Probabilistic deformation of contact geometry, diffusion processes and their quadratures*

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Abstract. Classical contact geometry is an odd dimensional analogue of symplectic geometry. We show that a natural probabilistic deformation of contact geometry, compatible with the very irregular trajectories of diffusion processes, allows to construct the stochastic version of a number of basic geometrical tools, like Liouville measure for example. Moreover, it provides an unified framework to understand the origin of explicit relations (cf. "quadrature") between diffusion processes, useful in many fields. Various applications are given, including one in stochastic finance.

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1 Introduction

In [1] (afterwards referred to as [Iso]) we have introduced a concept of "stochastic quadrature" for one dimensional processes solutions of stochastic differential equations (SDE):

$$dz(t) = \sqrt{\hbar}\, dw(t) + \widetilde{B}(z(t), t)dt \tag{1.1}$$

with respect to the increasing filtration $\mathcal{P}_t$ of the Brownian process $w(t)$. In Eq(1.1), $\hbar$ is a positive constant and the drift $\widetilde{B}$ is of the special form

$$\widetilde{B}(q,t) = \hbar \frac{\partial}{\partial q} ln\widetilde{\eta}(q,t) \tag{1.2}$$

for $\widetilde{\eta}$ a positive solution of

$$\hbar \frac{\partial \widetilde{\eta}}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2 \widetilde{\eta}}{\partial q^2} \equiv H_0 \widetilde{\eta}, \quad t \in \Sigma \tag{1.3}$$

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with Σ a compact or semi-infinite interval of $\mathbb{R}$.

The point of our construction of such stochastic quadratures was to provide an unified framework explaining the origin of explicit relations between some families of diffusion processes, often very useful in computations but hard to guess a priori.

A typical example of stochastic quadratures is Doob's familiar relation between the Brownian and the Ornstein-Uhlenbeck process $z(t)$ starting from x:

$$z(t) = e^{-\beta t}\left[x + w\left(\frac{1}{2\beta}\left(e^{2\beta t} - 1\right)\right)\right] \tag{1.4}$$

for β the multiplicative constant of the linear drift of $z(t)$.

Another motivation of [Iso] was more geometrical and aimed to answer a question apparently unrelated with our first point: how could we construct a natural stochastic symplectic geometry? Indeed, elementary as it sounds, the basic problem of the construction of a stochastic analogue of the classical Liouville measure, for example, does not seem to have found, as yet, a natural solution. This could be at the origin of conceptual difficulties when trying to construct probability measures for infinite dimensional symplectic dynamical systems.

Our understanding of the above adjective "natural" stems from the origin of the Brownian $w(t)$ itself: the construction in question should use nothing more than (the geometrical content of) Eq(1.3) itself.

Our purpose here is to show that the method introduced in [Iso] is considerably more general than we thought initially.

On one hand we can add perturbation potentials a and V to H_0 in the parabolic equation (1.3):

$$\begin{aligned} \hbar\frac{\partial\widetilde{\eta}}{\partial t} &= -\frac{\hbar^2}{2}\frac{\partial^2\widetilde{\eta}}{\partial q^2} + a(q)\frac{\partial\widetilde{\eta}}{\partial q} + V(q,t)\widetilde{\eta} \\ &\equiv H\widetilde{\eta} \end{aligned} \tag{1.5}$$

and our method will provide for free a maximal class of such potentials allowing to reduce the analysis of stochastic quadratures to the one of Eq(1.3). However, let us stress at once that the method advocated is in no way restricted to any special class of Hamiltonians.

It will appear that the underlying geometry is, in fact, more general than symplectic. It can be regarded as an $\hbar$-deformation of the classical contact geometry of elementary dynamical systems whose Hamiltonian is the "classical limit" of H in Eq(1.5).

Although this deformation is quite similar to the one expressing the transition from classical to quantum dynamics where $\hbar$ stands, of course, for Planck's constant (with the crucial difference that Schrödinger's equation is replaced here by its "Euclidean" or "imaginary time" parabolic counterpart (1.5), better suited to a probabilistic analysis) our results are relevant to any fields where one needs to seriously compute with diffusion processes.

The organization of this paper is the following:
§2 is a summary of the geometrical study of classical Hamilton-Jacobi equation and its relation with classical Hamiltonian and Lagrangian dynamics. What we shall need, specifically, is the contact geometrical approach to Hamilton-Jacobi equation (HJ), which is not very familiar, even in mathematical physics. Instead to present immediately its more elegant (but abstract) version in terms of ideal of differential forms, due to E. Cartan, we recall how the more traditional approaches are indeed founded on a special set of differential forms and how the geometry of HJ can be expressed in term of the Lie dragging of those forms along its symmetries.

Hamilton-Jacobi-Bellman equation (HJB) will be regarded here as a deformation of its classical counterpart and §3 is devoted to the analysis of its geometrical content, in Cartan's perspective. Theorem 3.1 describes the Lie dragging of the deformed ideal of forms associated with HJB, in terms of the coefficients of the infinitesimal symmetries of this PDE. It is the general result on the geometrical content of HJB valid for any (regular) potential V.

Although the statement of Theorem 3.1 seems, a priori, to have no relations whatsoever with stochastic analysis, one of the symmetries involved, as well as the deformed basic invariance identity for the action functional, strongly suggest what to do to find one, namely, in geometrical terms, a section of the base manifold of independent variables of HJB within the jet space for this equation.

Theorem 3.2 describes the main results of the probabilistic interpretation on the base integral submanifold of the jet space, in the perspective of our stochastic quadratures of diffusions.

§4 and §5 contain a list of explicit examples of quadratures resulting from the general results of §3. Although such quadratures can be computed, in principle, each time the determining equations (3.45) are solved, for any given V, a special (but large!) class of examples is directly accessible using exclusively the geometry of the free HJB equation (i.e $V{=}0$). We mention a Theorem of Rosencrans explaining how to do this reduction and use it for some explicit class of diffusions. Those readers familiar with the geometrical approach to the quantization problem will recognize, in the special status of the associated class of potentials, the probabilistic counterpart of the special status of the metaplectic representation in quantum physics.

Our last explicit example is inspired by recent results of Patie and Alili in stochastic finance [2, 12] and show how they can be reinterpreted in our geometrical perspective.

2 Elementary classical contact geometry

Since this theory cannot be regarded as common knowledge, we shall first summarize the part of classical contact geometry relevant to the geometrical approach to first order elementary dynamical systems. We shall limit ourselves to the one dimensional case $q \in \mathbb{R}$, only because our explicit examples will be in this class. For much more about contact geometry, cf. [3, Chapter 5; 4, Chapter 10]. Let us consider the Hamilton-Jacobi equation associated with an elementary mechanical system of given C^2 class energy (or Hamiltonian) $h(q,p,t)$, where q and p denote respectively the configuration (position) and momentum variables. Departing from the tradition we will choose:

$$-\frac{\partial S}{\partial t} + h\left(q, -\frac{\partial S}{\partial q}, t\right) = 0 \tag{2.1}$$

for a real valued function $S(q,t)$. Introducing the function of five variables

$$f(q,t,S,p,E) = -E + h(q,p,t) \tag{2.2}$$

we can regard Eq(2.1) as the partial differential equation of first order

$$f\left(q, t, S, -\frac{\partial S}{\partial q}, \frac{\partial S}{\partial t}\right) = 0 \tag{2.3}$$

The idea to define partial derivatives of the dependent variable of (2.1) as new variables is called "prolongation".

(Lie's) characteristic equations for Eq(2.3) are the following:

$$\dot{q} = \frac{\partial f}{\partial p},\ \dot{t} = -\frac{\partial f}{\partial E}$$

$$\dot{p} = -\frac{\partial f}{\partial q} - p\frac{\partial f}{\partial S},\ \dot{E} = \frac{\partial f}{\partial t} - E\frac{\partial f}{\partial S} \tag{2.4}$$

$$\dot{S} = f - p\frac{\partial f}{\partial p} - E\frac{\partial f}{\partial E} \tag{2.5}$$

where $\cdot$ denotes the derivative $\frac{d}{du}$ of the characteristics $\sigma(u) \equiv (q(u), t(u), S(u), p(u), E(u)), u \in \Sigma$, an interval of $\mathbb{R}$.

