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ON THE GEOMETRY OF THE HAMILTON-JACOBI-BELLMAN EQUATION

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Abstract. We show how a minimal deformation of the geometry of the classical Hamilton-Jacobi equation provides a probabilistic theory whose cornerstone is the Hamilton-Jacobi-Bellman equation. This is the basis for a novel dynamical system approach to Stochastic Analysis.

1. **Stochastic deformation of classical dynamical systems.** The geometrical study of the Hamilton-Jacobi theory lies at the heart of Analytical Mechanics.

Hamilton's original motivation to study light rays in terms of wave fronts, as related to Fermat's variational principle, became almost irrelevant (at least in Jacobi's perspective) when it was realized that the dynamical equations of motion of Analytical Mechanics have the same structure as those characterizing the critical orbits of Fermat's principle.

What we call today classical Hamilton-Jacobi theory was created by Jacobi during the first half of the 19th century. He introduced, in particular, the notion of complete solution of the Hamilton-Jacobi (HJ) equation and was thereby able to construct all solutions of Hamilton's canonical equations in phase space. This original and seemingly indirect integration method along with the new difficult problems it allowed to solve are at the origin of the theory of integrable systems, whose modern ramifications such as Algebraic Geometry are impressive.

Along the way, Jacobi developed the geometric theory of transformations (diffeomorphisms) of phase space that leave invariant the form of Hamilton's canonical equations. He proved, in fact, that a complete solution of the HJ equation generates such a transformation.

In this review article we wish to explain what a Stochastic Deformation of Jacobi's program should look like. Of course, there are many ways to introduce a random noise into Analytical Mechanics. A recent interesting possibility is partly inspired by works of J.-M. Bismut (cf. [4, 16] and references therein). We shall, therefore, try to show why the choice we advocate is natural.

We claim our way is natural because our deformation is essentially the same as that which makes Quantum Mechanics look as a deformation of Analytical Mechanics.

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The influence of the classical Hamilton-Jacobi theory in the original elaboration of Quantum Mechanics is very well documented (cf [9, 20] for example). Although slightly less obvious, it was also instrumental in Feynman's path integral approach.

Nowadays, though it may happen that one appeals to the underlying classical HJ model for the heuristics of the Hilbert space analysis regarding some specific issues of quantum mechanics [11], this is far from being usual.

Of course, quantum deformation cannot really be looked upon as a Stochastic Deformation since, for instance, Feynman's probability measures on the various path spaces involved in his approach are known to belong to the realm of mathematical fiction, even if many of his other original ideas can be formulated rigorously, cf. [1]. Nevertheless, the path integral method constitutes a beautiful structural guide for us.

The Hamilton-Jacobi-Bellman equation, interpreted as the proper stochastic deformation of the classical HJ equation, is associated with a new kind of intrinsically random dynamical systems whose properties can be described by the deformation of the corresponding classical geometric tools. The rules of the deformation are imposed by the nature of the underlying stochastic processes, in our case some specific diffusion processes for the sake of simplicity of our presentation.

We shall also show why our way provides a new approach to Stochastic Analysis, a field of Mathematics which has developed mostly without this basic reference to classical dynamical systems theory and which, therefore, may look a bit impenetrable to non-specialists.

The remaining part of our article will be organized as follows:

In Section 2 we summarize the part of Stochastic Analysis we need to grasp in order to understand what we mean by "Stochastic Deformation". Although this part is not very extensive, the aspects of Stochastic Analysis stressed there are certainly not the traditional ones in this field. This is due to the fact that for our deformation of classical dynamical systems, we are not really interested in the fine sample paths properties of the underlying stochastic processes but, instead, in some of their conditional expectations. In particular, all observables of our original classical dynamical system will be deformed into conditional expectations. The infinitesimal generator of the Markov processes and the associated concept of martingale will be the main tools of our approach. In Section 3 we describe the stochastic deformation of the classical Hamilton-Jacobi theory. Starting from a regularized notion of Action functional, we obtain a deformed almost sure characterization of its critical processes. The way we choose here is to deform Carathéodory's "Royal road to the classical calculus of variations" (Boerner, 1953), thereby justifying our interpretation of the Hamilton-Jacobi-Bellman equation. Then Cartan's differential form representation of the geometry of HJ is appropriately deformed as well and we indicate some probabilistic consequences. We conclude with an explicit example and some prospects of this stochastic deformation program.

2. **A very basic summary of stochastic calculus.** The most basic of all diffusion process is the Brownian motion, which was constructed mathematically by N. Wiener in 1923.

Given a $q \in \mathbb{R}^n$, consider the Gaussian kernel:

$$h_0(q,\tau,z) = (2\pi\hbar\tau)^{-n/2} \exp\left\{-\frac{|q-z|^2}{2\hbar\tau}\right\},\ z \in \mathbb{R}^n,\ \tau > 0 \tag{2.1}$$

where $\hbar$ is a positive constant.

For a k-discretization of the real line $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \cdots \leqslant \tau_k$, a measure on $(\mathbb{R}^n)^k$ is defined on Borel sets B_i by

$$m_{\tau_1\ldots\tau_k}(B_1 \times \cdots \times B_k) = \int_{B_1\times\cdots\times B_k} h_0(q,\tau_1,x_1)h_0(x_1,\tau_2-\tau_1,x_2)\ldots h_0(x_{k-1},\tau_k-\tau_{k-1},x_k)\,dx_1\ldots dx_k. \tag{2.2}$$

By Kolmogorov's extension theorem there is a probability space (Ω_q, σ, P_q) and a stochastic process $w(\tau)$ on $\Omega_q$whose finite dimensional distributions

$$P_q(w(\tau_1) \in B_1, \ldots, w(\tau_k) \in B_k)$$

coincide with (2.2). This is the Brownian motion with variance $\hbar$ starting from q. The paths $\tau \mapsto w(\tau)$ can be chosen to be continuous, so that any $\omega \in \Omega_q$ can be interpreted as the label of a path in $\Omega_q = \{\omega \in C(\mathbb{R}^+;\mathbb{R}^n)\, s.t\ \omega(0) = q\}$, the "Wiener space". This is, for instance, Feynman's notion in his path integral approach to Quantum Mechanics and it will be quite appropriate for us as well.

It follows from (2.1) that the Brownian motion is Gaussian. All its increments

$$w(\tau_2) - w(\tau_1), w(\tau_3) - w(\tau_2), \ldots, w(\tau_k) - w(\tau_{k-1})$$

are independent random variables since

$$E_q[(w(\tau_l) - w(\tau_{l-1}))(w(\tau_m) - w(\tau_{m-1}))] = 0 \text{ for } \tau_l < \tau_m$$

where E_q denotes the expectation computed with P_q.

In 1933 Paley, Wiener and Zygmund proved the most emblematic property of the Brownian motion $w(\cdot)$, namely that its paths are almost surely nowhere differentiable. All our processes $z(\tau)$ built from $w(\tau)$ will inherit this property, which will require a deformation, or regularization, of the most elementary tools of classical kinematics and dynamics.

