Bernstein Diffusions for a Class of Linear Parabolic Partial Differential Equations

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Abstract

In this article we prove the existence of Bernstein processes which we associate in a natural way with a class of non-autonomous linear parabolic initial- and final-boundary value problems defined in bounded convex subsets of Euclidean space of arbitrary dimension. Under certain conditions regarding their joint endpoint distributions, we also prove that such processes become reversible Markov diffusions. Furthermore we show that those diffusions satisfy two Itô equations for some suitably constructed Wiener processes, and from that analysis derive Feynman-Kac representations for the solutions to the given equations. We then illustrate some of our results by considering the heat equation with Neumann boundary conditions both in a one-dimensional bounded interval and in a two-dimensional disk.

1 Introduction and Outline

It is well known that Itô's theory of stochastic differential equations makes it possible to associate a Markov process to essentially any second-order elliptic differential operator, and that the fine properties of each one of these objects allow one to get precise information about the other. In particular, the knowledge of the behavior of such a diffusion typically leads to the discovery of new phenomena regarding the solutions to a host of elliptic and parabolic partial differential equations, ranging from Dirichlet and Neumann initial-boundary value problems to equations describing wave front propagation in periodic and random media. This is testified, for instance, by the many results and references in [9] and [12].

By the same token it is also possible to associate Markov processes with the Schrödinger equation of quantum physics in a variety of ways, as in [5] where the author's considerations rest on the principles of stochastic mechanics set forth in [23], and in [34] whose constructions are related to the properties of the stochastic variational calculus introduced in [33].

The diffusion processes constructed in [5] and [34] are, however, quite different from the more traditional dissipative diffusions in that they encode all the conservative properties of quantum mechanics. In particular, they exhibit the invariance under time reversal inherent in the Schrödinger equation and thereby maintain a perfect symmetry between past and future. More to the point, the Markov processes of [34] actually emerge as particular cases of reversible diffusions that belong to the larger class of the so-called reciprocal or Bernstein processes, whose theory was launched many years ago in [2] following Schrödinger's seminal contribution in [26]. The theory of Bernstein processes was subsequently further developed and systematically investigated in [18], and since then has played an important rôle in relating various fields such as the Malliavin calculus and Euclidean quantum mechanics, or Markov bridges with jumps and Lévy processes, to name only a few (see for instance [6], [7], [15], [24] and the references therein for a more complete account).

Given these facts there remains the interesting and intriguing question whether it is possible to associate reversible diffusion processes to parabolic equations of general form whose solutions typically display irreversible behavior, and to get new and nontrivial information out of this association. In his attempt to understand certain analogies between the properties of Brownian motion and quantum mechanics in the last section of [26], Schrödinger answers the question positively by analyzing a simple case through statistical arguments.

It is the purpose of this article to show that this can also be achieved by purely analytical means and indeed with a considerable degree of generality. Let $D \subset \mathbb{R}^d$ be a bounded open convex subset whose boundary is denoted by ∂D. We consider parabolic *initial-boundary value problems* of the form

$$\begin{aligned} \partial_t u(x,t) &= \frac{1}{2}\operatorname{div}_x\left(k(x,t)\nabla_x u(x,t)\right) - \left(l(x,t),\nabla_x u(x,t)\right)_{\mathbb{R}^d} - V(x,t)u(x,t), \\ &(x,t) \in D \times (0,T], \\ u(x,0) &= \varphi(x), \quad x \in D, \\ \frac{\partial u(x,t)}{\partial n_k(x,t)} &= 0, \quad (x,t) \in \partial D \times (0,T] \end{aligned} \tag{1}$$

with $T \in (0,+\infty)$ arbitrary, $(.,.)_{\mathbb{R}^d}$ the Euclidean inner product in $\mathbb{R}^d$, and where the last relation in (1) stands for the conormal derivative of u relative to the matrix-valued function k. Furthermore l, V and φ are an $\mathbb{R}^d$-valued vector-field and real-valued functions, respectively.

Let us now associate with (1) its *adjoint final-boundary value problem,* namely,

$$\begin{aligned} -\partial_t v(x,t) &= \frac{1}{2}\operatorname{div}_x\left(k(x,t)\nabla_x v(x,t)\right) + \operatorname{div}_x\left(v(x,t)l(x,t)\right) - V(x,t)v(x,t), \\ &(x,t) \in D \times [0,T), \\ v(x,T) &= \psi(x), \quad x \in D, \\ \frac{\partial v(x,t)}{\partial n_k(x,t)} &= 0, \quad (x,t) \in \partial D \times [0,T\ ), \end{aligned} \tag{2}$$

in order to build reversibility and eventually the Markov property into the theory we develop below.

We then organize the remaining part of this article in the following way: in Section 1 we prove the existence of Bernstein processes $Z_{\tau\in[0,T]}$ wandering in $\overline{D} := D \cup \partial D$, which we associate with (1) and (2) in a very natural way provided the coefficients therein be sufficiently smooth. We also show there that under certain conditions which pertain to their joint initial and final distributions, the processes in question become reversible Markov diffusions. Moreover, we can express their probability density as the product of the solutions to (1) and (2), and prove that they also satisfy two Itô stochastic differential equations for some suitably constructed Wiener processes. This last fact is not merely anecdotic, as it allows us eventually to get Feynman-Kac representations of those solutions in terms of $Z_{\tau\in[0,T]}$ in a perfectly symmetric manner, by invoking Itô's *backward* stochastic calculus in the case of (1), and quite independently the usual *forward* stochastic calculus in the case of (2). In Section 1 we also explain why the notion of reversible Markov diffusion which emerges from our considerations corresponds to a generalization of the classical notion of reversibility put forward in [20] and inspired by the last section of [26], which was many years later reformulated in [8] and [17]. In Section 2 we illustrate some of our results by means of two examples involving the simple heat equation in a one-dimensional bounded interval and in a two-dimensional disk, and in one of them the process we construct shares some of the features of a reflected diffusion. Finally, for the sake of completeness and for the convenience of the reader, we devote an appendix to reviewing the existence theory of weak solutions to (1) and (2) which provide the ingredients that are essential to our construction when the coefficients therein are indeed smooth enough.

Throughout this article we use the standard notations for all the functional spaces we need without any further comments, referring the reader for instance to [1], [10], [11] and [29]. We also use freely results on martingales and Wiener processes from [14].

2 A Class of Bernstein Diffusions

We are looking for processes $Z_{\tau\in[0,T]}$ whose natural state space is the compact convex set $\overline{D}$ endowed with the Borel σ-algebra $\mathcal{B}(\overline{D})$, and whose behavior corresponding to time values belonging to any subinterval $(s,t) \subset [0,T]$ is conditioned by the knowledge of Z_s and Z_t alone. This means that all past information gathered prior to time s is irrelevant, as is all future information accumulated after time t. The precise notion we need is the following (see [18] and some of the references therein for other equivalent formulations):

DEFINITION 1. *We say the $\overline{D}$-valued process $Z_{\tau\in[0,T]}$ defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a Bernstein process if the following conditional expectations satisfy the relation*

$$\mathbb{E}\left(h(Z_r)\left|\mathcal{F}_s^+ \vee \mathcal{F}_t^-\right.\right) = \mathbb{E}\left(h(Z_r)\left|Z_s, Z_t\right.\right) \tag{3}$$

for every bounded Borel measurable function $h : \overline{D} \mapsto \mathbb{R}$, and for all r, s, t satisfying $r \in (s,t) \subset [0,T]$. In (3), $\mathcal{F}_s^+$ denotes the σ-algebra generated by the Z_τ's for all $\tau \in [0,s]$, while $\mathcal{F}_t^-$ is that generated by the Z_τ's for all $\tau \in [t,T]$.

In the sequel we shall denote by $\mathcal{F}^+_{\tau\in[0,T]}$ the *increasing* filtration generated by the $\mathcal{F}_s^+$'s, and by $\mathcal{F}^-_{\tau\in[0,T]}$ the *decreasing* filtration generated by the $\mathcal{F}_t^-$'s.