Given the simplicity of Eqs (1.3) and (1.5), it will be sufficient to consider elementary Hamiltonians h of the form

$$h(q,p,t) = \frac{1}{2}p^2 + V(q,t) \tag{2.6}$$

so, using (2.2) the second equation of (2.4) implies that the parameter u can be identified with t, and therefore $\cdot$ can be regarded as $\frac{d}{dt}$.

The relation (2.2) shows that the four remaining equations split into Hamilton equations for h as in (2.6) and the (generalized) conservation of energy:

$$\dot{q} = \frac{\partial h}{\partial p} = p,\ \dot{p} = -\frac{\partial h}{\partial q} = -\frac{\partial V}{\partial q} \tag{2.7}$$

$$\dot{h} = \frac{\partial h}{\partial t} = \frac{\partial V}{\partial t}. \tag{2.8}$$

Regarding (2.5), we obtain, using again (2.7):

$$\begin{aligned} \dot{S} &= -\left(\frac{1}{2}\,\dot{q}^2 - V\right) \\ &= -L(\dot{q}, q, t) \end{aligned} \tag{2.9}$$

the right hand side defining (minus) the Lagrangian L of such a mechanical system. This means that, by integration along the characteristics, for $v \geq t$,

$$S(q,t) = \int_{q,t}^{q_v,v} L(\dot{q}(\tau), q(\tau), \tau) d\tau + S_v(q_v) \tag{2.10}$$

where a final condition $S(q(v), v) = S_v(q_v)$ has been introduced (A final condition and not an initial one as usual because we started, here, from Eq(2.1) instead of the usual Hamilton-Jacobi equation. Cf. [5] for a probabilistic interpretation).

Our starting space $\mathbb{R}^5$ of the variables (q, t, S, p, E) is called 1-jet and often denoted by J^1. According to (2.3), our Hamilton-Jacobi equation corresponds to the hypersurface $\varepsilon \equiv \{f = 0\}$ in J^1.

In Symplectic Geometry, the basic geometrical object is Liouville 2-form $\Omega = dp \wedge dq = d\omega$ where $\omega = pdq$, the Poincaré form. In the ("extended") cases where, like here, the conjugate variables $(t, -E)$ are also needed, one start, instead, from Poincaré-Cartan form:

$$\omega_{pc} = pdq - Edt. \tag{2.11}$$

In contrast with the symplectic case, the basic geometric object of contact geometry is the contact 1-form:

$$\omega = \omega_{pc} + dS. \tag{2.12}$$

For a given contact Hamiltonian $f \in C^\infty(J^1)$, the associated contact vector field is defined by Eqs (2.4–2.5):

$$\begin{aligned} X_f = \frac{\partial f}{\partial p}\frac{\partial}{\partial q} - \frac{\partial f}{\partial E}\frac{\partial}{\partial t} + \left(f - p\frac{\partial f}{\partial p} - E\frac{\partial f}{\partial E}\right)\frac{\partial}{\partial S} \\ - \left(\frac{\partial f}{\partial q} + p\frac{\partial f}{\partial S}\right)\frac{\partial}{\partial p} + \left(\frac{\partial f}{\partial t} - E\frac{\partial f}{\partial S}\right)\frac{\partial}{\partial E}. \end{aligned} \tag{2.13}$$

Notice that X_f can be regarded as an (extended) Hamiltonian vector field only when f does not depend on the variable S.

For f as in Eqs (2.2) and (2.6), in particular, we obtain

$$X_f = p\frac{\partial}{\partial q} + \frac{\partial}{\partial t} + \left(-\frac{1}{2}p^2 + V\right)\frac{\partial}{\partial S} - \frac{\partial V}{\partial q}\frac{\partial}{\partial p} + \frac{\partial V}{\partial t}\frac{\partial}{\partial E}. \tag{2.14}$$

Using definitions (2.12) and (2.13) it is clear that for any $f \in C^\infty(J^1)$,

$$\omega(X_f) = f \tag{2.15}$$

so that any contact Hamiltonian f can be defined that way.

The Lie algebra of contact vector fields is in bijective correspondence with a Lie algebra on $C^\infty(J^1)$, defined through the Lagrange (or Jacobi) bracket $\{.,.\}_L$:

$$\{f, g\}_L = \omega([X_g, X_f]). \tag{2.16}$$

As suggested by (2.13), it is only when f and g do not depend on the variable S that this bracket provides a Poisson structure (since, in general, it does not satisfy Leibniz rule).

A contact vector field X_n for Eq(2.3) is called an infinitesimal symmetry of this equation if and only if

$$\{f, n\}_L = 0 \text{ on } \varepsilon. \tag{2.17}$$

Let us come back to our elementary system defined by (2.2) and (2.6). If we look for symmetry Hamiltonians n of the simple form

$$n(q,p,t,E) = X(q,t)p - T(t)\,E - \phi(q,t) \tag{2.18}$$

for undetermined real valued coefficients X, T and ϕ, one checks that they are allowed, indeed, every time those coefficients solve the "determining" equations:

$$\begin{aligned} &\dot{T} = 2\frac{\partial X}{\partial q}, \quad \frac{\partial X}{\partial t} = \frac{\partial \phi}{\partial q} \\ &\frac{\partial \phi}{\partial t} = -\dot{T}V - X\frac{\partial V}{\partial q} - T\frac{\partial V}{\partial t}. \end{aligned} \tag{2.19}$$

Let us summarize the main classical results on Lie dragging along symmetries, relevant to the probabilistic deformation of our elementary systems.

$$\mathcal{L}_{X_n}(\omega_{pc}) = d\phi. \tag{2.20}$$

We give here the proof of this first relation, as an illustration.

By (2.13) when $f = n$, we have

$$\begin{aligned} X_n &= X\frac{\partial}{\partial q} + T\frac{\partial}{\partial t} - \phi\frac{\partial}{\partial S} + \left(-\frac{\partial X}{\partial q}p + \frac{\partial \phi}{\partial q}\right)\frac{\partial}{\partial p} \\ &+ \left(\frac{\partial X}{\partial t}p - \dot{T}E - \frac{\partial \phi}{\partial t}\right)\frac{\partial}{\partial E} \\ &\equiv X_n^q\frac{\partial}{\partial q} + X_n^t\frac{\partial}{\partial t} + X_n^S\frac{\partial}{\partial S} + X_n^p\frac{\partial}{\partial p} + X_n^E\frac{\partial}{\partial E} \end{aligned} \tag{2.21}$$

where we have introduced a notation for the components of X_n. So by definition (2.11), the properties of Lie derivative and the components of X_n,

$$\begin{aligned} \mathcal{L}_{X_n}(pdq - Edt) &= \mathcal{L}_{X_n}(p)dq + p\mathcal{L}_{X_n}(dq) - \mathcal{L}_{X_n}(E)dt - E\mathcal{L}_{X_n}(dt) \\ &= X_n^p dq + pdX_n^q - X_n^E dt - EdT \\ &= \left(-\frac{\partial X}{\partial q}p + \frac{\partial \phi}{\partial q}\right)dq + pdX(q,t) \\ &\quad - \left(\frac{\partial X}{\partial t}p - \dot{T}E - \frac{\partial \phi}{\partial t}\right)dt - E\dot{T}dt \\ &= d\phi(q,t) \end{aligned}$$

Since $\Omega = d\omega_{pc}$, it also follows that, formally, $\mathcal{L}_{X_n}(\Omega) = \mathcal{L}_{X_n}(d\omega_{pc}) = d\mathcal{L}_{X_n}(\omega) = dd\phi$ so

$$\mathcal{L}_{X_n}(\Omega) = 0. \tag{2.22}$$

Regarding the Lagrangian (2.9) of our system, we find, using (2.7) and (2.19),

$$\mathcal{L}_{X_n}(L) + L\dot{T} = \frac{\partial \phi}{\partial t} + p\frac{\partial \phi}{\partial q}. \tag{2.23}$$

As it is well known, the relation (2.23) is the basic invariance identity of the variational calculus for the functional defined on the r.h.s. of (2.10). Indeed, by (2.6), (2.7) and (2.9)

$$\begin{aligned} \int \omega_{pc} &= \int pdq - Edt \\ &= \int (p\dot{q} - h)dt = \int Ldt. \end{aligned} \tag{2.24}$$

Now we are going to generalize this whole geometrical picture of classical dynamics to the case where the (smooth) characteristics are replaced by solutions of SDE.

