of Bernstein processes will be finite [2].

6. The equations of motion

6.1. **The velocity.** Generalizing (3.9), let us consider the forward mean derivative in the sense of Nelson [17] along the Bernstein process:

$$D_t^+f(t,X_t)=\lim_{\epsilon\to 0}\frac{1}{\epsilon}E_t\big[f(t+\epsilon,X_{t+\epsilon})-f(t,X_t)\big].$$

Then, if $\Gamma(\phi,f)$ denotes $\nabla\phi.\nabla f$,

$$D_t^+f=\frac{\partial}{\partial t}f+\mathcal{L}f+\frac{1}{\phi}\Gamma(\phi,f),\quad \text{for } f\in\mathbb{D}\Big(\frac{\partial}{\partial t}+\mathcal{L}\Big).$$

From now on, we shall assume that we have a tangent space to E and a positive definite inner product $\langle,\rangle$ in each tangent space $T_x(E)$, turning $T_x(E)$ into a Hilbert space with o.n. basis $\{e_i(x)\}$. We also assume the existence of a derivative ∇ so that $\mathcal{L}f=\delta\nabla f$, where δ denotes the dual of the derivative in $L^2(E,\alpha)$ with respect to the metric $\langle,\rangle$. Then

$$D_t^+f=\frac{\partial}{\partial t}f+\mathcal{L}f+\frac{1}{\phi}\langle\nabla\phi,\nabla f\rangle.$$

When considering the divergence of a vector field Z, we can split it into two terms: one coming from the contribution of the measure in the integration by parts, another from the metric.

Written in the case of a Riemannian manifold with metric g this means, for f regular enough:

$$\int \langle Z, \nabla f\rangle \, d\alpha = \int g_{ij}\partial_{e_i} f Z^j \, d\alpha$$
$$= -\int f(\partial_{e_i} g_{ij})Z^j \, d\alpha - \int f\delta_\alpha Z \, d\alpha,$$

where we have denoted by $\delta_\alpha Z$ the remaining terms, not due to the derivation of the metric. For example, if $d\alpha = \rho\, dm$, for m the Riemann measure, we have:

$$\delta_\alpha Z = g_{ij}\partial_{e_i} Z^j + g_{ij} Z^j \partial_{e_i} \log \rho.$$

We remark that if a metric preserving the connection is known, then $\partial_{e_k} g_{ij}$ is given by the Christoffel symbols associated with the metric, namely $\partial_{e_k} g_{ij} = g_{lj}\Gamma^l_{ki} + g_{il}\Gamma^l_{kj}$.

Definition 6.1. The velocity of a Bernstein process is the vector field v defined by

$$\langle v, e_\rho\rangle = \delta_\alpha e_\rho + \langle \nabla \log \phi, e_\rho\rangle.$$

When the derivative of the metric in the tangent space is zero, i.e., when δ_α coincides with the divergence δ, then $v(X_t) = D_t^+ X_t$ is the forward mean derivative along the process X_t.

6.2. **The second order equations.** The equations of motion for the Bernstein processes will be given by the result of the computation of $D_t^+ v$, where v is the velocity and D_t^+ the mean forward derivative. In this subsection, we shall compute $D_t^+ v$ in the following cases:

6.2.1: $E = \mathbb{R}^d$, $d\alpha = \rho\, dx$.
6.2.2: E is a finite dimensional manifold with metric g, $d\alpha = dm$, the Riemann measure, and $\mathcal{L}$ is the Laplace-Beltrami operator.
6.2.3: E is the Wiener space $E = \mathcal{C}([0;1];\mathbb{R}^d)$, α the Wiener measure and $\mathcal{L}$ the Ornstein-Uhlenbeck operator.

6.2.1. In this case, $v^l = \delta e_l + \partial_l \log \phi$, i.e., $v = \nabla \log \rho + \nabla \log \phi$, and $\mathcal{L} = \Delta + \nabla \log \rho . \nabla$. By (5.1),

$$\frac{\partial}{\partial t} \log \phi = -\frac{\mathcal{L}\phi}{\phi} + V,$$

therefore we have:

$$D_t^+ v^l = -\partial_l\Big(\frac{\mathcal{L}\phi}{\phi}\Big) + \partial_l V + \mathcal{L}(\partial_l \log \phi) + \nabla \log \phi . \nabla(\partial_l \log \phi)$$
$$+ \mathcal{L}(\partial_l \log \rho) + \nabla \log \phi . \nabla(\partial_l \log \rho).$$