In order to construct such processes with relation to (1) and (2) we first recast Schrödinger's and Bernstein's ideas to fit the theory developed in [18]. Accordingly, the two main ingredients we need are transition density functions for the processes together with joint probability distributions for Z_0 and Z_T. As we shall see, this requires the existence of *classical, positive* solutions to (1) and (2), respectively, which in turn requires good smoothness properties of k, l, V, φ and ψ. In order to achieve this we assume that the boundary ∂D is of class $\mathcal{C}^{2+\alpha}$ for some $\alpha \in (0,1)$, and then impose the following hypotheses where $n(x)$ denotes the unit outer normal vector at $x \in \partial D$ (in all that follows we write c for all the irrelevant constants that occur in the various estimates unless we specify these constants otherwise, and we refer to [28] for a definition of the above concepts and for various properties of the spaces of Hölder continuous functions introduced here):

(K) The function $k : \overline{D} \times [0,T] \mapsto \mathbb{R}^{d^2}$ is such that for every $i, j \in \{1, ..., d\}$ we have $k_{i,j} = k_{j,i} \in \mathcal{C}^{\alpha,\frac{\alpha}{2}}(\overline{D} \times [0,T])$ and $\frac{\partial k_{i,j}}{\partial x_l} \in \mathcal{C}^{\alpha,\frac{\alpha}{2}}(\overline{D} \times [0,T])$ for all i, j, l. Moreover, the uniform ellipticity condition

$$(k(x,t)q, q)_{\mathbb{R}^d} \geq \underline{k}\, |q|^2$$

with $\underline{k} > 0$ holds for all $(x,t) \in \overline{D} \times [0,T]$ and all $q \in \mathbb{R}^d$, where $|.|$ denotes the Euclidean norm. Finally, the conormal vector-field $n_k(x,t) := k(x,t)n(x)$ is uniformly outward pointing, nowhere tangent to ∂D for every $t \in [0,T]$ and we have

$$(x,t) \mapsto \sum_{i=1}^{d} k_{i,j}(x,t) n_i(x) \in \mathcal{C}^{1+\alpha,\frac{1+\alpha}{2}}(\partial D \times [0,T])$$

for each j.

(L) For the components of the vector-field $l : \overline{D} \times [0,T] \mapsto \mathbb{R}^d$ we have $l_i, \frac{\partial l_i}{\partial x_j} \in \mathcal{C}^{\alpha,\frac{\alpha}{2}}(\overline{D} \times [0,T])$ for all $i, j \in \{1, ..., d\}$.

(V) The function $V : \overline{D} \times [0,T] \mapsto \mathbb{R}$ is such that $V \in \mathcal{C}^{\alpha,\frac{\alpha}{2}}(\overline{D} \times [0,T])$.

Thus, the above functions are all jointly Hölder continuous in the space-time variable (x,t).

Finally, φ and ψ ought to be smooth enough as well and compatible with the boundary conditions in (1) and (2):

(IF) We have $\varphi, \psi \in \mathcal{C}^{2+\alpha}(\overline{D})$ with φ satisfying the conormal boundary condition relative to k at $t = 0$, and ψ satisfying that condition at $t = T$.

It then follows from the classical theory of linear parabolic equations (see for instance [13], or more specifically Chapter 4 in [21] and Theorem 1 in [10]), which is, of course, a particular case of the variational approach reviewed in the appendix, that there exist evolution systems $U_A(t,s)_{0\leq s\leq t\leq T}$ and $U_A^*(t,s)_{0\leq s\leq t\leq T}$ in $L^2(D)$ given by

$$U_A(t,s)f(x) = \begin{cases} f(x) & \text{if } t = s, \\ \int_D dy g_A(x,t;y,s)f(y) & \text{if } t > s \end{cases} \tag{4}$$

and

$$U_A^*(t,s)f(x) = \begin{cases} f(x) & \text{if } t = s, \\ \int_D dy g_A^*(x,s;y,t)f(y) & \text{if } t > s \end{cases} \tag{5}$$

with g_A and g_A^* the parabolic Green functions associated with (1) and (2). These functions satisfy

$$g_A^*(x,s;y,t) = g_A(y,t;x,s) \tag{6}$$

for all $s,t \in [0,T]$ with $t > s$, and furthermore the functions defined by

$$u_\varphi(x,t) := \int_D dy g_A(x,t;y,0)\varphi(y), \quad t \in (0,T] \tag{7}$$

and

$$v_\psi(x,t) := \int_D dy g_A^*(x,t;y,T)\psi(y), \quad t \in [0,T)\,, \tag{8}$$

are indeed classical solutions to (1) and (2), respectively. More precisely we have the following result for them:

Proposition 1. *Assume that Hypotheses (K), (L), (V) and (IF) hold. Then the following statements are valid:*

(a) We have $u_\varphi, v_\psi \in \mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{D}\times[0,T])$ *with* u_φ *the unique classical solution to (1) and* v_ψ *the unique classical solution to (2).*

(b) If $\varphi > 0$, $\psi > 0$ *on* $\overline{D}$ *we have* $u_\varphi > 0$, $v_\psi > 0$ *on* $\overline{D}\times[0,T]$*, respectively.*

(c) If $\varphi > 0$ *on* $\overline{D}$ *we have* $g_A > 0$ *for all* $x, y \in \overline{D}$ *and all* $s,t \in [0,T]$ *with* $t > s$ *and furthermore* g_A *is jointly continuous in these variables. Moreover, this function is twice continuously differentiable in* x*, once continuously differentiable in* t *and satisfies*

$$\begin{aligned} &\partial_t g_A(x,t;y,s) = -A(t)g_A(x,t;y,s), \quad (x,t) \in D\times(s,T]\,, \\ &\frac{\partial g_A(x,t;y,s)}{\partial n_k(x,t)} = 0, \quad (x,t) \in \partial D\times(s,T]\,, \end{aligned} \tag{9}$$

where the elliptic differential operator

$$A(t) := -\frac{1}{2}\operatorname{div}(k(.,t)\nabla) + (l(.,t),\nabla)_{\mathbb{R}^d} + V(.,t) \tag{10}$$

corresponds to the right-hand side of (1). Finally, the heat kernel estimate

$$g_A(x,t;y,s) \leq c\,(t-s)^{-\frac{d}{2}} \exp\left[-c\frac{|x-y|^2}{t-s}\right] \tag{11}$$

holds.

(d) If $\psi > 0$ *on* $\overline{D}$ *we have* $g_A^* > 0$ *for all* $x, y \in \overline{D}$ *and all* $s, t \in [0, T]$ *with* $t > s$ *and furthermore* g_A^* *is jointly continuous in these variables. Moreover, this function is twice continuously differentiable in* x*, once continuously differentiable in* s *and satisfies*

$$-\partial_s g_A^*(x, s; y, t) = -A^*(s) g_A^*(x, s; y, t), \quad (x, s) \in D \times [0, t), \tag{12}$$
$$\frac{\partial g_A^*(x, s; y, t)}{\partial n_k(x, s)} = 0, \quad (x, s) \in \partial D \times [0, t),$$

where

$$A^*(s) := -\frac{1}{2} \operatorname{div}(k(., s)\nabla) - \operatorname{div}(\cdot l(., s)) + V(., s) \tag{13}$$

is the formal adjoint to (10) corresponding to the right-hand side of (2). Finally, the same heat kernel estimate

$$g_A^*(x, s; y, t) \leq c\,(t-s)^{-\frac{d}{2}} \exp\left[-c\frac{|x-y|^2}{t-s}\right] \tag{14}$$

as in (c) holds.

Hypotheses (K), (L), (V) and (IF) will be our standing hypotheses in the sequel.

Let us now consider the function

$$P(x, t; E, r; y, s) := \int_E dz p(x, t; z, r; y, s) \tag{15}$$

for every $E \in \mathcal{B}(\overline{D})$, where

$$p(x, t; z, r; y, s) := \frac{g_A(x, t; z, r) g_A(z, r; y, s)}{g_A(x, t; y, s)} \tag{16}$$

is well defined and positive for all $x, y, z \in \overline{D}$ and all r, s, t satisfying $r \in (s, t) \subset [0, T]$ when the first part of (c) in Proposition 1 holds. In addition, let μ be a positive integrable function on $\overline{D} \times \overline{D}$ such that

$$\mu(E \times F) := \int_{E \times F} dx dy \mu(x, y) \tag{17}$$

defines a probability measure on $\mathcal{B}(\overline{D}) \times \mathcal{B}(\overline{D})$. It is remarkable that the simultaneous knowledge of (15) and (17) is sufficient to guarantee the existence of a Bernstein process associated with (1). Indeed we have the following result:

Theorem 1. *Assume that the first part of (c) in Proposition 1 holds, and let* μ *be given by (17). Then there exists a probability space* $(\Omega, \mathcal{F}, \mathbb{P}_\mu)$ *and a* $\overline{D}$*-valued Bernstein process* $Z_{\tau \in [0,T]}$ *on* $(\Omega, \mathcal{F}, \mathbb{P}_\mu)$ *such that*

$$\mathbb{P}_\mu(Z_0 \in E_0, Z_T \in E_T) = \mu(E_0 \times E_T) \tag{18}$$

for all $E_0, E_T \in \mathcal{B}(\overline{D})$*. Furthermore we have*

$$\mathbb{P}_\mu\left(Z_r \in E \,|Z_s, Z_t\,\right) = P\left(Z_t, t; E, r; Z_s, s\right)$$

for every $E \in \mathcal{B}(\overline{D})$ *and all* r, s, t *satisfying* $r \in (s,t) \subset [0,T]$*, and moreover the finite-dimensional distributions are given by*

$$\begin{aligned}&\mathbb{P}_\mu\left(Z_0 \in E_0, Z_{t_1} \in E_1, ..., Z_{t_n} \in E_n, Z_T \in E_T\right)\\&= \int_{E_0\times E_T} dxdy\mu(x,y)\int_{E_1} dx_1 ... \int_{E_n} dx_n \prod_{i=1}^{n} p\left(y, T; x_i, t_i; x_{i-1}, t_{i-1}\right)\end{aligned} \tag{19}$$

where $x_0 = x$*, for all* $E_0, E_1, ..., E_n, E_T \in \mathcal{B}(\overline{D})$ *and all* $t_0, ..., t_n \in [0,T)$ *satisfying* $t_0 = 0 < t_1 < ... < t_n < T$*. Finally,* $\mathbb{P}_\mu$ *is the unique probability measure with these properties.*

Proof. The mapping $(x,y) \mapsto P\left(x,t;E,r;y,s\right)$ is evidently continuous on $\overline{D}\times\overline{D}$ for every $E \in \mathcal{B}(\overline{D})$ and all r,s,t satisfying $r \in (s,t) \subset [0,T]$. Moreover, the mapping $E \mapsto P\left(x,t;E,r;y,s\right)$ defines a probability measure on $\mathcal{B}(\overline{D})$ for all $x, y \in \overline{D}$ and all of those r,s,t; this is indeed a consequence of the composition law

$$U_A(t,s) = U_A(t,r)U_A(r,s) \tag{20}$$

pertaining to the evolution system (4), which translates as

$$g_A(x,t;y,s) = \int_D dz g_A(x,t;z,r)g_A(z,r;y,s) \tag{21}$$

for the corresponding Green function. We also have the relation

$$\begin{aligned}&\int_E dx p(x',t';x,t;y;s)P\left(x,t;F,r;y,s\right)\\&= \int_F dz p(x',t';z,r;y;s)P\left(x',t';E,t;z,r\right)\end{aligned} \tag{22}$$

for all $E, F \in \mathcal{B}(\overline{D})$, all $x', y \in \overline{D}$ and all r,s,t,t' satisfying $r \in (s,t) \subset (s,t') \subset [0,T]$. In order to see this we remark that the relation

$$\begin{aligned}&p(x',t';x,t;y,s)p(x,t;z,r;y,s)\\&= p(x',t';z,r;y,s)p(x',t';x,t;z,r)\end{aligned}$$

holds as an immediate consequence of (16), so that (22) obtains by integrating both sides of the preceding identity first with respect to z over F, and then the resulting expression with respect to x over E. The statement of the theorem then follows from a direct adaptation of the arguments in Section 2 of [18]. ■

Remark. The preceding result makes it clear that the probability of having $Z_r \in E$ is indeed solely conditioned by the past information at time s and the future information at time t. Furthermore, owing to (18) and (19) the probability measure μ clearly plays the rôle of a joint endpoint distribution for Z_0, Z_T, and we ought to note that there is *a priori* no reason why $Z_{\tau\in[0,T]}$ should be a Markov process when μ is arbitrary. However $Z_{\tau\in[0,T]}$ does become a reversible Markov diffusion in a sense we shall define shortly, for a very special class of endpoint distributions which we will identify. For this we first associate a *forward* Markov transition function with (2) in the following way:

Lemma 1. *Assume that the part of (b) in Proposition 1 relative to ψ and v_ψ holds, together with the first part of (d). Let us define the function*

$$M^*(x,s;E,t) := \int_E dy m^*(x,s;y,t) \tag{23}$$

for each $E \in \mathcal{B}(\overline{D})$, every $x \in \overline{D}$ and all $s,t \in [0,T]$ with $t > s$, where

$$m^*(x,s;y,t) := g_A^*(x,s;y,t)\frac{v_\psi(y,t)}{v_\psi(x,s)} \tag{24}$$

with v_ψ given by (8). Then (23) is the transition function of a forward Markov process in $\overline{D}$.

Proof. The mapping $x \mapsto M^*(x,s;E,t)$ is positive and continuous on $\overline{D}$, and furthermore $E \mapsto M^*(x,s;E,t)$ defines a probability measure on $\mathcal{B}(\overline{D})$. This last assertion follows from the relation

$$v_\psi(x,s) = \int_D dy g_A^*(x,s;y,t) v_\psi(y,t),$$

which, in turn, is a simple consequence of the composition law

$$U_A^*(t,s) = U_A^*(r,s)U_A^*(t,r)$$

for the evolution system (5), which gives

$$g_A^*(x,s;y,t) = \int_D dz g_A^*(x,s;z,r) g_A^*(z,r;y,t) \tag{25}$$

for the corresponding Green function. But then the Chapman-Kolmogorov relation

$$M^*(x,s;E,t) = \int_D dy m^*(x,s;y,r) M^*(y,r;E,t)$$

holds for (23), since (24) and (25) imply that

$$m^*(x,s;y,t) = \int_D dz m^*(x,s;z,r) m^*(z,r;y,t)$$

for all r,s,t such that $r \in (s,t) \subset [0,T]$. ■

In a completely symmetric way, we can also associate a *backward* Markov transition function with (1). Indeed we have the following result, whose proof is entirely similar to that of the preceding lemma and thereby omitted:

Lemma 2. *Assume that the part of (b) in Proposition 1 relative to φ and u_{φ} holds, together with the first part of (c). Let us define the function*

$$M\left(x,t;E,s\right) := \int_{E} dym(x,t;y,s) \tag{26}$$

for each $E \in \mathcal{B}(\overline{D})$, every $x \in \overline{D}$ and all $s,t \in [0,T]$ with $t > s$, where

$$m(x,t;y,s) := g_{A}(x,t;y,s)\frac{u_{\varphi}(y,s)}{u_{\varphi}(x,t)} \tag{27}$$

with u_{φ} given by (7). Then (26) is the transition function of a backward Markov process in $\overline{D}$.

The remarkable fact is that when $Z_{\tau\in[0,T]}$ is reversible in the sense of the definition below, it becomes a realization of the Markov processes we are alluding to in the preceding two lemmas. The precise notion we need is the following, where we recall that $\varphi > 0$, $\psi > 0$:

Definition 2. *We say the Bernstein process $Z_{\tau\in[0,T]}$ of Theorem 1 is reversible if the density of the joint probability measure (17) is of the form*

$$\mu(x,y) = \varphi(x)g_{A}(y,T;x,0)\psi(y) \tag{28}$$

where

$$\int_{D\times D} dxdy\varphi(x)g_{A}(y,T;x,0)\psi(y) = 1. \tag{29}$$

For the corresponding initial and final marginal distributions we then have

$$\mu_{0}(E) := \mu(E\times D) = \int_{E} dx\varphi(x)v_{\psi}(x,0)$$

and

$$\mu_{T}(F) := \mu(D\times F) = \int_{F} dy\psi(y)u_{\varphi}(y,T)$$

as a consequence of (8) and (7), respectively, with $\mu_{0}(D) = \mu_{T}(D) = 1$. Our definition is motivated by the following result, where we use the shorthand notation

$$\mu_{0}(x) = \varphi(x)v_{\psi}(x,0) \tag{30}$$

and

$$\mu_{T}(y) = \psi(y)u_{\varphi}(y,T) \tag{31}$$

for the marginal densities:

Theorem 2. *Assume that (b) and the first parts of (c) and (d) in Proposition 1 hold. Assume furthermore that μ is given by (17) and (28), and let $Z_{\tau\in[0,T]}$ be the corresponding Bernstein process associated with (1) in the sense of Theorem 1. Then the following statements hold for its finite-dimensional distributions:*

(a) We have

$$\mathbb{P}_{\mu}\left(Z_{0}\in E_{0},Z_{t_{1}}\in E_{1},...,Z_{t_{n}}\in E_{n}\right) = \int_{E_{0}}dx\mu_{0}(x)\int_{E_{1}}dx_{1}...\int_{E_{n}}dx_{n}\prod_{i=1}^{n}m^{*}\left(x_{i-1},t_{i-1};x_{i},t_{i}\right) \tag{32}$$

where m^{*} is given by (24) and $x_0 = x$, for all $E_0, E_1, ..., E_n \in \mathcal{B}(\overline{D})$ and all $t_0, ..., t_n \in [0,T)$ satisfying $t_0 = 0 < t_1 < ... < t_n < T$. Thus $Z_{\tau\in[0,T]}$ is a forward Markov process with transition function m^{*} and initial distribution density μ_0.