3 Probabilistic deformation of contact geometry

Let us come back to Eq(1.5). For the moment, we shall consider the case $a = 0$.

We shall start from the same nonlinear change of variable (whose origin dates back to E. Schrödinger, in the context of quantum mechanics, cf. [5]) as in the free case $V = 0$ treated in [Iso]:

$$S = -\hbar \, ln \, \eta, \tag{3.1}$$

for η a positive solution of Eq(1.5). Then S solves the following (Hamilton-Jacobi-Bellman) equation:

$$-\frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 - V - \frac{\hbar}{2}\frac{\partial^2 S}{\partial q^2} = 0 \tag{3.2}$$

interpreted here as a deformation of the classical PDE (2.1) for h as in (2.6).

Let us define, in analogy with the classical case,

$$B = -\frac{\partial S}{\partial q}, \quad E = -\frac{\partial S}{\partial t} \tag{3.3}$$

but with a few changes of signs (w.r.t (2.12)) due to the abovementioned Euclidean counterpart. In order to distinguish the 1-jet of HJB from its classical counterpart of §2, we denote by B the variable playing now the role of the classical momentum p. The geometrical content of Eq.(3.2) for the deformed 1-jet $J^1 = (q, t, S, B, E)$ is contained in the vanishing of differential forms:

$$\omega = Bdq + Edt + dS, \tag{3.4}$$

together with

$$\Omega = d\omega = dBdq + dEdt \tag{3.5}$$

where we drop the symbol $\wedge$ of exterior multiplication, for simplicity, and the two form β defining Hamilton-Jacobi-Bellman equation itself, namely

$$\beta = \left(E + \frac{1}{2}B^2 - V\right) dqdt + \frac{\hbar}{2} dBdt. \tag{3.6}$$

Let us recall that for given S, its "section" mapping lifts up the (q, t) base manifold of independent variables into the 5-dimensional jet space J^1 according to $\{q, t, S(q, t), \frac{\partial S}{\partial q}(q, t), \frac{\partial S}{\partial t}(q, t)\}$. All the "sectioned"forms pull back to zero onto the base 2-submanifold of J^1, which is called an integral submanifold.

Since $d\beta = (-dq + Bdt)d\omega$, it belongs to the ideal I of forms generated by ω, Ω and β. This one is therefore the smallest differential ideal containing ω and β. E. Cartan has shown that, in these conditions, the geometric representation of our PDE (3.2) can be completed (cf.[7], [6]).

Clearly, the ideal I contains the 3 fundamental ingredients we needed for our contact geometrical approach to classical dynamics. On the basis of our definition of Lagrangian, in the free case [Iso], and of the form of the Lagrangian for such elementary systems in Euclidean approaches to Feynman's ideas (cf. [5]), we infer that the Lagrangian underlying I should be (we shall prove it later on) the function of the independent variables B, q, t:

$$L(B, q, t) = \frac{1}{2}B^2 + V(q, t). \tag{3.7}$$

Let us denote by N a vector field on J^1 playing the role of a classical symmetry contact vector field X_n. In the context of our ideal I one should have:

$$\mathcal{L}_N(I) \subseteq I. \tag{3.8}$$

Such a N has been called, sometimes, isovector [6] and the theory of those in term of differential ideals is due, in essence, to E. Cartan [7].

Because of the linearity of Eq(1.5), the Lie algebra $\mathcal{G}$ of these isovectors contains an infinite-dimensional abelian ideal $\mathcal{J}$, with canonical supplement $\aleph$.

In the free case $V = a = 0$, $\aleph$ has dimension 6 and possesses a natural basis, each element of which corresponds to a symmetry of our system.

As in this free case, the probabilistic interpretation of our results rests on the fact that, on the integral 2-submanifold, the underlying continuous trajectories will be of the form $t \mapsto z(t)$, solution of stochastic differential equation (1.1).

It is well known that what plays, then, the role of the strong derivative along smooth (or "classical") paths is the infinitesimal generator of this diffusion ("Bernstein", cf. [5]) process:

$$\widetilde{D} = \frac{\partial}{\partial t} + \widetilde{B}\frac{\partial}{\partial q} + \frac{\hbar}{2}\frac{\partial^2}{\partial q^2} \tag{3.9}$$

where $\widetilde{B}$ was defined in (1.2).

Let us write an isovector N as

$$N = N^q\frac{\partial}{\partial q} + N^t\frac{\partial}{\partial t} + N^S\frac{\partial}{\partial S} + N^B\frac{\partial}{\partial B} + N^E\frac{\partial}{\partial E} \tag{3.10}$$

Then we have the

Theorem 3.1 *Along each isovector N of $\aleph = \left\{N \in \mathcal{G}\ s.t\ \frac{\partial N^S}{\partial S} = 0\right\}$ satisfying (3.8) for the ideal I generated by (ω, Ω, β) as defined before, and for the Lagrangian L associated with it, given by (3.7), we have*

$$(1) \quad \mathcal{L}_N(Bdq + Edt) = -dN^S \tag{3.11}$$

$$(2) \quad \mathcal{L}_N(\Omega) = 0 \tag{3.12}$$

$$(3) \quad \mathcal{L}_N(L) + L\frac{dN^t}{dt} = -DN^S \tag{3.13}$$

where $D \equiv \frac{\partial}{\partial t} + B\frac{\partial}{\partial q} + \frac{\hbar}{2}\frac{\partial^2}{\partial q^2}$.

A word of caution is needed, before the (tedious, algebrico-geometrical) proof of this Theorem.

If it was not for the last term of (3.13), one could suspect that the following proof has nothing to do with stochastic analysis. Note, however, that if we could look at D in (3.13), as an operator whose first term in q does not involve the independent variable $B \in J^1$ but the section $\widetilde{B} = \widetilde{B}(q,t) = -\frac{\partial \widetilde{S}}{\partial q}$, like in (3.9), then the relation of Theorem 3.1 with stochastic analysis would be clear. We shall show, in Theorem 3.2, that we are indeed allowed to do this and therefore to find a probabilistic interpretation on the integral 2-submanifold. The traditional notation $\sim$ for the sectioned geometrical objects is an anticipation of Theorem 3.2.

Proof: By definition $\mathcal{L}_N(\omega)$ is sum of multiples (with "Lagrange multipliers") of $\omega, d\omega$ and β. Necessarily, here,there is a function φ such that $\mathcal{L}_N(\omega) = \varphi\omega$. In analogy with (2.15), and following [6], let us define $F_N \equiv \omega(N) = BN^q + EN^t + N^S$ and consider

$$\begin{aligned} \varphi\omega - dF_N &= \mathcal{L}_N(\omega) - d(\omega(N)) \\ &= d\omega(N) \\ &= N^B dq - N^q dB + N^E dt - N^t dE. \end{aligned} \tag{3.14}$$

After substitution of (3.4), the identification of the coefficients of both sides provides $\varphi = \frac{\partial F_N}{\partial S}$, then

$$N^q = \frac{\partial F_N}{\partial B}, \quad N^t = \frac{\partial F_N}{\partial E}, \quad N^S = F_N - B\frac{\partial F_N}{\partial B} - E\frac{\partial F_N}{\partial E}$$
$$N^B = -\frac{\partial F_N}{\partial q} + B\frac{\partial F_N}{\partial S}, \quad N^E = -\frac{\partial F_N}{\partial t} + E\frac{\partial F_N}{\partial S} \tag{3.15}$$

which should be compared with the components of the (real time!) classical contact field (2.13) associated with a contact Hamiltonian F_N.