Now $(\mathcal{L}\partial_l - \partial_l \mathcal{L})F = -\langle \nabla F, \nabla(\partial_l \log \rho)\rangle$ and

$$\mathcal{L} \log \phi = \frac{\mathcal{L}\phi}{\phi} - |\nabla \log \phi|^2.$$

Therefore,

$$Dv^l = \partial_l V - \partial_l |\nabla \log \phi|^2 + \nabla \log \phi . \nabla(\partial_l \log \phi) + \mathcal{L}(\partial_l \log \rho)$$

and, finally,

$$D_t^+ v^l = \partial_l V + \mathcal{L}(\partial_l \log \rho).$$

For example, for the O.U. process, $D_t^+ v = D_t^+ D_t^+ X_t = \nabla V(X_t) + X_t$.

6.2.2. In the case of a Riemannian manifold there is, in general, a non trivial contribution from the derivative of the metric and $\delta_\alpha \neq \delta$. The velocity is $v = \nabla \log \phi$ and, in order to compute the equations of motion we use Weitzenböck's formula:

$$\Delta(\nabla \log \phi) = \nabla(\Delta \log \phi) + \mathrm{Ricci}(\nabla \log \phi).$$

On the other hand, $\partial/\partial t$ commutes with the derivative and from (5.1) as before one derives:

$$D_t^+ v = \nabla V + \mathrm{Ricci}\, v$$

(see also [22]). The additional Ricci term in the right hand side is unpleasant since it does not cancel at the formal limit of smooth trajectories, where only the physical forces should appear. On the other hand, the above definition of the forward mean derivative D_t^+ involves implicitly the choice of a stochastic parallel displacement along the trajectories of the Brownian motion on E. The choice adopted here is the one originally made by Itô [23], namely the Stratonovich interpretation of Levi-Civita classical parallel displacement, associated with the Laplace-Beltrami operator. Other choices are possible. One of them [24] replaces the standard Laplacian by

$$(\Delta X)^l = \nabla^k \nabla_k X^l + R^l_k X^l$$

acting on a vector field X. By Weitzenböck's formula, $-\Delta$ is the De Rham-Kodeira Laplacian on scalar and one-form [17]. Then the forward mean derivative becomes, in the notations of the fundamental example 3.3,

$$D_t^+ X^l = \frac{\partial X^l}{\partial t} + v_t^k \nabla_k X^l + \frac{\hbar}{2}(\Delta X)^l.$$

It follows that the equation of motion reduces indeed to

$$D_t^+ v = \nabla V.$$

As a matter of fact, as we are going to see, this equation is also more appropriated in the perspective of the study of the symmetries of Bernstein measures (cf. section 7).

6.2.3. In the Wiener space case there is no contribution from the metric and the velocity is given by

$$v(x) = -x + \nabla \log \phi(x),$$

where ∇ is taken as the gradient in the sense of Malliavin calculus [25].

Let us compute the equations of motion for the Ornstein-Uhlenbeck process ($\nabla \log \phi = 0$). We consider a "continuous basis" on the tangent space (i.e., on the Cameron-Martin space H) defined by the vectors $e_{\tau,l}(\xi) = \mathbf{1}_{\tau<\xi} e_l$, where e_l are the elements of the canonical basis in $\mathbb{R}^d$. These vectors constitute a basis of the tangent space in the sense that a vector field $Z : E \to H$ may be written as

$$Z(\xi) = \sum_l \int_0^1 \frac{d}{d\tau} Z_{\tau,l} e_{\tau,l}(\xi)\, d\tau.$$

The velocity is

$$v^l(x)(\tau) = -x^l(\tau) + (\nabla \log \phi(x))^l(\tau).$$

We want to compute $\overbrace{\delta\nabla}^{\mathcal{L}}(\delta \tilde{e}_{\tau,l})$, where $\tilde{e}_{\tau,l} = (\xi \wedge \tau) e_l$ (i.e., $\tilde{e}_{\tau,l}$ is the "primitive" of $e_{\tau,l}$).