(b) We have

$$\mathbb{P}_{\mu}\left(Z_{T}\in E_{T},Z_{t_{n}}\in E_{n},...,Z_{t_{1}}\in E_{1}\right) = \int_{E_{T}}dy\mu_{T}(y)\int_{E_{1}}dx_{1}...\int_{E_{n}}dx_{n}\prod_{i=1}^{n}m\left(x_{i+1},t_{i+1};x_{i},t_{i}\right) \tag{33}$$

where m is given by (27) and $x_{n+1} = y$, for all $E_T, E_1, ..., E_n \in \mathcal{B}(\overline{D})$ and all $t_1, ..., t_{n+1} \in (0,T]$ satisfying $T = t_{n+1} > t_n > ... > t_1 > 0$. Thus $Z_{\tau\in[0,T]}$ is also a backward Markov process with transition function m and final distribution density μ_T.

(c) We have

$$\mathbb{P}_{\mu}\left(Z_{t}\in E\right) = \int_{E}dx u_{\varphi}(x,t)v_{\psi}(x,t) \tag{34}$$

for each $E \in \mathcal{B}(\overline{D})$ and every $t \in [0,T]$, where u_φ and v_ψ are given by (7) and (8), respectively.

Proof. From (19) with $E_T = D$ and (16) we have

$$\begin{aligned}
&\mathbb{P}_{\mu}\left(Z_{0}\in E_{0},Z_{t_{1}}\in E_{1},...,Z_{t_{n}}\in E_{n}\right)\\
&= \int_{E_{0}\times D}dxdy\mu(x,y)\int_{E_{1}}dx_{1}...\int_{E_{n}}dx_{n}\prod_{i=1}^{n}\frac{g_{A}(y,T;x_{i},t_{i})g_{A}(x_{i},t_{i};x_{i-1},t_{i-1})}{g_{A}(y,T;x_{i-1},t_{i-1})}\\
&= \int_{E_{0}\times D}dxdy\varphi(x)\psi(y)\int_{E_{1}}dx_{1}...\int_{E_{n}}dx_{n}\prod_{i=1}^{n}g_{A}(x_{i},t_{i};x_{i-1},t_{i-1})\times g_{A}(y,T;x_{n},t_{n})\\
&= \int_{E_{0}}dx\varphi(x)\int_{E_{1}}dx_{1}...\int_{E_{n}}dx_{n}\prod_{i=1}^{n}g_{A}^{*}(x_{i-1},t_{i-1};x_{i},t_{i})\times v_{\psi}(x_{n},t_{n})
\end{aligned} \tag{35}$$

after the use of (28), the successive cancellation of the denominators in the above product and the use of (6) with (8). Moreover, because of (24) we may write

$$g_A^*(x_{i-1},t_{i-1};x_i,t_i)v_\psi(x_i,t_i) = v_\psi(x_{i-1},t_{i-1})m^*\left(x_{i-1},t_{i-1};x_i,t_i\right)$$

for every $i \in \{1,...,n\}$, so that the repeated application of this relation in the product on the very right-hand side of (35) leads to

$$\begin{aligned}
&\mathbb{P}_\mu\left(Z_0 \in E_0, Z_{t_1} \in E_1, ..., Z_{t_n} \in E_n\right) \\
&= \int_{E_0} dx\varphi(x)v_\psi(x,0)\int_{E_1} dx_1 ... \int_{E_n} dx_n \prod_{i=1}^{n} m^*\left(x_{i-1},t_{i-1};x_i,t_i\right),
\end{aligned}$$

which is the desired result. The proof of (33) follows from entirely similar arguments based on (7) and (27), with $E_0 = D$. Finally, (34) is a straightforward consequence of (32) with $E_0 = D$, or of (33) with $E_T = D$, in both cases with $n = 1$. ■

Remarks. (1) It follows from (2), (12), (13) and some lengthy calculations that the transition density (24) satisfies the parabolic partial differential equation

$$\begin{aligned}
&-\partial_s m^*(x,s;y,t) \\
&= \frac{1}{2}\operatorname{div}_x\left(k(x,s)\nabla_x m^*(x,s;y,t)\right) + \left(l(x,s),\nabla_x m^*(x,s;y,t)\right)_{\mathbb{R}^d} \\
&+ \left(k(x,s)\nabla_x \ln v_\psi(x,s),\nabla_x m^*(x,s;y,t)\right)_{\mathbb{R}^d}
\end{aligned} \tag{36}$$

relative to the variables $(x,s) \in D \times [0,t)$ of the past, along with the boundary condition

$$\frac{\partial m^*(x,s;y,t)}{\partial n_k(x,s)} = 0, \quad (x,s) \in \partial D \times [0,t). \tag{37}$$

In a similar way we infer from (1), (9) and (10) that the transition density (27) satisfies the equation

$$\begin{aligned}
&\partial_t m(x,t;y,s) \\
&= \frac{1}{2}\operatorname{div}_x\left(k(x,t)\nabla_x m(x,t;y,s)\right) - \left(l(x,t),\nabla_x m(x,t;y,s)\right)_{\mathbb{R}^d} \\
&+ \left(k(x,t)\nabla_x \ln u_\varphi(x,t),\nabla_x m(x,t;y,s)\right)_{\mathbb{R}^d}
\end{aligned} \tag{38}$$

with respect to the variables $(x,t) \in D \times (s,T]$ of the future, together with the boundary condition

$$\frac{\partial m(x,t;y,s)}{\partial n_k(x,t)} = 0, \quad (x,t) \in \partial D \times (s,T]. \tag{39}$$

The preceding relations suggest that we may think of the reversible Markov process of Theorem 2 as a process wandering in $\overline{D}$ which becomes reflected in the conormal direction whenever it hits the boundary ∂D. This way of looking at $Z_{\tau\in[0,T]}$ is reminiscent of the definition of the standard reflected Brownian motion given at the very beginning of [16], and we can indeed prove the reflection property we just alluded to in the first example of Section 2. Further below we also explain the appearance of the somewhat exotic logarithmic terms in (36) and (38), whose structure is intimately tied up with the specific form of (24) and (27).

(2) The considerations that lead to the statement of Theorem 2 show that the marginal densities (30) and (31) are entirely determined by φ and ψ. We could also have adopted the inverse point of view, namely, that of prescribing continuous $\mu_0 > 0$ and $\mu_T > 0$ satisfying the normalization conditions

$$\int_D dx\mu_0(x) = \int_D dx\mu_T(x) = 1,$$

and then considered the relations

$$\begin{aligned}\varphi(x)\int_D dzg_A(z,T;x,0)\psi(z) &= \mu_0(x),\\ \psi(y)\int_D dzg_A(y,T;z,0)\varphi(z) &= \mu_T(y)\end{aligned}\tag{40}$$

as a nonlinear inhomogeneous system of integral equations in the two unknowns φ and ψ. However, whereas it is true that such μ_0 and μ_T imply the existence of a unique solution to (40) consisting of continuous and positive functions φ and ψ as a consequence of the main theorem in [3], the theory developed in that article guarantees neither their regularity nor their the Hölder continuity, which will be so crucial to our considerations. It is in fact an interesting open problem whether the main result of [3] can be extended to cover such situations.

(3) Theorem 2 clearly illustrates a kind of reversibility of $Z_{\tau\in[0,T]}$ in that this process can run back and forth within $\overline{D}$, which *a posteriori* justifies the terminology of Definition 2. In particular, the probability density

$$\rho_\mu(x,t) := u_\varphi(x,t)v_\psi(x,t)\tag{41}$$

in (34) is expressed as the product of solutions to (1) and (2), which indeed brings in the two time directions in an explicit way. Therefore, at this stage it is natural to start exploring the possible connections that might exist between the preceding considerations and the notion of reversibility put forward in [20]. On the one hand, it follows from (6), (24) and (27) that the identity

$$m(y,t;x,s)\rho_\mu(y,t) = \rho_\mu(x,s)m^*(x,s;y,t)\tag{42}$$

holds for all $(x,y) \in \overline{D}\times\overline{D}$ and all $s,t\in[0,T]$ with $t>s$, which plays the rôle of (8) in [20]. From (42) we then immediately infer that

$$\rho_\mu(y,t) = \int_D dx\rho_\mu(x,s)m^*(x,s;y,t)$$

and

$$\rho_\mu(x,s) = \int_D dy m(y,t;x,s)\rho_\mu(y,t),$$

which generalize (7) in [20]. On the other hand, however, let us assume momentarily that $Z_{\tau\in[0,T]}$ is also reversible in the sense of [20], which means that

$$m(y,t;x,s) = m^*(y,s;x,t) \tag{43}$$

according to (9) of that article. From the preceding relation it then follows at once from (6) that the equality

$$g_A(y,t;x,s)\frac{u_\varphi(x,s)}{u_\varphi(y,t)} = g_A(x,t;y,s)\frac{v_\psi(x,t)}{v_\psi(y,s)}$$

is valid for all $(x,y) \in \overline{D} \times \overline{D}$ and all $s,t \in [0,T]$ with $t > s$, which implies that

$$\rho_\mu(x,t) = \rho_\mu(x,s)$$

for each $x \in \overline{D}$ and every $t \geq s$ by choosing $y = x$. Therefore ρ_μ must be independent of time, a very particular situation indeed which is almost never realized in our context with the exception of a few cases. Thus, whereas $Z_{\tau\in[0,T]}$ is a reversible Markov process in the sense of Theorem 2 thanks to the very specific form of the joint measures (28), it is in general not reversible according to [20]. We will dwell more on this further in this section when we have additional information about $Z_{\tau\in[0,T]}$.