By hypothesis $\mathcal{L}_N(\omega) \in I$ so $\mathcal{L}_N(d\omega) = d\mathcal{L}_N(\omega) \in dI \subset I$. Therefore the vector field N associated with F_N by (3.15) will be an Isovector of I if and only if $\mathcal{L}_N(\beta) \in I$. So there must be two 0-forms α and γ and a 1-form ε such that

$$\mathcal{L}_N(\beta) = \alpha\beta + \varepsilon\omega + \gamma d\omega. \tag{3.16}$$

As before, those Lagrange multipliers should be eliminated. Without loss of generality we can assume that ε has no dS term (cf. [6] p. 658). So it reduces to

$$\varepsilon = \mu dq + \lambda dt + \rho dB + \nu dE.$$

Explicitly, (3.16) means, after identification of the respective coefficients both sides:

$$-N^E - BN^B + N^t\frac{\partial V}{\partial t} + N^q\frac{\partial V}{\partial q} + \left(V - E - \frac{1}{2}B^2\right)\frac{\partial N^q}{\partial q}$$
$$+\left(V - E - \frac{1}{2}B^2\right)\frac{\partial N^t}{\partial t} - \frac{\hbar}{2}\frac{\partial N^B}{\partial q}$$
$$= \alpha\left(V - E - \frac{1}{2}B^2\right) + \lambda B - \mu E \tag{3.17}$$
$$\left(V - E - \frac{1}{2}B^2\right)\frac{\partial N^q}{\partial S} - \frac{1}{2}\hbar\frac{\partial N^B}{\partial S} = \lambda \tag{3.18}$$

$$\left(V - E - \frac{1}{2}B^2\right)\frac{\partial N^q}{\partial E} - \frac{1}{2}\hbar\frac{\partial N^B}{\partial E} = -\nu E - \gamma \tag{3.19}$$

$$\left(V - E - \frac{1}{2}B^2\right)\frac{\partial N^q}{\partial B} - \frac{1}{2}\hbar\frac{\partial N^B}{\partial B} - \frac{\hbar}{2}\frac{\partial N^t}{\partial t} = -\alpha\frac{1}{2}\hbar - \rho E \tag{3.20}$$

$$\left(E + \frac{1}{2}B^2 - V\right)\frac{\partial N^t}{\partial S} = \mu \tag{3.21}$$

$$\left(E + \frac{1}{2}B^2 - V\right)\frac{\partial N^t}{\partial E} = -\nu B \tag{3.22}$$

$$\left(E + \frac{1}{2}B^2 - V\right)\frac{\partial N^t}{\partial B} - \frac{\hbar}{2}\frac{\partial N^t}{\partial q} = -\rho B - \gamma \tag{3.23}$$

$$\nu = 0 \tag{3.24}$$

$$\frac{1}{2}\hbar\frac{\partial N^t}{\partial S} = \rho \tag{3.25}$$

$$-\frac{1}{2}\hbar\frac{\partial N^t}{\partial E} = 0. \tag{3.26}$$

So we are left with conditions (3.17), (3.19) and (3.26). From (3.26) we get $\partial N^t/\partial E = 0$ i.e (see (3.15)) $\frac{\partial^2 F_N}{\partial E^2} = 0$. Therefore, our contact Hamiltonian F_N is affine in E:

$$F_N = a + TE$$

where a and T may depend only upon (q, t, B, S). After substitution of this F in (3.16), using (3.23), (3.24) and (3.25) we can rewrite (3.19) as

$$\left(V - E - \frac{1}{2}B^2\right)\frac{\partial N^q}{\partial E} - \frac{\hbar}{2}\frac{\partial N^B}{\partial E} =$$
$$= -\frac{1}{2}\hbar\frac{\partial N^t}{\partial q} + \left(E + \frac{1}{2}B^2 - V\right)\frac{\partial N^t}{\partial B} + \frac{1}{2}\hbar B\frac{\partial N^t}{\partial S}.$$

Introducing the expressions (3.15) of $N^q, N^t, ...$, this reduces to

$$2\left(E + \frac{1}{2}B^2 + V\right)\frac{\partial T}{\partial B} = \hbar\frac{\partial T}{\partial q} - \hbar B\frac{\partial T}{\partial S}. \tag{3.27}$$

By identification of the coefficients of E, $\frac{\partial T}{\partial B} = 0$ and $\frac{\partial T}{\partial q} = B\frac{\partial T}{\partial S}$. By computing $\frac{\partial}{\partial B}$ of the last relation, we get $\frac{\partial T}{\partial S} = 0$. Finally

$$T = T_N(t) \tag{3.28}$$

is true for any $N \in \mathcal{G}$.

Now the relations (3.15) can be rewritten in terms of coefficients $a = a(q, t, S, B)$ and $T = T_N(t)$. Since $N^t = T_N(t)$, (3.21) implies $\mu = 0$ and (3.25) implies $\rho = 0$ and (3.23) implies $\gamma = 0$. Then (3.17) reduces to an identity involving the partial derivatives of a and T_N. By elimination of λ and

α using (3.18) and (3.20) in this identity, we can identify the coefficients of E^2 on both sides and get $\frac{\partial^2 a}{\partial B^2} = 0$, i.e

$$a = X_N B + h \tag{3.29}$$

where $X_N \equiv N^q$ and h can depend only upon (q, t, S). This means that the contact Hamiltonian F_N is also affine in B:

$$F_N = X_N B + T_N E + h. \tag{3.30}$$

The identification of the coefficients of E provides

$$2\frac{\partial X_N}{\partial q} - 2B\frac{\partial X_N}{\partial S} - \dot{T}_N = 0. \tag{3.31}$$

Since X_N cannot depend upon B, and $T_N = T_N(t)$ only, $\frac{\partial X_N}{\partial S} = 0$ i.e. $X_N = X_N(q, t)$. Moreover the last relation implies that

$$X_N(q, t) = \frac{1}{2}\dot{T}_N(t)q + l(t). \tag{3.32}$$

According to the definition of N^S (cf. (3.15)) and of the contact Hamiltonian F_N in (3.30), we have $N^S = h(q, t, S)$. So, for any $N \in \aleph, N^S$ depends only on (q, t). It is immediate to check that if $N_1, N_2 \in \aleph, [N_1, N_2](S)$ is a function of (q, t) so $\aleph$ is indeed a Lie subalgebra of $\mathcal{G}$.

We are now able to prove the claims of Theorem 3.1:

$$\begin{aligned} \mathcal{L}_N(\omega) &= \varphi\omega \\ &= \frac{\partial F_N}{\partial S}\omega = 0, \end{aligned}$$

whence

$$\mathcal{L}_N(d\omega) = 0. \tag{3.33}$$

Since, by (3.4), $\omega_{pc} = \omega - dS$, this reduces to

$$\begin{aligned} \mathcal{L}_N(Bdq + Edt) &= \mathcal{L}_N(\omega) - d(\mathcal{L}_N(S)) \\ &= -dN^S \end{aligned} \tag{3.34}$$

This proves (1). To prove (3), we need a more explicit version of N^S. Coming back to the reduced version of (3.17) providing (3.29), in the general setting where

$$N^S = h(q, t, S),$$

and substituting there what we deduced from (3.15) for F_N of the form (3.30), i.e.

$$
\begin{aligned}
&N^q \equiv X_N = \frac{1}{2}\dot{T}_N(t)q + l(t) \\
&N^t \equiv T_N(t) \\
&N^S = h(q,t,S) \\
&N^B = -\frac{1}{2}\dot{T}_N(t)B - \frac{\partial h}{\partial q} + B\frac{\partial h}{\partial S} \\
&N^E = -\left(\frac{1}{2}\ddot{T}_N q + \dot{l}\right)B - \dot{T}_N E - \frac{\partial h}{\partial t} + E\frac{\partial h}{\partial S},
\end{aligned}
\tag{3.15'}
$$

we obtain 3 equations corresponding to the identification of the coefficients of B^2, B and 1. Respectively,

$$\frac{\partial^2 h}{\partial S^2} = \frac{1}{\hbar}\frac{\partial h}{\partial S} \tag{3.35}$$

$$\frac{1}{2}\ddot{T}_N q + \dot{l} + \frac{\partial h}{\partial q} = \hbar\frac{\partial^2 h}{\partial S \partial q} \tag{3.36}$$

$$\frac{\partial h}{\partial t} + T_N\frac{\partial V}{\partial t} + \left(\frac{1}{2}\dot{T}_N q + l\right)\frac{\partial V}{\partial q} + \frac{\hbar}{2}\frac{\partial^2 h}{\partial q^2} = \left(\frac{\partial h}{\partial S} - \dot{T}_N\right)V. \tag{3.37}$$

We can express the solution of Eq(3.35) as

$$h(q,t,S) = \hbar\widetilde{\eta}(q,t)e^{\frac{1}{\hbar}S} - \phi(q,t). \tag{3.38}$$