For every smooth functional $f : E \to \mathbb{R}$, we have, by definition of the divergence δ:

$$E\big(\delta\nabla(\delta \tilde{e}_{\tau,l}) f\big) = E\big(\delta \tilde{e}_{\tau,l} . \delta \nabla f\big) = E\big(D_{\tilde{e}_{\tau,l}}(\delta \nabla f)\big).$$

The commutator between the derivative and the divergence [25] is given by

$$D_h(\delta\nabla f) = \delta D_h \nabla f + D_h f.$$

Therefore

$$\begin{aligned} E\big(\delta\nabla(\delta\tilde{e}_{\tau,l})f\big) &= E(\nabla f(\tau) \mid e_l)_{\mathbb{R}^n} \\ &= E\langle \nabla f, \tilde{e}_{\tau,l}\rangle_H \\ &= E\big(f\delta(\tilde{e}_{\tau,l})\big) = E(fx^l(\tau)). \end{aligned}$$

Since f is arbitrary, we deduce the following a.e. equation of motion for the Ornstein-Uhlenbeck process X_t:

$$D_t^+ v^{\tau,l}(X_t) = D_t^+ D_t^+ X_t^{\tau,l} = X_t^{\tau,l}, \quad \alpha - \text{a.e.}$$

For the general Bernstein process, we have to consider the supplementary terms

$$\Big(\frac{\partial}{\partial t} + \mathcal{L} + \nabla \log \phi.\nabla\Big)\big((\nabla \log \phi(x))^l(\tau)\big) - (\mathcal{L} + \nabla \log \phi.\nabla)(x^l(\tau)).$$

The computation is analogous to the one in 6.2.1. In this case the following commutation formula holds [25]:

$$(\mathcal{L}\nabla - \nabla\mathcal{L})F = -\nabla F;$$

therefore

$$\mathcal{L}(\nabla \log \phi(x)^l(\tau)) - (\nabla\mathcal{L} \log \phi(x))^l(\tau) = -(\nabla \log \phi(x))^l(\tau).$$

One finally obtains

$$\begin{aligned} D_t^+ v^{\tau,l}(x) &= D_t^+ D_t^+ x^l(\tau) \\ &= x^l(\tau) + (\nabla V(x))^l(\tau) \end{aligned}$$

i.e.,

$$D_t^+ v = D_t^+ D_t^+ x = x + \nabla V(x), \qquad \alpha \text{ a.e.}$$

It is easy to check that the time reversed version of all the equations of motion mentioned above hold as well. They involve the backward mean derivative D_t^- defined in the fundamental example 3.3 instead of D_t^+. Indeed, if, as in Definition 3.1, $\hat{X}_t = X_{1-t}$, then $D_t^+ \hat{X}_t = -DX_{1-t}$. For example, the time reversal of the last equation of motion is simply

$$D_t^- D_t^- x = x + \nabla V(x), \qquad \alpha \text{ a.e.}.$$

7. Symmetries of Bernstein processes and some open problems

As observed in sections 2 and 3, it is not the concept of entropy which is really essential for the relation between Bernstein processes and quantum mechanics but the two underlying concepts of action functionals (see the fundamental example 3.3). Besides their fundamental relation with Feynman's path integral approach, these actions carry, indeed, the crucial information on the symmetries of Bernstein measures.

Let us illustrate this point in the context of 6.2.2, namely when E is a Riemannian manifold with metric tensor g_{ij}. Let us consider the forward action functional $\mathcal{A}^+$ for the fundamental example 3.3:

$$\mathcal{A}^+(\mathbb{Q}) = K^+(\mathbb{Q}) + \mathbb{E}^{\mathbb{Q}} \int_0^1 V(X_s)\, ds.$$

More explicitly, using (3.10a) and replacing the interval $[0,1]$ of integration by $[t, t_1]$,

$$\mathcal{A}^+ = E \int_t^{t_1} \Big(\frac{1}{2}|D_s^+ X_s|^2 + V(X_s)\Big)\, ds, \tag{7.1}$$

where X_s is such that $\mathbb{P}_x(X_t = x) = 1$. This allows us to regard as well our functional (7.1) as a scalar function $S(x,t)$ on $E \times \mathbb{R}$. Then another variational principle inspired by stochastic control theory [26] shows that the drift of the unique minimal point of this action functional (in a large class of non-necessarily Markovian processes with diffusion coefficient fixed by the metric of E: $\mathbb{E}(dX_s^i dX_s^j \mid \mathcal{F}_s^0) = \hbar g^{ij} ds$; it is more illuminating here to reintroduce the Planck constant $\hbar \neq 1$) satisfies

$$D_t^+ X_t^i = -\nabla^i S(X_t, t). \tag{7.2}$$

As a matter of fact, we already know the explicit form of the function S in this case; it follows from our computations in the fundamental example 3.3 that

$$\begin{aligned} S &= -(\bar{S} + \bar{R}) \\ &= -\hbar \log \phi(q,t), \end{aligned} \tag{7.3}$$