Given a probability space (Ω, σ, P) it is necessary to introduce an increasing family $\mathcal{P}_t$, $t \geqslant 0$, of sub-algebras of σ, a "filtration", such that $\mathcal{P}_s \subset \mathcal{P}_t \subset \sigma$ for any $s < t$. A stochastic process $z(\tau)$ will be called adapted to $\mathcal{P}_\tau$ if $z(\tau)$ is $\mathcal{P}_\tau$-measurable, for all $\tau \geqslant 0$. The filtration $\mathcal{P}_t$ generated for instance by $\{z(s); 0 \leqslant s \leqslant t\}$ will represent the past information about its observation until time t.

One way to express Markov's property of a stochastic process $z(\tau)$ is to say that

$$P(z(\tau) \in B|\mathcal{P}_t) = P(z(\tau) \in B|z(t)) = P(t, z(t), \tau, B),\ t < \tau.$$

The right-hand side defines the transition distribution of the process and is a conditional probability, the condition being the fact that $z(t)$ is known. For our Brownian motion the density is $h_0(q, \tau - t, z)dz$.

For any measurable φ such that the following integral makes sense we can define

$$\int P(t, q, \tau, dz)\varphi(z) \equiv E_{q,t}[\varphi(z(\tau))] \quad t < \tau. \tag{2.3}$$

The right-hand side is, therefore, a conditional expectation which we can also write more explicitly as

$$E[\varphi(z(\tau))|z(t) = q].$$

Let us consider a $\mathbb{R}^n$-valued diffusion $z(\tau)$ built in term of the Brownian motion in the sense that it solves Itô's $(\mathcal{P}_\tau)$-stochastic differential equation (SDE)

$$dz(\tau) = B(z(\tau), \tau)d\tau + \hbar^{1/2}dw(\tau) \tag{2.4}$$

where B is a Lipschitz vector field.

Its Kolmogorov generating operator or, simply, its infinitesimal generator is defined by the following regularization of right derivative

$$Df(q,t) = \lim_{\Delta t \downarrow 0} E_{q,t}\left[\frac{f(z(t+\Delta t), t+\Delta t) - f(z(t),t)}{\Delta t}\right] \tag{2.5}$$

$$= \left(\frac{\partial}{\partial t} + B_i \nabla^i + \frac{\hbar}{2}\Delta\right) f(q,t) \tag{2.6}$$

for any $f : \mathbb{R}^n \times I \to \mathbb{R}$ in the domain $\mathcal{D}_D$, with continuous right-hand side of (2.6). Taking for f any component z_j, we observe that

$$Dz(t) = B(z(t),t) \tag{2.7}$$

so that the drift term in the SDE (2.4) is a mean forward derivative, knowing $z(\tau)$, while $\hbar$ provides the scale of fluctuations of $z(\tau)$ around its mean motion described by the drift.

Let $m(\tau)$ be adapted to the filtration $\mathcal{P}_\tau$. It is called a $(\mathcal{P}_\tau)$-martingale if $E[|m(\tau)|] < \infty$ for any $\tau \geqslant 0$ and

$$E[m(\tau)|\mathcal{P}_t] = m(t) \quad \text{for all} \quad \tau \geqslant t \geqslant 0. \tag{2.8}$$

When m is a function of a diffusion $z(\tau)$ solving SDE (2.4) the comparison of (2.8) with (2.5) shows that $f(z(t),t)$ is a $\mathcal{P}_t$ martingale if, and only if,

$$Df(z(t),t) = 0. \tag{2.9}$$

Notice that if, informally , $\hbar = 0$ in (2.4) then f is a martingale whenever it is a classical first integral of the limiting ODE (2.4). In the general case, of course, the condition (2.9) does not mean that f is a true numerical constant.

As an illustration, let us consider again the one-dimensional Brownian motion $z(\cdot) = w(\cdot)$ as in (2.4). From what we said before, it is easy to check that

$$Dw(t) = B(w(t),t) = 0 \tag{2.10}$$

but this means that the Brownian motion itself is a nontrivial martingale. A slightly more surprising martingale is

$$m(w(t),t) = w^2(t) - \hbar t. \tag{2.11}$$

Let us observe, in particular, that it is not the square of the above Brownian martingale which produces another martingale. In fact $Dw^2(t) = \hbar$. The irregularities of Brownian paths still manifest themselves in the $\hbar$-dependent extra term of (2.11), a nice anticipation of the stochastic deformation we are going to explore systematically afterwards.

Along the same vein, for any f as in (2.5) we consider the following expression

$$f(z(u),u) - f(q,t) - \int_t^u Df(z(\tau),\tau)d\tau. \tag{2.12}$$

It is clearly a $\mathcal{P}_u$-martingale (just act on (2.12) with D, where in fact the notation D_u could be more appropriate). Now we apply to (2.12) the conditional expectation $E_{q,t}[\cdot]$. The resulting relation

$$E_{q,t}\int_t^u Df(z(\tau),\tau)d\tau = E_{q,t}[f(z(u),u)] - f(q,t) \tag{2.13}$$

is called Dynkin's formula.

The best way to understand the consequences of the irregularities of the paths $\tau \mapsto z(\tau)$ regarding the solutions of (2.4) is to consider the notion of integral along them.

The integral in (2.12), for instance, can be understood in the familiar context of the Riemann-Stieltjes theory. But this is not the case with

$$\int_t^u p(z(\tau),\tau)dz(\tau) \tag{2.14}$$

independently of the regularity of p, an integral we will need afterwards.

Indeed, the paths $\tau \mapsto z(\tau)$, although continuous, are not of bounded variation. This means that in spite of what the notation (2.14) can suggest, there is no way to define such an integral in a pathwise way. The solution of this puzzle is precisely the purpose of K. Itô's stochastic differential and integral calculus [13]. As a matter of fact, we shall need to deal here mostly with expectations of integrals like (2.14) and therefore we will use a small part of this calculus. Itô has shown that there are three natural ways to make sense of (2.14) [14]. The first one is called forward (Itô's) stochastic integral with respect to the past filtration $\mathcal{P}_u$ and is defined as the limit in probability of the following time discretization

$$\int_t^u p \cdot dz(\tau) = \underset{\substack{\max|\tau_j-\tau_{j-1}|\to 0\\ 1\leqslant j\leqslant N}}{l\cdot i\cdot p} \sum_{j=1}^{N} p(z(\tau_{j-1}),\tau_{j-1})(z(\tau_j)-z(\tau_{j-1})). \tag{2.15}$$

This is the original Itô's stochastic integral. Let us recall that in sharp contrast to Riemann's integrals, it is essential for p in (2.15) to be evaluated at the past time τ_{j-1} when the increment $dz(\cdot)$ points toward the future.