(4) While a separable version of the process $Z_{\tau\in[0,T]}$ always exists under the above hypotheses, we may also assume that $Z_{\tau\in[0,T]}$ is continuous. Indeed for all $s,t \in [0,T]$ with $t > s$ and $\gamma \in (0,+\infty)$ we have

$$\mathbb{E}_\mu |Z_t - Z_s|^\gamma = \int_{D\times D} dxdy\mu_s(x)m^*(x,s;y,t)|y-x|^\gamma$$

where μ_s is the distribution density at time s, namely,

$$\mu_s(x) = \int_D dz\mu_0(z)m^*(z,0;x,s),$$

with $\mathbb{E}_\mu$ the expectation functional with respect to $\mathbb{P}_\mu$. Consequently, since $(x,s,y,t) \mapsto v_\psi^{-1}(x,s)v_\psi(y,t)$ is uniformly bounded on $\overline{D} \times [0,T] \times \overline{D} \times [0,T]$ and since

$$\int_D dx\mu_s(x) = 1,$$

we get the estimate

$$
\begin{aligned}
&\mathbb{E}_{\mu}\left|Z_t - Z_s\right|^{\gamma} \\
&\leq c \int_D dx \mu_s(x) \int_D dy g_A^*(x,s;y,t) \left|y-x\right|^{\gamma} \\
&\leq c\,(t-s)^{-\frac{d}{2}} \int_D dx \mu_s(x) \int_D dy \exp\left[-c\frac{\left|y-x\right|^2}{t-s}\right] \left|y-x\right|^{\gamma} \\
&\leq c\,(t-s)^{\frac{\gamma}{2}} \int_{\mathbb{R}^d} dy \exp\left[-c\left|y\right|^2\right] \left|y\right|^{\gamma} = c\,(t-s)^{\frac{\gamma}{2}}
\end{aligned}
\tag{44}
$$

according to (14), translation invariance on $\mathbb{R}^d$ and an elementary change of variables. Therefore, fixing $\gamma \in (2, +\infty)$ we obtain

$$
\mathbb{E}_{\mu}\left|Z_t - Z_s\right|^{\gamma} \leq c\,(t-s)^{1+\delta}
$$

with $\delta = \frac{\gamma-2}{2}$, so that the assertion follows from Kolmogorov's continuity conditions. In the sequel we shall thereby always assume that $Z_{\tau\in[0,T]}$ is separable and continuous.

While Theorem 2 shows that the process $Z_{\tau\in[0,T]}$ can run forward and backward in a Markovian manner within $\overline{D}$ for the very specific class of endpoint distributions (28), we now proceed to investigate its dynamical properties more in detail. Our first step in this direction is to show that $Z_{\tau\in[0,T]}$ is a reversible Markov diffusion in $\overline{D}$.

Lemma 3. *Assume that the same hypotheses as in Lemma 1 are valid. Then Lindeberg's condition*

$$
\lim_{t\to s_+} (t-s)^{-1} \int_{\{y\in D:|x-y|>\varepsilon\}} dy m^*(x,s;y,t) = 0 \tag{45}
$$

holds uniformly in $x \in D$ for each s and every sufficiently small $\varepsilon > 0$. In a similar way, if the same hypotheses as in Lemma 2 hold we have

$$
\lim_{s\to t_-} (t-s)^{-1} \int_{\{y\in D:|x-y|>\varepsilon\}} dy m(x,t;y,s) = 0 \tag{46}
$$

uniformly in $x \in D$ for each t and every sufficiently small $\varepsilon > 0$.

Proof. In order to get (45) we must prove that

$$
\lim_{t\to s_+} (t-s)^{-1} \int_{\{y\in D:|x-y|>\varepsilon\}} dy g_A^*(x,s;y,t) \frac{v_\psi(y,t)}{v_\psi(x,s)} = 0
$$

uniformly in $x \in D$ for each s according to (24). The key observation for this is that the positive function $(x,s,y,t) \mapsto v_\psi^{-1}(x,s) v_\psi(y,t)$ is uniformly bounded as a consequence of its smoothness on $\overline{D} \times [0,T] \times \overline{D} \times [0,T]$, so that estimate (14) leads to

$$
\begin{aligned}
0 &\leq (t-s)^{-1} \int_{\{y \in D : |x-y| > \varepsilon\}} dy g_A^*(x,s;y,t) \frac{v_\psi(y,t)}{v_\psi(x,s)} \\
&\leq c\,(t-s)^{-\frac{d+2}{2}} \int_{\{y \in D : |x-y| > \varepsilon\}} dy \exp\left[-c\frac{|y-x|^2}{t-s}\right] \\
&\leq c\,(t-s)^{-\frac{d+2}{2}} \exp\left[-c\frac{\varepsilon^2}{t-s}\right] \to 0
\end{aligned}
$$

as $t \to s_+$ uniformly in x since D is bounded, as desired. The proof of (46) is evidently identical, and based on (11). ■

The first half of the preceding lemma now allows us to prove the following result:

Proposition 2. *Assume that the same hypotheses as in Lemma 1 are valid. Then the following statements hold:*

(a) We have

$$
\begin{aligned}
&\lim_{t \to s_+} (t-s)^{-1} \int_{\{y \in D : |x-y| \leq \varepsilon\}} dy m^*(x,s;y,t)\,(y-x) \\
&= a^*(x,s) + k(x,s)\nabla_x \ln v_\psi(x,s)
\end{aligned} \tag{47}
$$

for each $x \in D$ and every s independently of any sufficiently small $\varepsilon > 0$, where the i^{th} component of the vector-field a^* is

$$
a_i^*(x,s) = \frac{1}{2}\operatorname{div}_x\left(k_i(x,s)\right) + l_i(x,s) \tag{48}
$$

for every $i \in \{1,...,d\}$, with $k_i(x,s)$ the i^{th} row or column of the symmetric matrix $k(x,s)$.

(b) We have

$$
\lim_{t \to s_+} (t-s)^{-1} \int_{\{y \in D : |x-y| \leq \varepsilon\}} dy m^*(x,s;y,t)\left((y-x) \otimes (y-x)\right) = k(x,s) \tag{49}
$$

for each $x \in D$ and every s, independently of any sufficiently small $\varepsilon > 0$.

Proof. Owing to (24) and (45) it is sufficient to prove that

$$
\begin{aligned}
&\lim_{t \to s_+} (t-s)^{-1} \int_D dy g_A^*(x,s;y,t) \frac{v_\psi(y,t)}{v_\psi(x,s)}\,(y-x) \\
&= a^*(x,s) + k(x,s)\nabla_x \ln v_\psi(x,s)
\end{aligned} \tag{50}
$$

and

$$
\lim_{t \to s_+} (t-s)^{-1} \int_D dy g_A^*(x,s;y,t) \frac{v_\psi(y,t)}{v_\psi(x,s)}\left((y-x) \otimes (y-x)\right) = k(x,s), \tag{51}
$$

respectively, since the functions $y \mapsto y-x$ and $y \mapsto (y-x) \otimes (y-x)$ are bounded on D. Thanks to the differentiability properties of v_ψ and the convexity of D we first write

$$\frac{v_\psi(y,t)}{v_\psi(x,s)} = 1 + \frac{(\nabla_x v_\psi(x,t), y-x)_{\mathbb{R}^d}}{v_\psi(x,s)} + \frac{\left(\mathsf{H}_{v_\psi}(x^*,t)\,(y-x)\,, y-x\right)_{\mathbb{R}^d}}{2v_\psi(x,s)} + (t-s)\frac{\partial_{s^*} v_\psi(x,s^*)}{v_\psi(x,s)} \tag{52}$$

as a consequence of the Taylor expansion for v_ψ, where H_{v_ψ} denotes the Hessian matrix relative to the spatial variable alone with x^* a point on the open line segment joining x and y, and where $s^* \in (s,t)$. The strategy of the proof then amounts to estimating the various contributions to (50) and (51) coming from (52).