Then (3.36) means

$$\frac{\partial \phi}{\partial q} = \frac{1}{2}\ddot{T}_N q + \ddot{l} \equiv \frac{\partial X_N}{\partial t} \tag{3.39}$$

therefore

$$\phi(q,t) = \frac{1}{4}\ddot{T}_N q^2 + \dot{l}q - \sigma(t). \tag{3.40}$$

After substitution of (3.38) in (3.37), we obtain an equation which splits into

$$\hbar\frac{\partial \widetilde{\eta}}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2 \widetilde{\eta}}{\partial q^2} + V\widetilde{\eta} \tag{3.41}$$

and

$$-\frac{1}{4}\dddot{T}_N q^2 - \ddot{l}q + \dot{\sigma} + T_N\frac{\partial V}{\partial t} + \frac{1}{2}\dot{T}_N q\frac{\partial V}{\partial q} + l\frac{\partial V}{\partial q} + \dot{T}_N V - \frac{\hbar}{4}\ddot{T}_N = 0. \tag{3.42}$$

To prove (3) recall that $N \in \aleph$ implies $N^S = h$ does not depend on S i.e. $\widetilde{\eta} = 0$ in the relation (3.38) or

$$h = -\phi(q,t). \tag{3.43}$$

Now, for L as in (3.7), since L is a function, i.e a 0-form,

$$\begin{aligned} \mathcal{L}_N(L) &= N(L) \\ &= N\left(\frac{1}{2}B^2 + V(q,t)\right) \\ &= BN^B + N^q\frac{\partial V}{\partial q} + N^t\frac{\partial V}{\partial t}. \end{aligned}$$

Introducing N^B, N^q, N^t given by (3.15') and $h = -\phi(q,t)$ as in (3.40), using (3.42) one verifies that

$$\mathcal{L}_N(L) + L\frac{dT_N}{dt} = \frac{\partial \phi}{\partial t} + B\frac{\partial \phi}{\partial q} + \frac{\hbar}{2}\frac{\partial^2 \phi}{\partial q^2} \equiv D\phi.$$

This is (3.13) for $\phi = -N^S$. □

Remarks:

1) In the course of the proof of Theorem 3.1, we have introduced new labels for the coefficients N^q, N^t and N^S (a priori functions of all the variables of J^1), namely $X_N(q,t)$, $T_N(t)$ and $-\phi_N(q,t)$. Our first reason for this is to turn easier the comparison with the classical expressions (2.18) and (2.19). The second one is to relate directly the formulation of our present results with the ones of the stochastic Noether Theorem, proved in [8] without using the jet space J^1 and its 2-integral submanifold of independent variables (q,t), where the probabilistic interpretation $z_t = q$ will be valid. Denoting by F_N the symmetry contact Hamiltonian associated with the isovector $N \in \aleph$, it follows from (3.30), (3.43) and our calculations above that

$$\begin{aligned} F_N &= F_N(q,B,t,E) \\ &= X_N(q,t)B + T_N(t)E - \phi_N(q,t). \end{aligned} \tag{3.44}$$

This is the (Euclidean) deformation of the classical symmetry contact Hamiltonian denoted by $n(q,p,t,E)$ in (2.18). To get a better notion of the deformation in question, let us recall that, in the context of the stochastic Noether Theorem for such systems, we had found the following "determining" relations between the coefficients X_N, T_N and ϕ_N: (cf. [8], Lemma 3.5, in one dimension)

$$\begin{aligned} &\dot{T}_N = 2\frac{\partial X_N}{\partial q} \\ &\frac{\partial X_N}{\partial t} = \frac{\partial \phi_N}{\partial q} \\ &\frac{\partial \phi_N}{\partial t} + \frac{\hbar}{2}\Delta\phi_N = \dot{T}_N V + X_N\frac{\partial V}{\partial q} + T_N\frac{\partial V}{\partial t}. \end{aligned} \tag{3.45}$$

By (3.15') here, the first relation is true. The second one was already checked in (3.39). Now the derivative $\frac{\partial}{\partial q}$ of (3.42) coincides with the integrability condition $\frac{\partial^2 \phi_N}{\partial t \partial q} = \frac{\partial^2 \phi_N}{\partial q \partial t}$ in the relations above:

$$\frac{\partial^2 X_N}{\partial t^2} - 3\frac{\partial X_N}{\partial q}\frac{\partial V}{\partial q} - X_N \frac{\partial^2 V}{\partial q^2} - T_N \frac{\partial^2 V}{\partial q \partial t} = 0, \tag{3.46}$$

after substitution of $X_N = \frac{1}{2}\dot{T}_N q + l(t)$ and $T = T_N(t)$. So the determining relations resulting from our analysis are the deformations of the classical relations (2.19). Except for the change of signs of Euclidean origin, the only deformation involves, in fact, the Laplacian of ϕ_N.

Integrability condition (3.46) is useful computationaly. For a given V, it provides easily coefficients X and T allowed for symmetries.

2) If we did not know from the start that, behind the ideal I, there is the parabolic equation (3.41), the above calculation would have proved it.

Let N be an isovector for Eq(1.5). We consider here again the case $a = 0$.

By construction $e^{\alpha N}, \alpha \in \mathbb{R}$, maps (q, t, S, B, E) to $(q_\alpha, t_\alpha, S_\alpha, B_\alpha, E_\alpha)$. Defining $\widetilde{\eta}_\alpha$ by

$$e^{-\frac{1}{\hbar}S_\alpha} = \widetilde{\eta}_\alpha(q_\alpha, t_\alpha)$$

it is known that $\widetilde{\eta}_\alpha$ solves the same PDE as $\widetilde{\eta}(q,t)$ (this is the definition of its symmetry group). If we denote by $e^{\alpha \widetilde{N}} : \widetilde{\eta} \mapsto \widetilde{\eta}_\alpha$ the associated one-parameter group, we obtain the following homomorphism of Lie algebras associated with the section within J^1 mentioned in the introduction:

$$\begin{aligned} N &= N^q \frac{\partial}{\partial q} + N^t \frac{\partial}{\partial t} + N^S \frac{\partial}{\partial S} + N^B \frac{\partial}{\partial B} + N^E \frac{\partial}{\partial E} \\ \longrightarrow -\widetilde{N} &= +N^t \frac{\partial}{\partial t} + N^q \frac{\partial}{\partial q} - \frac{1}{\hbar} N^S \end{aligned} \tag{3.47}$$

where the last formula means that for each regular function $f(q,t)$,

$$\widetilde{N} f(q,t) = -N^t \frac{\partial f}{\partial t} - N^q \frac{\partial f}{\partial q} + \frac{1}{\hbar} N^S f.$$

Let us recall that, by definition, such a mapping preserves all the operations in the less complicated Lie algebra defined on the integral submanifold. Our notation $\sim$ is the same as the one used for sectioning differential forms in an ideal (cf. [6, 7]). We will use it now to give the probabilistic interpretation of Theorem 3.1 on the integral submanifold:

Theorem 3.2 *Let $z_t, t \in \Sigma$, be a solution of the SDE* (1.1) *built in term of positive solutions of Eq.*(1.5) *with $H = -\frac{\hbar^2}{2}\Delta + V(q,t)$ and for any regular V in the Kato class (cf.* [5, 11]*). The probabilistic counterparts of the characteristic equations* (2.7)–(2.9) *for the associated classical system are given, in term of the generator* (3.9)*, by*

(1) $\widetilde{D}z = \widetilde{B}$

(2) $\widetilde{D}\widetilde{B} = \frac{\partial V}{\partial q}$

(3) $\widetilde{D}\widetilde{E} = \frac{\partial V}{\partial t}$

(4) $\widetilde{D}\widetilde{S} = -\left(\frac{1}{2}\widetilde{B}^2 + V\right) \equiv -L$ or

(5) $\widetilde{S}(z_t, t) = E_t \int_t^v L(\widetilde{B}_\tau, z_\tau, \tau)d\tau + E_t\widetilde{S}_v(z_v)$

(6) $\quad\quad = E_t \int_t^u \widetilde{B} \circ dz_\tau + \widetilde{E}d\tau + E_t\widetilde{S}_v(Z_v).$

The invariance identity (3) of Theorem 3.1 can rewritten in terms of $N^q = X_N$ and $N^t = T_N$ as

(7) $X_N \frac{\partial L}{\partial q} + T_N \frac{\partial L}{\partial t} + (\widetilde{D}X_N - \widetilde{B}\dot{T}_N)\frac{\partial L}{\partial \widetilde{B}} + L\dot{T}_N = \widetilde{D}\phi_N.$

Proof: This Theorem summarizes properties of diffusions (1.1) which have been proved, along the years, without knowledge of their contact geometrical background (cf. [5, 8]). According to (3.1) and (3.3), we have

$$\widetilde{B}(q,t) = \hbar\nabla\ ln\ \widetilde{\eta}(q,t)$$
$$\widetilde{E}(q,t) = \hbar\partial_t\ ln\ \widetilde{\eta}(q,t)$$

for $\widetilde{\eta}$ a positive solution of the parabolic equation (1.5) (with $a = 0$). The relation (1) to (4) result from direct computation with the generator $\widetilde{D}$ (cf. (3.9)) of $z(t)$.