To understand the consequences, consider Itô's famous formula for the above Brownian motion $z(\tau)=w(\tau)$. If $G(z)$ is a $C^2(\mathbb{R})$-function with compact support this formula says that

$$G(w(u))-G(w(t)) = \int_t^u G'(w(\tau))dw(\tau) + \frac{\hbar}{2}\int_t^u G''(w(\tau))d\tau. \tag{2.16}$$

Let $G(z)$ be a primitive of $g(z)$ such that $G(q)=0$. Then (2.16) allows us to rewrite the stochastic integral more traditionally as

$$\int_t^u g(w(\tau))dw(\tau) = G(w(u)) - \frac{\hbar}{2}\int_t^u g'(w(\tau))d\tau. \tag{2.17}$$

Consider $G(z)=\frac{z^2-q^2}{2}$, suitable for $g(z)=z$. Then

$$\int_t^u w(\tau)dw(\tau) = \frac{w^2(u)-q^2}{2} - \frac{\hbar}{2}(u-t).$$

This relation is generally formulated for the so-called "standard" Brownian motion starting at $t=0$ from the origin $q=0$ as

$$\int_0^u w(\tau)dw(\tau) = \frac{1}{2}(w^2(u)-\hbar u). \tag{2.18}$$

Now, according to (2.11), the right-hand side is a $\mathcal{P}_u$-martingale. This observation is general and describes the fundamental property of Itô's integral: as a function of its upper time limit $\bullet$ under the integrability condition $\int_t^u E|g|^2 d\tau < \infty$, for all $u<T$ in (2.17), it is a $\mathcal{P}_\bullet$ martingale. Fortunately, when we have to handle only expectations of stochastic integral expressions, the mean forward derivative of

the underlying process provides the needed information, without dealing with the delicate features of Itô's calculus.

Indeed, under the same kind of integrability condition for p as before, the expectation of (2.14) becomes the Riemann integral

$$E\int_t^u p(z(\tau))dz(\tau) = E\int_t^u p(z(\tau))Dz(\tau)d\tau. \tag{2.19}$$

This follows from the properties of a conditional expectation like (2.5) with respect to the absolute expectation denoted by E.

As we said before, in definition (2.15) the increment $dz(\tau)$ points toward the future. This is at the origin of the martingale property of this Itô's $\mathcal{P}$-stochastic integral. Since the coefficients in (2.4) can be regarded as conditional expectations, given the past configuration $z(\tau)$, this aspect lies at the heart of Itô's calculus considered here as an $(\hbar)$-deformation of Leibniz's classical calculus.

On the other hand, for the same kind of reasons, it is also clear that an integral like (2.15) cannot be time-symmetric. There is another notion of stochastic integral which preserve the time-symmetry but which loses the fundamental martingale property. In geometrical contexts it has, however, proved to be useful. Itô calls "Fisk-Stratonovich" this integral, which is defined by

$$\int_t^u p\circ dz(\tau) = \underset{\substack{\max|\tau_j-\tau_{j-1}|\to 0\\ 1\leqslant j\leqslant N}}{l\cdot i\cdot p}\sum_{j=1}^N \frac{1}{2}(p(z(\tau_{j-1}),\tau_{j-1})+ p(z(\tau_j),\tau_j))(z(\tau_j)-z(\tau_{j-1})) \tag{2.20}$$

rather than by (2.15).

Itô has also indicated the differential relation between this symmetric integral and his past stochastic definition (2.15), namely,

$$\int_t^u p\circ dz(\tau) = \int_t^u p\cdot dz(\tau) + \frac{\hbar}{2}\nabla\cdot p\, d\tau. \tag{2.21}$$

In particular, when $p(z) = z$, for $z(\tau) = w(\tau)$ as before, we obtain

$$\int_t^u w(\tau)\circ dw(\tau) = \int_t^u w(\tau)\cdot dw(\tau) + \frac{\hbar}{2}(u-t) \tag{2.22}$$

$$= \frac{w^2(u)}{2} - \frac{q^2}{2}. \tag{2.23}$$

But this means that Stratonovich's integral satisfies the chain rule of Leibniz's classical calculus. This is, in fact, almost its only advantage over Itô's integral (2.15).

Given what we said about the relation between the time asymmetric Itô stochastic integral and Stratonovich's integral, it is hard to avoid the issue of time reversal for diffusion processes. This is still a largely ignored part of Stochastic Analysis even by specialists of the field, so we are bound to insist on it: in a program of stochastic deformation of elementary classical dynamical systems, this aspect is indeed important since most of them are invariant under time reversal.

Here, the fastest way to proceed is to observe that Itô gave a third definition besides (2.14) and (2.20) (cf [14]). It is the mirror image of (2.15) but for a decreasing filtration $\mathcal{F}_t$, representing the future information about our system after time t.

Physically, $\mathcal{F}_t$ can be interpreted as a time reversal of the past filtration $\mathcal{P}_t$. Then Itô's definition of this "backward" stochastic integral is

$$\int_t^u p * dz(\tau) = \underset{\substack{\max|\tau_j-\tau_{j-1}|\to 0\\ 1\leqslant j\leqslant N}}{l\cdot i\cdot p} \sum_{j=1}^{N} p(z(\tau_j),\tau_j)(z(\tau_j)-z(\tau_{j-1})), \tag{2.24}$$

which should be compared with (2.15). Its relation with the symmetric integral transforms (2.21) into

$$\int_t^u p\circ dz(\tau) = \int_t^u p * dz(\tau) - \frac{\hbar}{2}\nabla\cdot p\, d\tau. \tag{2.25}$$

So the deformation term changes its sign with respect to (2.21). Of course, by construction when $\hbar = 0$ the three concepts of stochastic integrals coincide with the Riemann's notion.

For much more about Stochastic Analysis and, in particular, its geometrical content, cf. [13], [18].

3. **Stochastic deformation of the Hamilton-Jacobi theory.**

Along the paths of any diffusion process built in term of the Brownian motion, the classical concept of action functional is clearly inappropriate. Indeed it follows from properties of the Brownian motion that this classical action becomes divergent. However, the tools for its regularization have already been prepared.

In connection with the main theme of these notes, we shall restrict ourselves to the elementary functional, the action functional

$$J[z(\cdot)] = E_{q,t}\int_t^u L(z(\tau), Dz(\tau))d\tau \tag{3.1}$$

for a Lagrangian $L:\mathbb{R}^{2n}\to\mathbb{R}$ of the form $L(q,\dot q) = \frac{1}{2}|\dot q|^2 + V(q)$.

The domain of J, $\mathcal{D}_J$, is a set of $\mathbb{R}^n$-valued processes $z(\cdot)$ on (Ω,σ,P) starting from $q : z_t = q$, adapted to an increasing filtration $\mathcal{P}_\tau$, $\tau\in[t,u]\subset\mathbb{R}$, with a fixed covariance $I\hbar$, where $\hbar$ is a positive constant and I the $n\times n$ identity matrix. Note that even non-Markovian processes are admissible. It is only required that they admit a regularized time derivative $Dz(\tau)$ such that $E_{q,t}\int_t^u |Dz(\tau)|^2 d\tau < \infty$.

It is assumed that Dynkin's formula holds for a large enough subset in the domain of D, Kolmogorov's generating operator for Markovian processes $z(\cdot)$ in $\mathcal{D}_J$, of the general form

$$Df = \frac{\partial f}{\partial t} + v\nabla f + \frac{\hbar}{2}\Delta f \tag{3.2}$$

for any $f:\mathbb{R}^n\times I\to\mathbb{R}$, continuous along with its partial derivatives up to the second order. The constant $\hbar > 0$ will be our deformation parameter. It follows that D is an uniformly parabolic partial differential operator. Its first-order term v is the as yet unspecified drift of $z(\cdot)$.

Recall that, by construction of (3.2), $Dz = v$.

Finally, the "potential" $V:\mathbb{R}^n\to\mathbb{R}$ will be a regular element of the Kato class (cf. [7]).