In order to explain the appearance of the vector-field a^* on the right-hand side of (50) we begin by showing that

$$\lim_{t \to s_+} (t-s)^{-1} \int_D dy g_A^*(x,s;y,t)\,(y_i - x_i) = \frac{1}{2}\operatorname{div}_x\left(k_i(x,s)\right) + l_i(x,s) \tag{53}$$

for every $i \in \{1,...,d\}$. Let us define the function $f_i : D \mapsto \mathbb{R}$ by $f_i(y) := y_i - x_i$; since $f_i(x) = 0$ we have

$$\begin{aligned} &\lim_{t \to s_+} (t-s)^{-1} \int_D dy g_A^*(x,s;y,t)\,(y_i - x_i) \\ &= \lim_{t \to s_+} (t-s)^{-1} \left(\int_D dy g_A^*(x,s;y,t) f_i(y) - f_i(x)\right) \\ &= \left(-A^*(s) f_i\right)(x) = \frac{1}{2}\operatorname{div}_x\left(k_i(x,s)\right) + l_i(x,s) \end{aligned}$$

according to (5) and an elementary calculation based on (13), so that (53) holds. A similar calculation with the function $f_{i,j} : D \mapsto \mathbb{R}$ given by $f_{i,j}(y) := (y_i - x_i)\,(y_j - x_j)$ leads to

$$\lim_{t \to s_+} (t-s)^{-1} \int_D dy g_A^*(x,s;y,t)\,(y_i - x_i)\,(y_j - x_j) = \left(-A^*(s) f_{i,j}\right)(x) = k_{i,j}(x,s) \tag{54}$$

for all $i,j \in \{1,...,d\}$ since $f_{i,j}(x) = 0$, which allows us to evaluate the contribution of the gradient term in (52). Indeed, if we substitute that term into the

i^{th}component of the left-hand side of (50) and use (54) we obtain

$$\lim_{t\to s_+}(t-s)^{-1}\int\limits_D dyg_A^*(x,s;y,t)\frac{(\nabla_x v_\psi(x,t),y-x)_{\mathbb{R}^d}}{v_\psi(x,s)}(y_i-x_i)$$
$$=\sum_{j=1}^d\frac{\partial_{x_j}v_\psi(x,s)}{v_\psi(x,s)}\lim_{t\to s_+}(t-s)^{-1}\int_D dyg_A^*(x,s;y,t)(y_i-x_i)(y_j-x_j)$$
$$=\frac{(k(x,s)\nabla_x v_\psi(x,s))_i}{v_\psi(x,s)} \tag{55}$$

for every $i\in\{1,...,d\}$, which is the i^{th}component of the second term on the right-hand side of (50). Therefore, in order to get (50) it remains to prove that there are no contributions coming from the third and fourth terms on the right-hand side of (52).

Regarding the third term we first observe that

$$\lim_{t\to s_+}(t-s)^{-1}\int_D dyg_A^*(x,s;y,t)\,|y-x|^3=0$$

uniformly in $x\in D$ for every s, since from (14) we obtain

$$0\leq(t-s)^{-1}\int_D dyg_A^*(x,s;y,t)\,|y-x|^3$$
$$\leq c\,(t-s)^{-\frac{d+2}{2}}\int_D dy\exp\left[-c\frac{|y-x|^2}{t-s}\right]|y-x|^3$$
$$\leq c\,(t-s)^{-\frac{d+2}{2}}\int_{\mathbb{R}^d}dy\exp\left[-c\frac{|y|^2}{t-s}\right]|y|^3$$
$$=c(t-s)^{\frac{1}{2}}\int_{\mathbb{R}^d}dy\exp\left[-c\,|y|^2\right]|y|^3\to 0 \tag{56}$$

as $t\to s_+$, again by translation invariance and the same change of variables as in (44). Consequently we have *a fortiori* the estimate

$$0\leq(t-s)^{-1}\int_D dyg_A^*(x,s;y,t)\left|\left(\mathsf{H}_{v_\psi}(x^*,t)\,(y-x)\,,y-x\right)_{\mathbb{R}^d}(y_i-x_i)\right|$$
$$\leq c\,(t-s)^{-1}\int_D dyg_A^*(x,s;y,t)\,|y-x|^3\to 0$$

as $t\to s_+$ since the matrix-norm of $\mathsf{H}_{v_\psi}(x^*,t)$ is uniformly bounded on the compact cylinder $\overline{D}\times[0,T]$, from which we infer that

$$\lim_{t\to s_+}(t-s)^{-1}\int\limits_D dyg_A^*(x,s;y,t)\frac{\left(\mathsf{H}_{v_\psi}(x^*,t)\,(y-x)\,,y-x\right)_{\mathbb{R}^d}}{2v_\psi(x,s)}(y_i-x_i)=0$$

for each s and every $i \in \{1, ..., d\}$, as desired.

Finally, there is no contribution from the fourth term either since from its substitution into the i^{th}component of the left-hand side of (50), the cancellation of the time increments and (5) we get

$$\begin{aligned}&\frac{\partial_t v_\psi(x,s)}{v_\psi(x,s)} \lim_{t\to s_+} \int_D dy g_A^*(x,s;y,t) f_i(y) \\ &= \frac{\partial_t v_\psi(x,s)}{v_\psi(x,s)} f_i(x) = 0.\end{aligned}$$

The proof of (51) is entirely similar, the unique non-vanishing contribution being determined by the constant term on the right-hand side of (52) via (54). ■

The second half of Lemma 3 allows us to obtain a similar result for the backward Markov process of Lemma 2. We omit the proof, which is based on (9)-(11) of Proposition 1, Lindeberg's condition (46), and is thereby identical to that of the preceding proposition.

Proposition 3. *Assume that the same hypotheses as in Lemma 2 are valid. Then the following statements hold:*

(a) We have

$$\begin{aligned}&\lim_{s\to t_-} (t-s)^{-1} \int_{\{y\in D:|x-y|\leq\varepsilon\}} dy m(x,t;y,s)\,(x-y) \\ &= a(x,t) - k(x,t)\nabla_x \ln u_\varphi(x,t) \qquad (57)\end{aligned}$$

for each $x \in D$ and every t independently of any sufficiently small $\varepsilon > 0$, where the i^{th}component of the vector-field a is

$$a_i(x,t) = -\frac{1}{2}\operatorname{div}_x (k_i(x,t)) + l_i(x,t) \qquad (58)$$

for every $i \in \{1, ..., d\}$, with $k_i(x,t)$ as in Proposition 2.

(b) We have

$$\lim_{s\to t_-} (t-s)^{-1} \int_{\{y\in D:|x-y|\leq\varepsilon\}} dy m(x,t;y,s)\,((x-y)\otimes(x-y)) = k(x,t) \qquad (59)$$

for each $x \in D$ and every t, independently of any sufficiently small $\varepsilon > 0$.

Thus, both Propositions 2 and 3 show that the process $Z_{\tau\in[0,T]}$ of Theorem 2 is indeed a reversible Markov diffusion whose coefficients are determined by (47), (49), (57) and (59), respectively. In the sequel we shall denote by

$$b^*(x,t) := a^*(x,t) + k(x,t)\nabla_x \ln v_\psi(x,t) \qquad (60)$$

and

$$b(x,t) := a(x,t) - k(x,t)\nabla_x \ln u_\varphi(x,t) \qquad (61)$$

the drift terms (47) and (57), respectively.