Eq(5) follows from (4) by Itô-Dynkin formula, under integrability condition.

The geometrical definition (6) of the action $\widetilde{S}$ in term of the Poincaré-Cartan form ω_{pc} of (3.4) needs to be clarified. If E_t denotes the conditional expectation $E[...|z(t)]$, we have

$$E_t \int_t^v \widetilde{\omega}_{pc} \equiv E_t \int_t^v \widetilde{B} \circ dz(\tau) + \widetilde{E}d\tau$$

where $\circ$ denotes (Fisk) Stratonovich integral in the sense of Itô [9]

$$= \hbar E_t \int_t^v (\nabla\ ln\ \widetilde{\eta}(z_\tau, \tau) \circ dz(\tau) + \partial_\tau\ ln\ \widetilde{\eta}(z(t), \tau)d\tau)$$
$$= \hbar E_t \int_t^v d(ln\ \widetilde{\eta}(z(\tau), \tau))$$
$$= \hbar E_t\ ln\, \widetilde{\eta}(z(v), v) - \hbar\ ln\ \widetilde{\eta}(z(t), t)$$
$$= \widetilde{S}(z(t), t) - E_t\widetilde{S}(z(v)).$$

Regarding (7), observe that, using (3.9) and the determining equations (3.45),

$$\widetilde{D}X_N - \widetilde{B}\dot{T}_N = \frac{\partial \phi_N}{\partial q} - \widetilde{B}\frac{\partial X_N}{\partial q}$$

and so (7) reduces indeed to (3.13). □

As classically, the invariance identity (7) can be regarded as a basic formula of a (stochastic) calculus of variations. Such a calculus already exists, cf. [11], and allows to obtain directly some of the abovementioned results (but without the geometrical insight).

4 Perturbation of the free case and examples

As indicated in the introduction, Theorem 3.1 and Theorem 3.2 are true for any regular potential V (in the Kato class). As soon as Eq.(3.45) can be solved, an isovector N (and therefore $\widetilde{N}$) is determined and can be used to relate explicitly solutions of heat equations i.e, for us, to obtain a quadrature of diffusion processes. But there is a special class of potentials V, in Theorem 3.1 and Theorem 3.2, for which all such computations can be done explicitly without using more than the isovectors of the free heat equation ($a = V = 0$). This important class plays the role, in our probabilistic framework, of the quadratic class associated with the metaplectic representation in quantum theory (cf., for instance, [15]). Let us start from a positive solution η_χ of the free equation, such that

$$\eta_\chi(q,0) = \chi(q) > 0. \tag{4.1}$$

The above observation can be expressed via the following result of Rosencrans:

Theorem 4.1 [10] *If we denote, for a given $\widetilde{N}$ as in (3.47), $(e^{\alpha\widetilde{N}}\eta_\chi)(q,t)$ by $\rho_N(q,t,\alpha)$ and $\rho_N(q,o,\alpha)$ by $\eta^N(q,\alpha)$, then η^N solves*

$$\begin{cases} \hbar\frac{\partial\eta^N}{\partial\alpha} = N^t(q,0)\frac{\hbar^2}{2}\frac{\partial^2\eta^N}{\partial q^2} - N^q(q,0)\hbar\frac{\partial\eta^N}{\partial q} - N^S(q,0)\eta^N \\ \eta^N(q,0) = \chi(q) \end{cases} \tag{4.2}$$

So, choosing an isovector N of the free equation s.t $N^q(q,0) = -\frac{1}{\hbar}(aq+b)$, $N^t(q,0) = -1$ and $N^S(q,0) = -(cq^2+dq+f)$, for a,b,c,d,f real constants then (denoting again the parameter by t):

$$\begin{cases} \hbar\frac{\partial\eta^N}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2\eta^N}{\partial q^2} + (aq+b)\frac{\partial\eta^N}{\partial q} + (cq^2+dq+f)\eta^N \\ \eta^N(q,0) = \chi(q) \end{cases} \tag{4.3}$$

Let us consider some examples in this quadratic class:

1) Linear potential $\mathbf{V(q) = \lambda q}$, $\lambda \in \mathbb{R}$

This is the case $a = b = c = f = 0, d = \lambda$ of Theorem 4.1. Then

$$\eta_\chi^N(q,t) = e^{-\frac{1}{\hbar}\left(\frac{\lambda^2}{6}t^3 - \lambda t q\right)} \eta_\chi\left(q - \lambda\frac{t^2}{2}, t\right) \tag{4.4}$$

solves Eq(4.3) if η_χ solves the free equation. The drift $\widetilde{B}_V$ of the diffusion $z_V(t)$ associated with this perturbation is, by (1.2),

$$\begin{aligned} \widetilde{B}_V(q,t) &= \hbar\frac{\partial}{\partial q} ln\, \eta_\chi^N(q,t) \\ &= \lambda t + \widetilde{B}\left(q - \lambda\frac{t^2}{2}, t\right) \end{aligned} \tag{4.5}$$

for $\widetilde{B}$ the drift coming from the free equation. The relation between the two families of diffusions is, as expected, the deterministic translation:

$$z_V(t) = z(t) + \lambda\frac{t^2}{2}. \tag{4.6}$$

2) Quadratic and first order linear perturbations

This is the case $a = \beta\hbar, b = 0, c = \frac{1}{2}\beta^2, d = 0, f = -\frac{\hbar}{2}\beta$ in Theorem 4.1, corresponding to (a one dimensional version of) $A(q) = \beta q$ and $V(q) = \frac{1}{2}\beta^2 q^2 - \frac{\hbar}{2}\beta$ in [11, p.71]. It was shown there that the relevant parabolic equation is $\hbar\frac{\partial\eta^N}{\partial t} = H\eta^N$ with

$$\begin{aligned} H &= -\frac{\hbar^2}{2}\left(\nabla - \frac{A}{\hbar}\right)^2 + V \\ &= -\frac{\hbar^2}{2}\Delta + \hbar\beta q\nabla \end{aligned} \tag{4.7}$$

and that the associated drift is of the form

$$\widetilde{B}_V(q,t) = \hbar\frac{\partial}{\partial q}\; ln\; \eta_\chi^N(q,t) - A(q). \tag{4.8}$$

Using the relation between η_χ^N and the free solution:

$$\eta_\chi^N(q,t) = \eta_\chi\left(e^{\beta t}q, \frac{1}{2\beta}(e^{2\beta t} - 1)\right) \tag{4.9}$$

the relation between drifts becomes

$$\widetilde{B}_V(q,t) = e^{\beta t}\widetilde{B}\left(e^{\beta t}q, \frac{1}{2\beta}(e^{2\beta t} - 1)\right) - \beta q.$$

After time integration we get

$$z_V(t) = e^{-\beta t}\left[c(\omega) + z\left(\frac{1}{2\beta}(e^{2\beta t} - 1)\right)\right] \tag{4.10}$$

where $c(\omega)$ is an arbitrary random constant.

The simplest illustration is to start from $\chi = 1$. Then $\eta_\chi = 1$ and $z(t)$ is the Wiener $\sqrt{\hbar}w(t)$ itself, whose drift $\widetilde{B} = 0$. Then $\widetilde{B}_V = -\beta q$ so z_V solves

$$dz_V(t) = -\beta z_V(t)dt + \sqrt{\hbar}dw(t). \tag{4.11}$$

This means that such a perturbation contains Doob's relation (1.4) as a very special case.