Under these conditions, which in fact can be considerably weakened, the problem to find a minimal or simply critical diffusion for the action J is well defined, and we have the

Theorem 3.1. *The critical point of the Action (3.1) in $\mathcal{D}_J$ is a diffusion process solving almost surely the system of deformed Euler-Lagrange equations*

$$D\frac{\partial L}{\partial Dz} - \frac{\partial L}{\partial z} = 0, \qquad t \leqslant \tau \leqslant u, \tag{3.3}$$

together with $z(t) = q$, $Dz(u) = 0$. Namely, for the above mentioned elementary Lagrangian we have

$$DDz(\tau) = \nabla V(z(\tau)) \tag{3.4}$$

with the same boundary conditions.

Let us observe that the idea of such a deformed least Action Principle is due to K. Yasue [22]. As often with new ideas, [22] was weakened by serious technical difficulties, which were partly due to the unspecified class of diffusions used in carrying out the variation. The class of critical processes needed here, distinct from Yasue's class, was introduced in [23] and has since then been called Bernstein class or class of reciprocal processes.

Instead of giving the proof of (3.3) which can be found, for instance, in [7] for a wider class of Lagrangians, we are going to propose an alternative and more geometric demonstration, inspired by Carathéodory's "royal road to the classical calculus of variations".

We shall assume that we can define a vector momentum by

$$p = \frac{\partial L}{\partial Dz}(z, Dz) \tag{3.5}$$

in such a way that (3.5) can be solved for Dz, i.e that L is strongly non-degenerate. For our elementary case, of course, p coincides with Dz.

Suppose that we are given a scalar C^2-function $S = S(q,t)$ in an open set G of $\mathbb{R}^n \times \mathbb{R}$. For $z(\cdot)$ any diffusion as before, let us introduce, using (3.2),

$$DS = \frac{\partial S}{\partial \tau} + Dz\nabla S + \frac{\hbar}{2}\Delta S. \tag{3.6}$$

With this we can consider another Lagrangian for the Action (3.1) namely,

$$\widehat{L}(z, Dz, \tau) = L(z, Dz) + \frac{\partial S}{\partial \tau} + Dz\cdot\nabla S + \frac{\hbar}{2}\Delta S. \tag{3.7}$$

Proposition 1. *The Action $\widehat{J}$, built with the Lagrangian $\widehat{L}$, is equivalent to J in the sense that its Euler-Lagrange equations are the same. Equivalently, DS can be called a deformed null Lagrangian.*

The proof amounts to checking that for $L(z, Dz, \tau) = \frac{\partial S}{\partial \tau}(z,\tau) + Dz\cdot\nabla S(z,\tau) + \frac{\hbar}{2}\Delta S(z,\tau)$, Eq.(3.3) is identically satisfied. Since, by (3.2) for $v = Dz$, we have $D\nabla S = \frac{\partial}{\partial \tau}\nabla S + Dz\Delta S + \frac{\hbar}{2}\Delta\nabla S$, this is indeed the case. As a matter of fact, by Dynkin's formula,

$$E_{qt}\int_t^u DS(z(\tau),\tau)d\tau = E_{qt}S(z(u),u) - S(q,t)$$

and it is easy to check that the use of $\widehat{L}$ instead of L changes only the final boundary condition of Eq (3.3) into

$$Dz(u) = -\nabla S(z(u),u). \tag{3.8}$$

Let us introduce the deformation of what Carathéodory call a "Geodesic field" (other expressions might be "Mayer field" or "Extremal field", cf. [10]).

This will be the data of a vector field $B = B(z, \tau)$ at each point of a finite region G, where "vector field" has to be understood in the sense of our deformed (almost sure) ODE. Namely

$$Dz = B(z, \tau) \tag{3.9}$$

for $z(\cdot)$ admitting a generator like (3.2).

Then we require that, for an appropriate scalar function S, as in (3.7), $\widehat{L}\,(z, Dz, \tau) \geqslant 0$, the equality holding true only when $Dz = B(z, \tau)$. So, after an integration along a diffusion path $\tau \mapsto z(\tau)$, $t \leqslant \tau \leqslant u$, between $q \in \mathbb{R}^n$ and a fixed random variable $z(u)$, the path γ such that (3.9) holds is a minimum of $\widehat{J}$ and therefore, by the proposition, of J as well.

Since $\widehat{L}$ assumes a minimal value for Dz as in (3.9), $\frac{\partial \hat{L}}{\partial Dz} = 0$ for these. But by (3.7) and (3.5) we have

$$\frac{\partial \widehat{L}}{\partial Dz} = \frac{\partial L}{\partial Dz} + \nabla S \equiv p + \nabla S \tag{3.10}$$

so that for a Mayer field we set

$$p = -\nabla S. \tag{3.11}$$

Since L is non-degenerate, the $Dz = B$ needed for fields are, therefore, determined and for our elementary Lagrangian $Dz = B(z, \tau) = p = -\nabla S(z, \tau)$.

Since $\widehat{L} = 0$ on the minimum,

$$L(z, B, \tau) = -\frac{\partial S}{\partial \tau} - B\nabla S - \frac{\hbar}{2}\Delta S. \tag{3.12}$$

We shall define the energy function h by

$$h = -\frac{\partial S}{\partial \tau} \tag{3.13}$$

For our elementary Lagrangian, (3.12) amounts to

$$\frac{\partial S}{\partial \tau} - \frac{1}{2}|\nabla S|^2 + \frac{\hbar}{2}\Delta S + V = 0. \tag{3.14}$$

This is the Hamilton-Jacobi-Bellman equation (HJB) for the scalar function $S = S(z, \tau)$. A solution of HJB provides a vector field p (3.11) and, therefore, for non-degenerate L, the field B, (3.9), defining a Mayer field.

Today it would be more common to say that in a simply connected set $G \subset \mathbb{R}^n \times \mathbb{R}$, the graph $\Sigma = \{(t, z, p(z, t)), (z, t) \in G\}$ defines a deformed Lagrangian sub-manifold if there is a C^1-scalar function S there such that both (3.11) and (3.14) hold true.

When $\hbar = 0$ this reduces to the classical definition for one of the two (adjoint with respect to the time parameter cf [6]) Hamilton-Jacobi equations underlying the classical dynamical system.

Equation (3.14) is well-known in the stochastic optimal control literature (cf. [8] for instance). As can be guessed from Theorem 3.1, the specific contribution of our stochastic deformation perspective will be to provide a more comprehensive connection with the whole structure of classical dynamical system theory.

If p is more regular than continuous, say $p \in C^2(G)$, then a (deformed) Mayer field, or a Lagrangian manifold, can be characterized by the integrability conditions following from (3.11) and (3.14) (elementary case):

$$\frac{\partial p^i}{\partial z_k} = \frac{\partial p^k}{\partial z_i} \qquad 1 \leqslant i,\, k \leqslant n \tag{3.15}$$

and

$$\frac{\partial p^i}{\partial \tau} + p^k \nabla^i p_k + \frac{\hbar}{2} \nabla^i \nabla^k p_k = \nabla^i V. \tag{3.16}$$

According to (3.2) the l.h.s of (3.16) coincides with Dp^i but, then, the deformed Euler-Lagrange equation (3.3) is satisfied. This is why some authors, including sometimes Carathéodory himself, refer to such fields as "Extremal fields" when $\hbar = 0$.