Remark. We now explain why the specific form of (60) and (61) suggests that our notion of reversibility and the results obtained thus far may be interpreted as corresponding to a generalization of the notion of reversibility defined in [20] which has its origins in the last section of [26], and why our notion is suitable for the description of non-stationary Markov processes such as $Z_{\tau\in[0,T]}$. In order to see this it is best to refer to the reformulation of the main result of [20] as encoded in Relation (2.9) of [8]: given a time-homogeneous forward Markov diffusion wandering in $\mathbb{R}^d$, a probability measure which is absolutely continuous with respect to Lebesgue measure and defined by a smooth positive density ρ is reversible with respect to that diffusion if, and only if, the generator of the diffusion is of the form

$$\begin{aligned} &Lf(x) \\ &= \frac{1}{2\rho(x)} \operatorname{div}_x (\rho(x)k(x)\nabla_x f(x)) \\ &= \frac{1}{2} \operatorname{div}_x (k(x)\nabla_x f(x)) + \frac{1}{2} (k(x)\nabla_x \ln \rho(x), \nabla_x f(x))_{\mathbb{R}^d} \end{aligned} \tag{62}$$

for a suitable class of f's, where k is the associated symmetric, positive definite and time-independent diffusion matrix. But this leads at once to

$$b_{DSF}(x) := a_{DSF}(x) + \frac{1}{2}k(x)\nabla_x \ln \rho(x) \tag{63}$$

for the corresponding drift, with the i^{th}component of the vector-field a_{DSF} given by

$$a_{DSF,i}(x) := \frac{1}{2} \operatorname{div}_x (k_i(x)) \tag{64}$$

for every $i \in \{1, ..., d\}$ where $k_i(x)$ is the i^{th} row or column of $k(x)$. Indeed this follows immediately from (62) and the relation

$$\begin{aligned} &\frac{1}{2} \operatorname{div}_x (k(x)\nabla_x f(x)) \\ &= \frac{1}{2} (k(x)\nabla_x, \nabla_x f(x))_{\mathbb{R}^d} \;\; + (a_{DSF}(x), \nabla_x f(x))_{\mathbb{R}^d} \,, \end{aligned}$$

so that in the end reversibility in the sense of [8] or [20] is equivalent to a drift of the form (63). But it is then plain that (60) and (61) have the very same structure as (63), and furthermore that

$$\begin{aligned} &\frac{1}{2} (b^*(x,t) - b(x,t)) \\ &= \hat{a}(x,t) + \frac{1}{2}k(x,t)\nabla_x \ln \rho_\mu(x,t) \end{aligned} \tag{65}$$

where $\rho_\mu(x,t)$ is given by (41) and

$$\hat{a}_i(x,t) = \frac{1}{2} \operatorname{div}_x (k_i(x,t)) \tag{66}$$

for every i, with (65) and (66) formally identical to (63) and (64), respectively. It is, therefore, deemed appropriate to say that the reversibility of $Z_{\tau\in[0,T]}$ as illustrated by the statement of Theorem 2 and due to the very specific form of the joint probability measures (28) is a natural generalization of the notion defined in [8] and [20]. For another discussion of reversibility in a more geometric context we refer the reader to Section 4 in Chapter 5 of [17].

We now proceed by proving that there exist two vector-valued Wiener processes $W^*_{\tau\in[0,T]}$ and $W_{\tau\in[0,T]}$, indeed one for each one of the filtrations $\mathcal{F}^+_{\tau\in[0,T]}$ and $\mathcal{F}^-_{\tau\in[0,T]}$ we defined at the very beginning of this section, which will eventually allow us to consider $Z_{\tau\in[0,T]}$ as a forward and backward Itô diffusion whenever the drifts (60) and (61) do not vanish identically and simultaneously. We begin with the following preparatory result:

Lemma 4. *Assume that the hypotheses of Lemma 1 are valid and let us define the process*

$$Y^*_t := Z_t - Z_0 - \int_0^t d\tau b^*\left(Z_\tau, \tau\right) \tag{67}$$

for every $t \in [0,T]$, where $Z_{\tau\in[0,T]}$ is considered as the forward Markov diffusion of Theorem 2. Then $Y^*_{\tau\in[0,T]}$ is a continuous, square-integrable martingale with respect to $\mathcal{F}^+_{\tau\in[0,T]}$. Under the hypotheses of Lemma 2 a similar statement holds for the process

$$Y_t := Z_t - Z_T + \int_t^T d\tau b\left(Z_\tau, \tau\right) \tag{68}$$

with respect to $\mathcal{F}^-_{\tau\in[0,T]}$, with $Z_{\tau\in[0,T]}$ considered as the backward Markov diffusion of Theorem 2.

Proof. We prove (67) by first observing that

$$\sup_{t\in[0,T]} \mathbb{E}_\mu \left|Z_t\right|^2 < +\infty \tag{69}$$

where

$$\mathbb{E}_\mu \left|Z_t\right|^2 = \int_{D\times D} dxdy\mu_0(x)m^*(x,0;y,t)\left|y\right|^2.$$

Indeed we have

$$\begin{aligned}
&\mathbb{E}_\mu \left|Z_t\right|^2 \\
&\leq c\int_D dx\mu_0(x)\int_D dy g^*_A(x,0;y,t) \\
&\leq ct^{-\frac{d}{2}}\int_D dx\mu_0(x)\int_{\mathbb{R}^d} dy \exp\left[-c\frac{\left|y\right|^2}{t}\right] \leq c < +\infty
\end{aligned}$$

uniformly in $t \in [0,T]$ according to (14), since $(x,y,t) \mapsto v_{\psi}^{-1}(x,0)v_{\psi}(y,t)\left|y\right|^2$ is uniformly bounded on $\overline{D} \times \overline{D} \times [0,T]$. Therefore we have

$$\sup_{t\in[0,T]} \mathbb{E}_{\mu}\left|Y_t^*\right|^2 < +\infty \tag{70}$$

as a consequence of the uniform boundedness of $b^*(x,t)$ given by (60).

In order to prove the statement of the lemma, it is sufficient to show that the scalar-valued process

$$y_t^* := \left(Y_t^*, q\right)_{\mathbb{R}^d} = \left(Z_t - Z_0, q\right)_{\mathbb{R}^d} - \int_0^t d\tau \left(b^*\left(Z_\tau,\tau\right),q\right)_{\mathbb{R}^d} \tag{71}$$

is a continuous, square-integrable martingale for every $q \in \mathbb{R}^d$. While the continuity is clear according to the remark following Theorem 2, the fact that

$$\sup_{t\in[0,T]} \mathbb{E}_{\mu}\left|y_t^*\right|^2 < +\infty$$

is an immediate consequence of (70) and (71). Therefore, it remains to show that the equality

$$\mathbb{E}_{\mu}\left(y_t^* \left|\mathcal{F}_s^+\right.\right) = y_s^* \tag{72}$$

holds $\mathbb{P}_{\mu}$-a.s. for all $t \geq s$, and for this we need only prove that the right-hand derivative of (72) with respect to t vanishes, namely, that

$$\lim_{r\to t_+} (r-t)^{-1}\,\mathbb{E}_{\mu}\left(y_r^* - y_t^* \left|\mathcal{F}_s^+\right.\right) = 0 \tag{73}$$

$\mathbb{P}_{\mu}$-a.s. for each t. Owing to the basic properties of conditional expectations, this amounts to proving that

$$\lim_{r\to t_+} (r-t)^{-1}\,\mathbb{E}_{\mu}\left(\mathbb{E}_{\mu}\left(y_r^* - y_t^* \left|\mathcal{F}_t^+\right.\right)\left|\mathcal{F}_s^+\right.\right) = 0 \tag{74}$$

for each t since $\mathcal{F}_s^+ \subseteq \mathcal{F}_t^+$.

In order to get (74) we first show that

$$\lim_{r\to t_+} (r-t)^{-1}\,\mathbb{E}_{\mu}\left(y_r^* - y_t^* \left|\mathcal{F}_t^+\right.\right) = 0. \tag{75}$$

We have

$$\begin{aligned} &\mathbb{E}_{\mu}\left(y_r^* - y_t^* \left|\mathcal{F}_t^+\right.\right) \\ &= \mathbb{E}_{\mu}\left(\left(Z_r - Z_t, q\right)_{\mathbb{R}^d}\left|\mathcal{F}_t^+\right.\right) - \mathbb{E}_{\mu}\left(\int_t^r d\tau\left(b^*\left(Z_\tau,\tau\right),q\right)_{\mathbb{R}^d}\left|\mathcal{F}_t^+\right.\right) \end{aligned} \tag{76}$$

according to (71), with

$$\mathbb{E}_{\mu}\left(\left(Z_r - Z_t, q\right)_{\mathbb{R}^d}\left|\mathcal{F}_t^+\right.\right) \tag{77}$$

$$= \mathbb{E}_{\mu}\left(\left(Z_r - Z_t, q\right)_{\mathbb{R}^d}\left|Z_t\right.\right) = \int_D dy m^*\left(Z_t, t; y, r\right)\left(y - Z_t, q\right)_{\mathbb{R}^d} \tag{78}$$

and

$$\begin{aligned}
&\mathbb{E}_\mu\left(\int_t^r d\tau\left(b^*\left(Z_\tau,\tau\right),q\right)_{\mathbb{R}^d}\left|\mathcal{F}_t^+\right.\right) \\
&=\int_t^r d\tau\mathbb{E}_\mu\left(\left(b^*\left(Z_\tau,\tau\right),q\right)_{\mathbb{R}^d}\left|Z_t\right.\right) \\
&=\int_t^r d\tau\int_D dy m^*(Z_t,t;y,\tau)\left(b^*\left(y,\tau\right),q\right)_{\mathbb{R}^d}. \qquad (79)
\end{aligned}$$