Let us stress again that it would be a mistake to understand Theorem 3.1 and Theorem 3.2 as meaning that stochastic quadratures of diffusion processes are available only when the perturbations are quadratic polynomials.

Consider for example:

3) $\mathbf{V(q)} = \frac{\alpha^2}{\mathbf{q}^2}$, $\alpha \in \mathbb{R}$

It is easy to verify that the integrability condition (3.46) is satisfied by the following pairs of coefficients (X_N, T_N):

$$\begin{aligned} X_N &= 0 \quad & T_N &= 0 \\ X_N &= 0 \quad & T_N &= 1 \\ X_N &= \frac{1}{2}q, \quad & T_N &= t \\ X_N &= -qt, \quad & T_N &= -t^2 \end{aligned} \tag{4.12}$$

The associated coefficients ϕ_N follow easily from (3.45). Using the notations of [Iso] (cf. (7) p.190) the list (4.12) corresponds, respectively, to the isovectors N_3, N_1, N_2 and N_6. There are no other symmetries for this V. But we can still express the diffusion $z_V(t)$ in term of some (here 4) of the isovectors of the free, 6 dimensional Lie algebra $\aleph$ of §3. So, at the expense of reducing the dimension of $\aleph$, many stochastic quadratures are available beyond the quadratic class of potentials involved in Theorem 4.1.

5 A transformation of diffusions relevant to first crossing problems and mathematical finance

As mentioned in the introduction, our approach allows us, for example, to recover and extend explicit results established by Patie [12] and Alili-Patie [2] with a view to the computation of some option prices. Here we set $a = V = 0$ in the above general geometrical structure. Still, this application is not trivial: it requires the use of two of the free isovectors computed in [Iso], namely

$$\begin{aligned} N_4 &= 2t\frac{\partial}{\partial t} + q\frac{\partial}{\partial q} - 2E\frac{\partial}{\partial E} - B\frac{\partial}{\partial B} \\ N_6 &= 2t^2\frac{\partial}{\partial t} + 2qt\frac{\partial}{\partial q} + (\hbar t - q^2)\frac{\partial}{\partial S} - (2qB + 4tE + \hbar)\frac{\partial}{\partial E} + 2(q - tB)\frac{\partial}{\partial B}. \end{aligned}$$

For $\alpha > 0$ and $\beta \in \mathbb{R}$, let us set $\mu = -ln(\alpha)$ and $\lambda = -\alpha\beta$, and define:

$$R_{\alpha,\beta} \underset{\text{def}}{\equiv} e^{\mu N_4} e^{-\frac{\lambda}{2} N_6}, \tag{5.1}$$

$R_{\alpha,\beta}$ is a differential operator on the space $\mathcal{C}^\infty(M) = \mathcal{C}^\infty(\mathbb{R}^5)$ of smooth functions of (t, q, S, E, B), and it maps the subspace of smooth functions of (t, q) into itself. We extended $R_{\alpha,\beta}$ to $(\mathcal{C}^\infty(\mathbb{R}^5))^2$ by setting:

$$\forall (f, g) \in \mathcal{C}^\infty(\mathbb{R}^5)^2 \quad R_{\alpha,\beta}(f, g) = (R_{\alpha,\beta}(f), R_{\alpha,\beta}(g)).$$

It follows from case 2) of [Iso], p.201, that, for each $f \in \mathcal{C}^\infty(\mathbb{R}^2)$, $e^{-\frac{\lambda}{2} N_6}(f) = g$ is given by:

$$g(t, q) = f\left(\frac{t}{1 + \lambda t}, \frac{q}{1 + \lambda t}\right). \tag{5.2}$$

Also, $e^{\mu N_4}$ maps g to h, where

$$h(t, q) = g\left(e^{2\mu} t, e^{\mu} q\right) \tag{5.3}$$

(cf. case 1) of [1], p.201). Therefore $R_{\alpha,\beta} \underset{\text{def}}{\equiv} e^{\mu N_4} e^{-\frac{\lambda}{2} N_6}$ maps f to h, where

$$\begin{aligned} h(t, q) &= g\left(e^{2\mu} t, e^{\mu} q\right) \\ &= f\left(\frac{e^{2\mu} t}{1 + \lambda e^{2\mu} t}, \frac{e^{\mu} q}{1 + \lambda e^{2\mu} t}\right) \\ &= f\left(\frac{\frac{1}{\alpha^2} t}{1 - \alpha\beta \frac{1}{\alpha^2} t}, \frac{\frac{1}{\alpha} q}{1 - \alpha\beta \frac{1}{\alpha^2} t}\right) \\ &= f\left(\frac{t}{\alpha(\alpha - \beta t)}, \frac{q}{\alpha - \beta t}\right), \end{aligned} \tag{5.4}$$

i.e.

$$R_{\alpha,\beta}(f)(t, q) = f\left(\varphi_{\alpha,\beta}(t), \psi_{\alpha,\beta}(t, q)\right)$$

where

$$\varphi_{\alpha,\beta}(t) \underset{\text{def}}{\equiv} \frac{t}{\alpha(\alpha - \beta t)}$$

and

$$\psi_{\alpha,\beta}(t, q) \underset{\text{def}}{\equiv} \frac{q}{\alpha - \beta t}.$$

Proposition 5.1 *Let $z(\cdot)$ denote a Bernstein diffusion solving* (1.1) *and let us define $z_{\alpha,\beta}$ by:*

$$z_{\alpha,\beta}(\varphi_{\alpha,\beta}(t)) = \psi_{\alpha,\beta}(t, z(t)). \tag{5.5}$$

Then $z_{\alpha,\beta} = S^{(\alpha,\beta)}(z)$, where $S^{(\alpha,\beta)} : \mathcal{C}(\mathbb{R}^+, \mathbb{R}) \longrightarrow \mathcal{C}([0, T], \mathbb{R})$, with $T = -(\alpha\beta)^{-1}$ for $\beta < 0$ and $+\infty$ otherwise, is Patie's transformation in [12] p.49.

Proof: Let us set $t_{\alpha,\beta} = \varphi_{\alpha,\beta}(t)$, then $t_{\alpha,\beta} = \frac{t}{\alpha(\alpha-\beta t)}$ whence

$$\begin{aligned} 1+\alpha\beta t_{\alpha,\beta} &= 1+\frac{\beta t}{\alpha-\beta t} \\ &= \frac{\alpha}{\alpha-\beta t} \end{aligned}$$

and:

$$\begin{aligned} z_{\alpha,\beta}(t_{\alpha,\beta}) &= z_{\alpha,\beta}(\varphi_{\alpha,\beta}(t)) \\ &= \psi_{\alpha,\beta}(t, z(t)) \\ &= (\alpha-\beta t)^{-1} z(t) \\ &= \alpha^{-1}(1+\alpha\beta t_{\alpha,\beta}) z(t) \\ &= \alpha^{-1}(1+\alpha\beta t_{\alpha,\beta}) z\left(\frac{\alpha^2 t_{\alpha,\beta}}{1+\alpha\beta t_{\alpha,\beta}}\right). \end{aligned}$$

Now by definition ([12], p.49),

$$S^{(\alpha,\beta)}(\omega)(\tau) = \frac{1+\alpha\beta\tau}{\alpha}\,\omega\left(\frac{\alpha^2\tau}{1+\alpha\beta\tau}\right)$$

so

$$z_{\alpha,\beta}(t_{\alpha,\beta}) = S^{(\alpha,\beta)}(z)(t_{\alpha,\beta}), \text{ i.e. } z_{\alpha,\beta} = S^{(\alpha,\beta)}(z).$$

□

Corollary 5.2 *When $\alpha = 1, z_{1,\beta} = S^{(\beta)}(z)$, where $S^{(\beta)} \underset{\text{def}}{\equiv} S^{(1,\beta)}$ is defined as in [2], §2. In particular, for $\beta < 0$ and $\eta_\chi = \chi = 1, z(t) = \sqrt{\hbar}\, w(t)$, as in Ex 2) §4 and $z_{1,\beta}(t) = \sqrt{\hbar}\, w_{0,0}^{0,-\beta^{-1}}(t)$, a Brownian bridge.*

Proof: When $\alpha = 1$, $\mu = 0$ we have $R_{\alpha,\beta} = e^{-\frac{\lambda}{2}N_6}$ and $z_{\alpha,\beta}(\tau) = (1+\beta\tau)z\left(\frac{\tau}{1+\beta\tau}\right) \equiv S^{(\beta)}(z)(\tau)$ by [2], §1 p.226. Since $R_{\alpha,\beta} = e^{-\frac{\lambda}{2}N_6} = e^{\frac{\beta}{2}N_6}$ the statement about the Brownian Bridge follows from [Iso] p.201 (setting $\alpha = -\beta$).