In particular, this also means that the solutions to Eqs (3.15)-(3.16) allow us to compute the Action (3.1) along the paths of a deformed Mayer field using only the scalar function S defining this field. Indeed, if follows from (3.12) that $L(z, Dz, \tau) = -DS(z, \tau)$ and therefore, by Dynkin's formula

$$J[z(\cdot)] = E_{qt} \int_t^u L d\tau = S(q, t) - E_{qt} S(z(u), u). \tag{3.17}$$

Before we proceed, we should explain why there are really well-defined diffusion processes behind the deformation of classical dynamics described above. And we should use them to have a better geometric insight into the origin of this deformation.

The HJB equation (3.14) is a non-linear uniformly parabolic PDE. Although a lot is known about its classical (regular) solution and even considerably more since the advent of the notion of viscosity solution (cf [8]), which is a more appropriate concept since typical solutions are not smooth, it is clear that explicitly solvable HJB equations are scarce.

There are circumstances where such a deformation of classical dynamics is considerably simpler to handle. Interestingly enough, they are the same as for the most famous of all deformation theories, namely, the quantum theory advocated by E. Schrödinger in 1926 (cf [20, 6]). Let us make the following change of variable in our elementary equation (3.14), without loss of generality:

$$S(z, \tau) = -\hbar \log \eta(z, \tau). \tag{3.18}$$

Then η should be a positive solution of the linear equation

$$\hbar \frac{\partial \eta}{\partial \tau} = H \eta \tag{3.19}$$

where H is the quantum Hamiltonian operator

$$H = -\frac{\hbar^2}{2} \Delta + V.$$

Let us consider a solution of Eq (3.19) for $\tau \leqslant u$ and a final positive boundary condition $\eta_u(z)$. Then it was shown in the eighties (cf. [6], [7], [23] and references therein) that there is indeed a diffusion $z(\cdot) \in \mathcal{D}_J$, minimizing the Action J (3.1), built in term of the Brownian motion, with a drift and energy functions given by (3.11), (3.14) and (3.18), namely

$$B(z(\tau), \tau) = -\nabla S(z(\tau), \tau) = \hbar \nabla \log \eta(z(\tau), \tau), \tag{3.20}$$

$$\begin{aligned} h(z(\tau), \tau) &= -\partial_\tau S(z(\tau), \tau) = \hbar \partial_\tau \log \eta(z(\tau), \tau), \\ &= \left(-\frac{1}{2} B^2 - \frac{\hbar}{2} \nabla \cdot B + V \right) (z(\tau), \tau). \end{aligned} \tag{3.21}$$

Geometrically, it should be clear by now that conditions (3.15) and (3.16) deform the classical claim that the Poincaré-Cartan one-form, whose section in the (z, τ)

space is

$$\widetilde{\omega}_{pc} = p^i(z,\tau)dz_i - h(z,\tau)d\tau,$$

is closed on the set G, as well as simply connected by hypothesis so that $\widetilde{\omega}_{pc}$ is exact: there exists a scalar field S on G such that $\widetilde{\omega} = dS$, which is a solution of the classical HJ equation 3.14 for $\hbar = 0$.

Along classical extremals γ we must have, informally,

$$\int_\gamma \widetilde{\omega}_{pc} = \int_t^u L d\tau = S_t - S_u.$$

The deformation of this relation, implicit in (3.17), is as follows:

Proposition 2. *Along an extremal $z(\cdot)$ of our Action functional (3.1) (in the elementary case) we have*

$$\begin{aligned} J[z(\cdot)] &= E_{qt}\int_t^u \widetilde{\omega}_{pc} = E_{qt}\int_t^u p\circ dz(\tau) + h d\tau \qquad (3.22)\\ &= E_{qt}\int_t^u \left\{\frac{1}{2}|Dz(\tau)|^2 + V(z(\tau))\right\} d\tau \end{aligned}$$

where $\circ$ denotes Stratonovich's stochastic integral.

Indeed, using $p = Dz = B = -\nabla S$ and the relation (2.21) between Itô's forward integral and Stratonovich's integral, $\widetilde{\omega}$ can be rewritten as the Riemann integral

$$\begin{aligned} E_{qt}\int_t^u B\circ dz(\tau) &= E_{qt}\int_t^u B\cdot dz(\tau) + \frac{\hbar}{2}\nabla\cdot B d\tau\\ &= E_{qt}\int_t^u \left\{BDz(\tau) + \frac{\hbar}{2}\nabla\cdot B\right\} d\tau. \end{aligned}$$

Together with definitions (3.20) and (3.21), this gives

$$\begin{aligned} E_{qt}\int_t^u \widetilde{\omega} &= E_{qt}\int_t^u \left\{\frac{1}{2}|Dz(\tau)|^2 + V(z(\tau))\right\} d\tau\\ &= -E_{qt}\int_t^u DS(z(\tau),\tau)d\tau = S(q,t) - E_{qt}S(z(u),u). \end{aligned}$$

So the deformation (3.16) of the classical exactness condition for $\widetilde{\omega}_{pc}$ comes from Stratonovich's integral in its deformation (3.22).

The most important result on the geometry of a classical Hamilton-Jacobi equation is the one describing its symmetries. E. Cartan found a especially elegant but unfortunately not well-known formulation of it in terms of "Closed Differential Ideal of Forms" (cf., for instance, [5]). Cartan's notion of a differential ideal is a dual of Frobenius Theorem. Integral submanifolds of the ideal generated by the forms are solutions of the Hamilton-Jacobi equation. Let us summarize Cartan's approach for our classical elementary case:

Proposition 3. *The Hamilton-Jacobi equation can be represented by the following ideal I_{HJ} of differential forms:*

$$I_{HJ}\left\{\begin{array}{ll} \omega = p^i dz_i - E d\tau + dS, & (3.23)\\ \Omega = d\omega = dp^i dz_i - dE d\tau, & (3.24)\\ \beta = \left(E - \frac{1}{2}p^2 - V(q)\right) dz_i d\tau, & (3.25) \end{array}\right.$$

defined on a $2n+3$ space of independent variables (z_i, p^i, τ, E, S), $1 \leqslant i \leqslant n$.

To recover HJ consider the solution $\mathbb{R}^{n+1}$-submanifold where the scalar S become a function of z and τ (this is called "sectioning" and denoted by a $\sim$). Then pullback all forms to zero ("annuling"). For example $\widetilde{\omega} = 0$ provides the classical version of (3.11) and (3.13) (up to a sign convention) and $\widetilde{\Omega} = 0$ the integrability conditions ensuring the exactness of $\widetilde{\omega}$. Then $\widetilde{\beta} = 0$ is the HJ equation itself: $\frac{\partial S}{\partial \tau} - \frac{1}{2}|\nabla S|^2 - V = 0$.

Definition (3.23) shows that it is not only the Poincaré-Cartan form ω_{pc} that is relevant to HJ's geometry. Comparing with (3.12) and (3.22) this should not come to us as a surprise. The proper framework is Contact Geometry, (cf [3]), defined on the odd-dimensional manifold $\mathbb{R}^{2n+3}$ and ω in (3.23) is, indeed, a contact form.