where, by (5.1), ϕ is a positive solution of the heat equation

$$\hbar \frac{\partial \phi}{\partial t} + \frac{\hbar^2}{2} \nabla^j \nabla_j \phi - V(q,t)\phi = 0, \tag{7.4}$$

since the Laplace-Beltrami operator is the generator of $(P_t)_{t \geq 0}$ in this situation. It follows immediately from (7.3) that S solves a non-linear uniformly parabolic partial differential equation known as the Hamilton-Jacobi-Bellman equation [26]:

$$-\frac{\partial S}{\partial t} + \frac{1}{2}|\nabla S|^2 - \frac{\hbar}{2}\nabla^i \nabla_i S - V = 0, \qquad (q,t) \in M \times \mathbb{R}. \tag{7.5}$$

This equation was the original motivation for the development, by Crandall and Lions, of their method of “viscosity solutions” [27]. After reintroduction of the constant $\hbar$, notice that it follows from (7.2) and (7.3) that the velocity v defined in 6.2.2 is

$$v = -\nabla S. \tag{7.6}$$

Let us define as well the energy by

$$\begin{aligned} E &= -\frac{\partial S}{\partial t} \\ &= -\frac{1}{2} v^i v_i - \frac{\hbar}{2} \nabla^i v_i + V, \end{aligned} \tag{7.7}$$

where the second expression results from the Hamilton-Jacobi-Bellman equation and from (7.6).

Proposition 7.1. *For the fundamental example 3.3, the α − a.e. equations of motion of the Bernstein process are*

$$\begin{aligned} D_t^+ v^i &= \nabla^i V \\ D_t^+ E &= \frac{\partial V}{\partial t}. \end{aligned} \tag{7.8}$$

Proof. Indeed, we already know the first equation from 6.2.2, with D_t^+ the forward mean derivative on vector field. On scalar fields such as the energy E, the De Rahm-Kodeira Laplacian of D_t^+ reduces to the Laplace-Beltrami operator. Taking $\frac{\partial}{\partial t}$ of the second expression of (7.7) and using the integrability condition $\frac{\partial}{\partial t} v_i = \nabla_i E$. we verify the second equation of (7.8). □

Let us consider the local group of transformations of the equation (7.4) generated by

$$N = X^i(q,t)\nabla_i + T(t)\frac{\partial}{\partial t} + \frac{1}{\hbar}\varphi(q,t), \tag{7.9}$$

where the vector field X and scalar fields T and φ are real and analytic. Conditions on X, T and φ insuring that N generates a symmetry group of equation (7.4), i.e., transforming solutions of (7.4) to other solutions, are know as the defining equations of this group [29]. When they are satisfied, one shows the following

Theorem 7.2. [30, 31] *Suppose that the action functional* (7.1) *is invariant under the abovementioned Lie groups of transformations, i.e.,* $\forall[t_0, t_1]$ *in the time interval of existence of the Bernstein process,*

$$E_{t_0}\int_{t_0}^{t_1}\Big(\frac{1}{2}|D_s^+X_s|^2+V(X_s)\Big)\,ds$$
$$=E_{\tau_1}\int_{\tau_0}^{\tau_1}\Big(\frac{1}{2}|D_\tau^+Q_\tau|^2+V(Q_\tau)\Big)\,d\tau-\alpha E_{t_1}\int_{t_0}^{t_1}D_s^+\varphi(X_s,s)\,ds\,+o(\alpha). \quad (7.10)$$

In (7.10), Q_τ *denotes the one-parameter family of diffusions resulting from the change of space-time variables associated with* (7.9), *i.e.,*

$$Q=q+\alpha X(q,s)+o(\alpha), \qquad \tau=s+T(s)+o(\alpha),$$

for X, T *and* φ *solving the determining equations of the symmetry group of equation* (7.4).

Then, along the Bernstein diffusion X_s

$$D_s^+(v_iX^i+ET-\varphi)(X_s,s)\equiv D_s^+M_s(X_s)=0, \quad a.s., \quad (7.11)$$

i.e., M_s *is a* $\mathbb{P}_{\mu_{t_0}}$ *continuous local martingale over* $\mathcal{F}_s^{t_0}$.

Let us now specialize this result to the simplest (flat) realization of the fundamental example, namely $E=\mathbb{R}$, $d\alpha=dq$, $V=0$, i.e. (7.4) of the form $\hbar\partial\phi/\partial t+(\hbar^2/2)\Delta\phi=0$.