□

From the structure of the Lie algebra $\mathcal{H}$ follows:

Theorem 5.3 *For all $\alpha > 0$, $\alpha' > 0$, β and β', one has:*

$$R_{\alpha',\beta'} \circ R_{\alpha,\beta} = R_{\alpha'',\beta''}$$

where

$$\alpha'' \underset{\text{def}}{\equiv} \alpha\alpha'$$

and

$$\beta'' \underset{\text{def}}{\equiv} \alpha\beta' + \frac{\beta}{\alpha'}.$$

Proof: From $[N_4, N_6] = 2N_6$ (see [Iso],) follows:

$$N_6 N_4 = (N_4 - 2I) N_6,$$

whence, by an easy induction over m:

$$\forall m \in \mathbb{N} \quad N_6 {N_4}^m = (N_4 - 2I)^m N_6,$$

whence:

$$N_6 \, e^{\mu N_4} = e^{-2\mu} \, e^{\mu N_4} \, N_6.$$

Now another easy induction yields:

$$\begin{array}{l} \forall n \in \mathbb{N} \\ N_6^n \, e^{\mu N_4} = e^{-2n\mu} \, e^{\mu N_4} \, N_6^n, \end{array}$$

whence:

$$e^{-\frac{\lambda'}{2} N_6} \, e^{\mu N_4} N = e^{\mu N_4} \, e^{-\frac{\lambda'}{2} \, e^{-2\mu} N_6}.$$

It follows that:

$$\begin{array}{rcl} R_{\alpha',\beta'} \circ R_{\alpha,\beta} & = & e^{\mu' N_4} \, e^{-\frac{\lambda'}{2} N_6} \, e^{\mu N_4} \, e^{-\frac{\lambda}{2} N_6} \\ & = & e^{\mu' N_4} \, e^{\mu N_4} \, e^{-\frac{\lambda'}{2} \, e^{-2\mu} N_6} \, e^{-\frac{\lambda}{2} N_6} \\ & = & e^{\mu'' N_4} \, e^{-\frac{\lambda''}{2} N_6} \end{array}$$

where

$$\begin{array}{rcl} \mu'' & = & \mu + \mu' \\ & = & -ln(\alpha) - ln(\alpha') \\ & = & -ln(\alpha'') \end{array}$$

and:

$$\begin{array}{rcl} \lambda'' & = & \lambda' \, e^{-2\mu} + \lambda \\ & = & (-\alpha'\beta') \, e^{2ln\alpha} - \alpha\beta \\ & = & -\alpha\alpha' \left(\alpha\beta' + \dfrac{\beta}{\alpha'} \right) \\ & = & -\alpha''\beta''. \end{array}$$

Therefore

$$R_{\alpha',\beta'} \circ R_{\alpha,\beta} = R_{\alpha'',\beta''}.$$

□

Corollary 5.4 *One has:*

$$S^{(\alpha,\beta)} \circ S^{(\alpha',\beta')} = S^{(\alpha'',\beta'')}.$$

Proof: By definition,

$$\begin{aligned}\varphi_{\alpha'',\beta''}(t) &= R_{\alpha'',\beta''}(t)\\ &= R_{\alpha',\beta'}(R_{\alpha,\beta}(t))\\ &= R_{\alpha',\beta'}(\varphi_{\alpha,\beta}(t))\\ &= \varphi_{\alpha,\beta}(\varphi_{\alpha',\beta'}(t))\end{aligned}$$

and

$$\begin{aligned}\psi_{\alpha'',\beta''}(t,q) &= R_{\alpha'',\beta''}(q)\\ &= R_{\alpha',\beta'}(R_{\alpha,\beta}(q))\\ &= R_{\alpha',\beta'}(\psi_{\alpha,\beta}(t,q))\\ &= \psi_{\alpha,\beta}(\varphi_{\alpha',\beta'}(t),\ \psi_{\alpha',\beta'}(t,q))\end{aligned}$$

as well as

$$\begin{aligned}S^{\alpha'',\beta''}(z)(\varphi_{\alpha'',\beta''}(t)) &= \psi_{\alpha'',\beta''}(t,z(t))\\ &= \psi_{\alpha,\beta}\left(\varphi_{\alpha',\beta'}(t),\psi_{\alpha',\beta'}(t,z(t))\right)\\ &= \psi_{\alpha,\beta}(\varphi_{\alpha',\beta'}(t),\ S^{(\alpha',\beta')}(z)(\varphi_{\alpha',\beta'}(t)))\\ &= S^{(\alpha,\beta)}(S^{(\alpha',\beta')}(z))[\varphi_{\alpha,\beta}(\varphi_{\alpha',\beta'}(t))]\\ &= (S^{(\alpha,\beta)}\circ S^{(\alpha',\beta')})(z)[\varphi_{\alpha'',\beta''}(t)],\end{aligned}$$

whence

$$S^{(\alpha'',\beta'')}(z) = (S^{(\alpha,\beta)}\circ S^{(\alpha',\beta')})(z)$$

i.e.

$$S^{(\alpha,\beta)}\circ S^{(\alpha',\beta')} = S^{\left(\alpha\alpha',\alpha\beta'+\frac{\beta}{\alpha'}\right)}$$

i.e. Patie's formula ([12], p.60), modulo correction of a misprint. □

Remark 5.5 *The above formula can also be obtained in an elementary fashion:*

$$\begin{aligned}((S^{(\alpha,\beta)}\circ S^{(\alpha',\beta')})(\omega))(\tau) &= (S^{(\alpha,\beta)}(S^{(\alpha'\beta')}(\omega)))(\tau)\\ &= \frac{1+\alpha\beta\tau}{\alpha}(S^{(\alpha',\beta')}(\omega))\left(\frac{\alpha^2\tau}{1+\alpha\beta\tau}\right)\\ &= \frac{1+\alpha\beta\tau}{\alpha}\,\frac{1+\alpha'\beta'\frac{\alpha^2\tau}{1+\alpha\beta\tau}}{\alpha'}\,\omega\left(\frac{{\alpha'}^2\frac{\alpha^2\tau}{1+\alpha\beta\tau}}{1+\alpha'\beta'\frac{\alpha^2\tau}{1+\alpha\beta\tau}}\right)\\ &= \frac{1+\alpha''\beta''\tau}{\alpha''}\,\omega\left(\frac{{\alpha''}^2\tau}{1+\alpha''\beta''\tau}\right)\\ &= \left(S^{(\alpha'',\beta'')}(\omega)\right)(\tau).\end{aligned}$$

Corollary 5.6 ([2]) $S^{(\beta)}$ satisfies the semigroup property.

Proof: One applies Corollary 5.4 with $\alpha = \alpha' = 1$; in this case, $\alpha'' = 1$ and $\beta'' = \beta + \beta'$ whence:

$$S^{(\beta+\beta')} = S^{(\beta)} \circ S^{(\beta')}.$$

□

6 Conclusion

The method used here is considerably more general than our present results.

Of course, it still holds if we start from the n dimensional heat equation instead of Eq(1.3). All the results of Theorem 3.1 and 3.2, §3 are preserved. In fact, the method is the same when the underlying stochastic processes (and therefore operator H_0 of Eq(1.3)) are more general than diffusions. In particular, the method can be adapted to the class of diffusion processes with jumps introduced in [13].

Finally, let us observe that the analogy alluded to, in the introduction, between our stochastic deformation and the quantization of elementary dynamical systems, is not superficial. For example, the stochastic Noether Theorem associated with the isovectors N provides, after coming back from Eq(3.41) to Schrödinger equation, quantum first integrals in the L^2 sense. Even in the free one dimensional case, this list of constants is strictly larger than the one known by traditional means in quantum mechanics (cf. [5] and [16]).

Part of the stochastic deformation strategy illustrated here could also be relevant to a more general one, inspired by ideas of dynamical systems [14].

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