A symmetry of the HJ equation is defined by a ("contact") Hamiltonian vector field

$$N = N_i^z \frac{\partial}{\partial z_i} + N^\tau \frac{\partial}{\partial \tau} + N^S \frac{\partial}{\partial S} + N_i^p \frac{\partial}{\partial p^i} + N^E \frac{\partial}{\partial E} \tag{3.26}$$

whose coefficients $N^\bullet$ are determined by the invariance condition of I_{HJ} under a N-variation, namely

$$\mathcal{L}_N(I_{HJ}) \subseteq I_{HJ}, \tag{3.27}$$

where $\mathcal{L}_N$ denotes a Lie derivative along N.

The deformation of I_{HJ} appropriate to the HJB equation (3.14) (elementary case) is the following ideal

$$I_{HJB} \begin{cases} \omega = B^i dz_i + E d\tau + dS \equiv \omega_{pc} + dS, & (3.28) \\ \Omega = dB^i dz_i + dE d\tau, & (3.29) \\ \beta = \left(E + \frac{1}{2}|B|^2 - V\right) dz^i d\tau + \frac{\hbar}{2} dB^i dt. & (3.30) \end{cases}$$

¿From $\widetilde{\omega} = 0$ we recover the first relations of (3.20) and (3.21). The only deformation term in (3.30) accounts for the $\frac{\hbar}{2}\Delta S$ in (3.14).

Defining a symmetry for I_{HJB} as in (3.27) we obtain a Lie algebra that contains, in particular, an infinite-dimensional sub-algebra $\mathfrak{a}$ whose origin is precisely Schrödinger's change of variable (3.18), the key to our probabilistic interpretation. However, it is the supplement of $\mathfrak{a}$ which provide the computational information on HJB.

We shall simply mention the effect of a N-variation on the main geometrical tools underlying the HJB equation. Even when $n = 1$ the following results are quite tedious to check (cf [17]).

Theorem 3.2. *Along a N-variation of the form (3.26), the ideal I_{HJB} and the Lagrangian L of Action (3.1) satisfy*

1. $\mathcal{L}_N(\omega_{pc}) = -dN^S$.
2. $\mathcal{L}_N(\Omega) = 0$.
3. $\mathcal{L}_N(L) + L\frac{dN^\tau}{d\tau} = -DN^S$.

The proof shows, in particular, that it is sufficient to consider symmetry contact Hamiltonians functions of the form

$$N(z, \tau, S, B, E) = N^z(z, \tau)B + N^\tau(\tau)E + N^S(z, \tau) \tag{3.31}$$

with specific relations between the coefficients N^z, N^τ and N^S (sometimes called "Determining Equations" in Lie group theory). For our elementary case, these are

$$\begin{aligned} \frac{dN^\tau}{d\tau} &= 2\frac{\partial N_i^z}{\partial z_i} \quad \text{(no summation)}, \\ \frac{\partial N_i^z}{\partial \tau} &= -\frac{\partial N^s}{\partial z_i}, \\ -\frac{\partial N^s}{\partial \tau} - \frac{\hbar}{2}\Delta N^s &= \frac{dN^\tau}{d\tau}V + N_i^z\frac{\partial V}{\partial z_i} + T\frac{\partial V}{\partial \tau}, \\ \frac{\partial N_i^z}{\partial z_j} + \frac{\partial N_j^z}{\partial z_i} &= 0 \qquad i \neq j. \end{aligned} \tag{3.32}$$

It is only after sectioning on the solution submanifold (z, τ) that B and E can be identified respectively with (3.20) and (3.21), and that the algebraic claims of the Theorem can be read as probabilistic statements along the paths $\tau \mapsto z(\tau)$ of the diffusion critical for $J[z(\cdot)]$. The third relation then becomes the fundamental invariance formula for our deformed calculus of variations, implying Noether's Theorem (cf [6], [21]). Let us recall this last result, which we will need later on.

Theorem 3.3. ***Stochastic Noether's Theorem***. *In the elementary case, for any coefficients N^z, N^τ, N^s solving the determining equations (3.32), the following scalar field*

$$m(z(\tau), \tau) = (N^z B + N^\tau h + N^s)(z(\tau), \tau)$$

computed along critical points of the Action (3.1), with drift and energy given by (3.20) and (3.21), is a $\mathcal{P}_\tau$-martingale.

All the geometry of the HJB equation is, therefore, encoded in Theorem 3.2. The invariance formula 3), in particular, contains an amazing quantity of geometrical information on the diffusions that are critical for the Action J.

Even if, as shown by the previous probabilistic interpretation of $\widetilde{\omega}_{pc}$, some of this information requires an additional stochastic analysis, we are guided by the fact that the whole framework deforms in a natural way what is known about the geometry of the classical Hamilton-Jacobi equation.

4. **Examples and prospects.**

Example 1. Here, $n = 1$, $V = e + \frac{k^2}{2} = cste$, $L(q, \dot{q}) = \frac{1}{2}|\dot{q}|^2 + \frac{k^2}{2} + e$, in the elementary Action (3.1):

$$J[z(\cdot)] = E_{qt} \int_t^u \left\{ \frac{1}{2}|Dz(\tau)|^2 + \frac{k^2}{2} + e \right\} d\tau - \hbar E_{qt}\left[\log \cos h \left(\frac{k \cdot z(u)}{\hbar} \right) \right]. \tag{4.1}$$

Let us pick an auxiliary Lagrangian $\widehat{L}$ (3.7) of the form

$$\widehat{L}(z, Dz, \tau) = L(z, Dz) + DS(z, \tau) \tag{4.2}$$

where, as suggested by (3.8) and the final boundary condition of Action (4.1),

$$S(z, \tau) = -\hbar \log \cos h \left(\frac{k \cdot z}{\hbar} \right) + e(u - \tau). \tag{4.3}$$

Using Itô's formula (3.6),

$$DS(z(\tau), \tau) = -e - Dz(\tau) k tgh \left(\frac{k \cdot z(\tau)}{\hbar} \right) - \frac{k^2}{2} \cos h^{-2} \left(\frac{k \cdot z(\tau)}{\hbar} \right). \tag{4.4}$$

After an integration over time, Dynkin's formula shows that the boundary condition of (4.1) can be rewritten as

$$E_{q,t}\int_t^u DS(z(\tau),\tau)d\tau + S(q,t). \tag{4.5}$$

So the Action for the auxiliary Lagrangian $\widehat{L}$ reduces to

$$\widehat{J}[z(\cdot)] = E_{qt}\int_t^u \frac{1}{2}\left(Dz(\tau) - ktgh\left(\frac{k\cdot z(\tau)}{\hbar}\right)\right)^2 d\tau + S(q,t) \tag{4.6}$$

whose minimum is clearly reached on

$$Dz(\tau) = -\nabla S(z(\tau),\tau) = B(z(\tau),\tau) = ktgh\left(\frac{k\cdot z(\tau)}{\hbar}\right). \tag{4.7}$$