Here is the list of the martingales M_t^i, $i=1,\dots,6$, associated by Noether's Theorem with the symmetry group of the one-dimensional free heat equation [30]:

$$\begin{aligned}&M_t^1=1, \qquad M_t^2=E(X_t,t), \qquad M_t^3=v(X_t,t),\\&M_t^4=v(X_t,t).t-X_t, \qquad M_t^5=v(X_t,t).X_t+E(X_t,t).2t\\&M_t^6=v(X_t,t).X_t.t+E(X_t,t)t^2-\frac{1}{2}(X_t^2-\hbar t),\end{aligned} \quad (7.12)$$

where the velocity and energy are given respectively by

$$v(q,t)=\hbar\frac{\nabla\phi}{\phi}(q,t) \quad (7.13)$$

and

$$E(q,t)=-\Big(\frac{1}{2}v^2+\frac{\hbar}{2}\nabla v\Big)(q,t). \quad (7.14)$$

Consider the corresponding situation in quantum mechanics, i.e., for the free Schrödinger's equation $i\hbar\partial\psi/\partial t+(\hbar^2/2)\Delta\psi=0$ in $L^2(\mathbb{R})$, instead of the abovementioned free heat equation. Then the construction of Bernstein processes, relying heavily on the positivity of the integral kernel of the Brownian Motion transition semigroup, falls down [32]. Consider, however, the solution of the (Heisenberg's) free quantum equation of motion. In this case, since the operator equations of motion are linear, the free Hamiltonian $H=-(\hbar^2/2)\Delta$ generates the classical automorphism [33] and therefore

$$\begin{pmatrix}Q(t)\\P(t)\end{pmatrix}=\begin{pmatrix}1&t\\0&1\end{pmatrix}\begin{pmatrix}Q\\P\end{pmatrix}, \quad (7.15)$$

where $Q:\psi(q)\to q\psi(q)$ and $P:\psi(q)\to -i\frac{\partial}{\partial q}\psi(q)$ denote respectively the Cartesian position and momentum observables, self-adjoints on $\mathcal{D}(H)\subset L^2(\mathbb{R},dq)$. (Let us recall that if ψ_t denotes the solution $U_t\psi_0$ of Schrödinger's equation in terms of the one-parameter unitary group of evolution $U_t = \exp((-i/\hbar)Ht)$ then $Q(t) = U_t^{-1}QU_t$ and $P(t) = U_t^{-1}PU_t$.)

We claim that the probabilistic counterpart of the solution (7.15) in Euclidean Quantum Mechanics is simply

$$\begin{pmatrix} X_t \\ D_t^+ X_t \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M_t^4 \\ M_t^3 \end{pmatrix}, \tag{7.16}$$

where M_t^3 and M_t^4 are two of the martingales of (7.12) associated by Noether's Theorem with this free evolution. Here we have

$$D_t^+ = \frac{\partial}{\partial t} + v\frac{\partial}{\partial q} + \frac{\hbar}{2}\frac{\partial^2}{\partial q^2} \tag{7.17}$$

for v as in (7.13). Since M_t^3 is a martingale, it is clear that $D_t^+ D_t^+ X_t \equiv D_t^+ v = 0$. Since the energy E is another one (M_t^2 according to (7.12)), $D_t^+ E(X_t, t) = 0$ so that the equations of motion (7.8) for $V = 0$ are satisfied. Also notice that the two martingales involved in (7.16) play the role of initial conditions of (7.15) (read classically or quantically) and that the concept of solution involved here is interestingly distinct from the usual one of a SDE.

The fact that, for classical dynamical systems (i.e., in EQM perspective, the singular limit $\hbar = 0$), the free equation of motion and its solution contain the complete geometric essence of the idea of integrability [34] suggests the following

Conjecture. *For any system governed by a.e. equations of motion of the form* (7.8) *with a special class of scalar potentials V, one can define a concept of stochastic integrability so that, using the martingales predicted by Noether's Theorem, we have existence and uniqueness of their solution, on the model of* (7.16)*.*

Such a result would involve various extensions of basic concepts of the theory of smooth (classical) dynamical systems, very natural from the point of view of the quantization. Indeed, as mentioned in the introduction, the whole point of Euclidean quantum mechanics is to shed a new light on the relations between regular quantum physics and probability theory. In particular, it has been shown [32], in the context of what we call here the fundamental example 3.3, that our stochastic approach to symmetries of Bernstein measures translates into a quantum result richer than those known via Hilbert space methods (to prove this result within the full generality of the present paper is another open problem).

In other words, the point of EQM is to show that, instead of being an artificial way to approach quantum physics, the theory of probability and stochastic analysis may help us to have a deeper understanding of quantum reality.

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