It follows that the generator of our critical diffusion is of the form (3.2) for $v = B$, namely,

$$D = \frac{\partial}{\partial\tau} + ktgh\left(\frac{k\cdot z}{\hbar}\right)\nabla + \frac{\hbar}{2}\Delta \tag{4.8}$$

and that its energy function (cf (3.21)) is a true constant

$$\begin{aligned} h &= -\frac{1}{2}B^2 - \frac{\hbar}{2}\nabla\cdot B + e + \frac{k^2}{2} \\ &= e. \end{aligned} \tag{4.9}$$

This critical diffusion is built from the positive solution $\eta(z,\tau) = \cos h\left(\frac{k\cdot z}{\hbar}\right)$ $\exp\left\{-\frac{e}{\hbar}(u-\tau)\right\}$ of the heat equation (3.19) for the quantum, free Hamilton operator:

$$H = -\frac{\hbar^2}{2}\Delta + \frac{k^2}{2} + e. \tag{4.10}$$

Coming back to HJB, it is clear that we have achieved a stochastic deformation of a one-dimensional classical Lagrangian submanifold at the energy level e, where $p(z) = -\nabla S$, since S of (4.3) solves the deformed time-independent HJB equation

$$-\frac{1}{2}|\nabla S|^2 + \frac{\hbar}{2}\Delta S + V = e. \tag{4.11}$$

For a general time-dependent situation, the underlying η solves, generally, the time-dependent heat equation (3.19). By (3.20), $Dz(\tau) = B(z(\tau),\tau)$ is an explicit function of the time parameter, in contrast to the above "stationary" example. This is where a serious qualitative difference between the classical dynamics founded on HJ and its deformation, associated with HJB, appears. It follows from Itô's calculus that, along generic diffusion paths $\tau \mapsto z(\tau)$, there is much more than the trivial constants annuled by the generator D of (3.2). Any $\mathcal{P}_\tau$-martingale $m(z(\tau),\tau)$ also satisfies $Dm = 0$ (cf §2). This is actually what makes our stochastic deformation interesting. For example, as seen in Theorem 3.3, the deformation of the classical Noether's Theorem conclusion is the existence of a collection of martingales of the critical process $z(\tau)$. This collection allows us, given such a process, to construct one-parameter families of diffusions which are critical for the same Action functional.

Example 2. To illustrate this aspect, consider the Action with the same free Lagrangian $L = \frac{1}{2}|\dot{q}|^2$ as before, without the additive constant of (4.1) which is dynamically irrelevant for the deformed Euler-Lagrange equation (3.3). The simplest of its critical diffusions corresponds to the choice of the trivial boundary condition

$$S(z(u),u) = -\hbar\log\eta(z(u),u) = cste. \tag{4.12}$$

We can choose $\eta(z,u)=1$, so as to cancel out the boundary term of the Action. Since in this case the solution of Eq (3.19) remains $\eta(z,\tau)=1$ for all $\tau\leqslant u$, the corresponding critical diffusion is characterized by a vanishing drift B and an energy h (cf (3.20)-(3.21)). This process is, therefore, a Brownian motion denoted by $w(\tau)$ (for simplicity, here, the conditioning is $q=t=0$).

On the other hand, given our free Lagrangian $L=\frac{1}{2}B^2$, it is easy to check that $N^z(z,\tau)=2z\tau$, $N^\tau(\tau)=2\tau^2$, $N^S(z,\tau)=z^2-\hbar\tau$ satisfy the invariance relation 3) of Theorem 3.2, and therefore the abovementioned determining equation for $V=0$.

It follows therefore from Noether's Theorem 3.3 that

$$m(z(\tau),\tau)=B(z(\tau),\tau)N^z(z(\tau),\tau)+h(z(\tau),\tau)N^\tau+N^S(z(\tau),\tau) \tag{4.13}$$

is a martingale of any critical diffusion of our free Action functional. In particular for the Wiener process $B=h=0$ we are left with our favorite Brownian martingale

$$m(w(\tau),\tau)=w^2(\tau)-\hbar\tau, \tag{4.14}$$

allowing us to construct the one-parameter family of diffusions

$$z^\alpha(\tau)=(1-\alpha\tau)w\left(\frac{\tau}{1-\alpha\tau}\right)\quad 0\leqslant\tau<\alpha^{-1}. \tag{4.15}$$

For any $\alpha>0$, $z^\alpha(\tau)$ is another critical diffusion of the free Action, coming back to the origin at time α^{-1} and which is called, for this reason, a "Brownian Bridge". In particular, the energy function h of $z^\alpha(\tau)$ is no longer a true constant, like in the stationary case considered before, but a martingale of $z^\alpha(\tau)$.

Here is a probabilistic way to understand the origin of (4.15):

Let us apply D to the left-hand side, after recalling that D acts as a classical time derivative on (deterministic) functions of bounded variation of time, or on time changed processes, and that our Brownian w has zero drift; we obtain

$$Dz^\alpha(\tau)=-\alpha w\left(\frac{\tau}{1-\alpha\tau}\right)\equiv-\alpha\frac{z^\alpha(\tau)}{1-\alpha\tau}.$$

So the drift of the Brownian Bridge z^α is $B^\alpha(q,\tau)=-\frac{\alpha q}{1-\alpha\tau}$, which is singular at $\tau=\alpha^{-1}$, as it should.

Moreover, it is geometrically much more revealing to start from any positive solution η of the free heat equation underlying our free Action, instead of the trivial one $\eta(z,\tau)=1$, for all $\tau\leqslant u$, characterizing the Brownian w.

Let us call $z(\tau)$ the diffusion built from such a η via (3.20). Then the relation between critical diffusions

$$z^\alpha(\tau)=(1-\alpha\tau)z\left(\frac{\tau}{1-\alpha\tau}\right) \tag{4.16}$$

generalizing (4.15) is nothing but the expression of the existence of the following symmetry group of our free heat equation, namely,

$$\eta^\alpha(z,\tau)=(1-\alpha\tau)^{-1}\exp\left\{\frac{-\alpha z^2}{2\hbar(1-\alpha\tau)}\right\}\eta\left(\frac{z}{1-\alpha\tau},\frac{\tau}{1-\alpha\tau}\right)\quad\text{for all }\alpha\in\mathbb{R},$$

meaning that if η is solution so is η^α. It follows from (3.20) that the associated relation between the drifts is

$$B^\alpha(z,\tau)=\frac{-\alpha z}{1-\alpha\tau}+\frac{1}{1-\alpha\tau}B\left(\frac{z}{1-\alpha\tau},\frac{\tau}{1-\alpha\tau}\right),$$

manifestly generalizing in a dynamical systems context the relation regarding the Brownian motion.

Our dynamical system perspective shows that the whole family (4.16) of diffusions solving (3.3) admits the same symmetry as the Wiener diffusion.

Here, the advantage of using systematically the linear structure controlling the dynamics of our process should be obvious. The classical Lie group structure of the underlying heat equation is transferred to the diffusions built from any positive solution of this linear equation.

This kind of relations between critical processes of the same Action functional can be regarded as a manifestation of integrability for the underlying HJB equation.

The prospect, open by Noether's Theorem (3.3), to construct a natural deformation of the classical concept of integrable system should not be underestimated. Not only would it make the HJB equation even more popular and useful (the difficulty in computing its solutions is notorious in the Stochastic Optimal Control literature), but it would also give rise to natural deformations of a number of geometrical tools of classical dynamical system theory (Liouville Theorem, Action-Angle variables, etc...)

Before concluding we should clarify the claim we made at the beginning, namely, that our deformation is the "same as the quantum deformation".

Indeed, what was said there seems to have very little connection with the quantization of our elementary classical dynamical systems.

The first reason is that, as is well-known from Feynman's path integral approach (and in mathematics from the work of R.H. Cameron in 1960), there are no diffusion processes, or, equivalently, no well-defined probability measures on paths space associated with the quantization of such systems. Indeed, in our case, the positivity of the kernel h_0 (2.1) is fundamental for the existence of all probability measures used there.

Nevertheless, we claim that the geometry of HJB studied above is as close as possible to the one underlying quantization. Our diffusions $z(\cdot)$, critical for the Action 3.1, play the role of the Heisenberg quantum observable of position, denoted by $Q(\cdot)$.

In fact, our Stochastic Noether's Theorem 3.3, in its more general formulation, can be "translated" into Hilbert space using the fact that $\tau \to -it$ in (3.19) provides Schrödinger equation ([2]). Such a transformation, although fatal to the probabilistic interpretation of our deformation, is harmless regarding its geometric content. It is simple to check, for instance, that the Hilbert space counterpart of our free martingale (4.14) is the observation that

$$Q^2(t) + i\hbar t \tag{4.17}$$

is a first integral of the quantum system with Hamiltonian $H = -\frac{\hbar^2}{2}\Delta$.

Remarkably enough, the resulting list of quantum first integrals found in that indirect way is strictly larger than the one given in textbooks [2], even for elementary systems.

Regarded as a probabilistic counterpart of Quantum Mechanics, the stochastic deformation described above is known, in Physics, as "Euclidean Quantum Mechanics" (cf., for a recent illustration, [15]).

What do we understand better in quantum mechanics by having this "Euclidean" approach in mind?

The example of the free quantum first integral (4.16) shows what should be expected.

Stochastic Analysis was historically developed without input of Quantum Physics, although the list of failed mathematical attempts to fundamentally connect these two fields, starting with Norbert Wiener himself (N. Wiener, A. Siegel, Phys. Rev. 91 (1953), 1551), is quite impressive. On the quantum side the success of Feynman's path integral strategy, in spite of its hopeless original mathematical structure, suggests that a rigorous (probabilistic) counterpart is still waiting to be discovered (cf. [1],[6]).

The conceptual basis of both theories is so different, however, that such a task is quite ambitious. But it is also why it is interesting: both theories focus on very distinct concepts and some imagination is, therefore, needed to establish the missing connections.

For example, the role of time is considerably more subtle in Stochastic Analysis than in Quantum Mechanics, where its status is the same as in Newtonian mechanics. Even discounting the basic rule of random times, on the probabilistic side, (without the slightest quantum counterpart) the time parameter plays a key role as soon as the Wiener process is defined, notably to obtain the martingales associated with its invariance properties, the simplest one being precisely (4.14). In Quantum Mechanics, the symmetries of the theory are dealt with by unitary group of operators on Hilbert space [12]. Explicitly time-dependent observables are not so easy to analyze in this perspective. This is probably why the quantum version of the basic collection of Wiener martingales, starting with (4.16), was not related, as it should have been earlier, with its probabilistic counterpart. So the answer to the above mentioned question is that Quantum Mechanics can see, along the same way, its foundations enriched by the comparison with Stochastic Analysis. Let us illustrate why the reciprocal claim is also true.

A basic difficulty in trying to establish the connection between these two fields has to do with time-reversal symmetry. The elementary classical dynamical systems considered before are time-reversible. And none of the probabilistic tools we used and none of the conclusions we reached seem to be. How can we claim, as in the Introduction, that our deformation is essentially the same as quantization?

Conceptually, the answer to this apparent puzzle is that, even in Quantum Mechanics, the counterpart of the linear equation (3.19) is not sufficient for the probabilistic interpretation. In particular, the absolute probability that the position of the system, in a state $\psi(z,t) \in L^2$, lies inside an interval Δ, is given by a product form density

$$\int_\Delta \psi\overline{\psi}(z,t)dz$$

where $\overline{\psi}$ denotes the complex conjugate of ψ. The counterpart, for our deformation, is that the absolute probability of our processes $z(\tau)$ is of the product form

$$\int_\Delta \eta\eta^*(z,\tau)d\tau \tag{4.18}$$

where, in addition to the solution of Eq. (3.19), a positive solution η^* of its "adjoint" heat equation

$$-\hbar\frac{\partial \eta^*}{\partial \tau} = H\eta^* \tag{4.19}$$

is required. (4.18) has been called the Euclidean Born interpretation (cf. [23],[6]).

Clearly, the relation between the equations for η and η^* can be interpreted as time-reversal. In the same way as an increasing filtration $\mathcal{P}_\tau$ (the past of $z(\tau)$)

is associated with (3.19), a decreasing one, $\mathcal{F}_\tau$, contains its future information, relevant to (4.19). All calculations made here have a $\mathcal{F}_\tau$-counterpart, probabilistic versions of the complex conjugation.

The processes (or measures) involved in our construction are invariant under time-reversal, in this sense, but any choice of filtration creates an "arrow of time" in their description.

Technically, the transition of $\mathcal{P}_\tau$ to $\mathcal{F}_\tau$ corresponds to the one from Itô's original stochastic calculus to the (less familiar) backward version. It was, however, prepared by Itô's definition of the backward stochastic integral (2.24), whose expectation provides the following counterpart of (2.19) namely,

$$E\int_t^u p_*(z(\tau))dz(\tau) = E\int_t^u p_*(z(\tau))D_*z(\tau)d\tau \tag{4.20}$$

where D_* is the backward generator of $z(\tau)$ defined, as expected from (2.24), by the following regularization of left derivative

$$D_*f(q,t) = \lim_{\Delta t\downarrow 0} E_{q,t}\left[\frac{f(z(t),t) - f(z(t-\Delta t), t-\Delta t)}{\Delta t}\right].$$

If needed, the time-reversible diffusions $z(\tau)$, originally introduced in [23] under the name of "Bernstein processes", can be described, symmetrically as in (2.4), as solutions of a $\mathcal{F}_\tau$-SDE whose coefficients are computed in terms of the solution of (4.19), cf [6, 7].

So we are entitled to say that the Stochastic Deformation method advocated here is indeed inspired by Quantization and allows us to transfer ideas back and forth between the two fields.

A main goal of the research program sketched in this paper is to deform Jacobi's theory of complete solution for the classical Hamilton-Jacobi equation and his way to use it for solving classical equations of motion.

It seems that such a result should also have interesting consequences regarding our present understanding of Stochastic Analysis, which has developed without resorting systematically to an underlying linear structure, fundamental to the quantization procedure.

As a final illustration, let us mention that the deformation presented here is essentially independent of the nature of the underlying processes (or, equivalently, of their generator D). It has been shown, for instance, that it works as well for Lévy processes, a large class of independent increments processes including Poisson - as well as Wiener processes [19]. Each quantum "representation" provides, in fact, its own class of underlying Bernstein processes and associated Hamilton-Jacobi-Bellman equations.

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