Now from (50), (60) and (77) we get

$$\lim_{r\to t_+}(r-t)^{-1}\,\mathbb{E}_\mu\left(\left(Z_r-Z_t,q\right)_{\mathbb{R}^d}\left|\mathcal{F}_t^+\right.\right)=\left(b^*\left(Z_t,t\right),q\right)_{\mathbb{R}^d} \qquad (80)$$

$\mathbb{P}_\mu$-a.s.. Furthermore, we also claim that

$$\lim_{r\to t_+}(r-t)^{-1}\int_t^r d\tau\int_D dy m^*(Z_t,t;y,\tau)\left(b^*\left(y,\tau\right),q\right)_{\mathbb{R}^d}=\left(b^*\left(Z_t,t\right),q\right)_{\mathbb{R}^d} \qquad (81)$$

$\mathbb{P}_\mu$-a.s. or, equivalently, that

$$\lim_{r\to t_+}(r-t)^{-1}\int_t^r d\tau\int_D dy m^*(Z_t,t;y,\tau)\left(b^*\left(y,\tau\right)-b^*\left(Z_t,t\right),q\right)_{\mathbb{R}^d}=0. \qquad (82)$$

The crucial fact about proving (82) is that the drift-term (60) satisfies the Hölder continuity estimate

$$\left|b^*\left(y,\tau\right)-b^*\left(Z_t,t\right)\right|\leq c\left(\left|y-Z_t\right|^\alpha+\left|\tau-t\right|^{\frac{\alpha}{2}}\right),$$

which is an easy consequence of the Hölder properties of the diffusion matrix k and those of the vector field l stated in Hypotheses (K) and (L), together with (a) of Proposition 1 regarding v_ψ. Consequently, by using the very same kind of estimates as we did in the proofs of (44) and (56) we obtain successively

$$\begin{aligned}
0&\leq(r-t)^{-1}\int_t^r d\tau\int_D dy m^*(Z_t,t;y,\tau)\left|b^*\left(y,\tau\right)-b^*\left(Z_t,t\right)\right| \\
&\leq c\,(r-t)^{-1}\int_t^r d\tau\,(\tau-t)^{-\frac{d}{2}}\int_{\mathbb{R}^d}dy\exp\left[-c\frac{|y|^2}{\tau-t}\right]\left(|y|^\alpha+|\tau-t|^{\frac{\alpha}{2}}\right) \\
&\leq c\,(r-t)^{-1}\int_t^r d\tau\,(\tau-t)^{\frac{\alpha}{2}}\int_{\mathbb{R}^d}dy\exp\left[-c\,|y|^2\right]|y|^\alpha \\
&+c\,(r-t)^{-1}\int_t^r d\tau\,(\tau-t)^{\frac{\alpha}{2}}\int_{\mathbb{R}^d}dy\exp\left[-c\,|y|^2\right] \\
&\leq c\,(r-t)^{\frac{\alpha}{2}}\to 0
\end{aligned}$$

as $r\to t_+$, which indeed proves (82). The combination of (76) with (79)-(82) then gives (75).

It remains to prove that (74) is a consequence of (75). On the one hand we have

$$\left| \int_D dym^* \left(Z_t, t; y, r \right) \left(y - Z_t, q \right)_{\mathbb{R}^d} \right| \leq c \left(r - t \right) \tag{83}$$

$\mathbb{P}_\mu$-a.s. for some positive, non-random and finite constant c since (80) holds and the right-hand side of (80) is uniformly bounded. On the other hand, we also have

$$\left| \int_t^r d\tau \int_D dym^* (Z_t, t; y, \tau) \left(b^* \left(y, \tau \right), q \right)_{\mathbb{R}^d} \right| \leq c \left(r - t \right) \tag{84}$$

because of (81). Consequently, it follows from (76)-(79) and (83), (84) that

$$\left| \mathbb{E}_\mu \left(y_r^* - y_t^* \left| \mathcal{F}_t^+ \right. \right) \right| \leq c \left(r - t \right)$$

$\mathbb{P}_\mu$-a.s. for some suitable non-random and finite constant c and all $r, t \in [0, T]$ with $r \geq t$. Therefore, (75) indeed implies (74) by dominated convergence, so that (73) obtains and thereby the martingale property (72). The proof of the statement concerning the continuous process (68) follows from similar arguments, by proving that

$$\sup_{t \in [0,T]} \mathbb{E}_\mu \left| Y_t \right|^2 < +\infty$$

and that the relation

$$\mathbb{E}_\mu \left(y_s \left| \mathcal{F}_t^- \right. \right) = y_t$$

holds $\mathbb{P}_\mu$-a.s. for all $s \leq t$, where

$$y_t := \left(Z_t - Z_T, q \right)_{\mathbb{R}^d} + \int_t^T d\tau \left(b \left(Z_\tau, \tau \right), q \right)_{\mathbb{R}^d}$$

and $\mathcal{F}_t^- \subseteq \mathcal{F}_s^-$. ■

Remark. Let us assume momentarily that $b^* = b = 0$ identically. We then infer from (67) and (68) that $Z_{\tau \in [0,T]}$ is a continuous, square-integrable martingale for both filtrations $\mathcal{F}^+_{\tau \in [0,T]}$ and $\mathcal{F}^-_{\tau \in [0,T]}$ simultaneously. Consequently we have

$$\mathbb{E}_\mu \left(\left(Z_t - Z_s, q \right)^2_{\mathbb{R}^d} \left| \mathcal{F}_s^+ \right. \right) = \mathbb{E}_\mu \left(\left(Z_t, q \right)^2_{\mathbb{R}^d} \left| \mathcal{F}_s^+ \right. \right) - \left(Z_s, q \right)^2_{\mathbb{R}^d}$$

$\mathbb{P}_\mu$-a.s. for all $t \geq s$ and every $q \in \mathbb{R}^d$, and at the same time

$$\mathbb{E}_\mu \left(\left(Z_t - Z_s, q \right)^2_{\mathbb{R}^d} \left| \mathcal{F}_t^- \right. \right) = \mathbb{E}_\mu \left(\left(Z_s, q \right)^2_{\mathbb{R}^d} \left| \mathcal{F}_t^- \right. \right) - \left(Z_t, q \right)^2_{\mathbb{R}^d} .$$

By averaging both expressions we then obtain

$$\begin{aligned} &\mathbb{E}_\mu \left(Z_t - Z_s, q \right)^2_{\mathbb{R}^d} \\ &= \mathbb{E}_\mu \left(Z_t, q \right)^2_{\mathbb{R}^d} - \mathbb{E}_\mu \left(Z_s, q \right)^2_{\mathbb{R}^d} \\ &= \mathbb{E}_\mu \left(Z_s, q \right)^2_{\mathbb{R}^d} - \mathbb{E}_\mu \left(Z_t, q \right)^2_{\mathbb{R}^d} = 0 \end{aligned}$$

for every $q \in \mathbb{R}^d$, so that $Z_t = Z_s$ $\mathbb{P}_\mu$-a.s. for all $t \geq s$. Conversely, if the function $\tau \mapsto Z_\tau$ is constant $\mathbb{P}_\mu$-a.s. we obviously have $b^* = b = 0$ identically. From now on we will exclude this exceptional case from our considerations for reasons that will become apparent from the statement of Theorem 3 below and its proof.

It is worth emphasizing the fact that the existence of the Wiener processes we are looking for is conditioned by that of the reversible Markov diffusion $Z_{\tau\in[0,T]}$ and not the other way around, as there is absolutely nothing stochastic in Problems (1) and (2). Furthermore, from a technical point of view all the stochastic integrals defined below involve continuous square-integrable martingales as integrators, as for instance in Chapter 3 of [19]. Then, the precise result about $Z_{\tau\in[0,T]}$ being a reversible Itô diffusion is the following, in which we write $k^{\frac{1}{2}}(x,t)$ for the positive square root of the diffusion matrix $k(x,t)$ and assume that $\tau \mapsto Z_\tau$ is not constant $\mathbb{P}_\mu$-a.s. according to the preceding remark.

Theorem 3. *Assume that the hypotheses of Lemmas 1 and 2 are valid. Assume furthermore that μ is given by (17) and (28), and let $Z_{\tau\in[0,T]}$ be the reversible Markov diffusion of Theorem 2. Then the following statements hold:*

(a) There exists a d-dimensional Wiener process $W^*_{\tau\in[0,T]}$ such that the relation

$$Z_t = Z_0 + \int_0^t d\tau b^*(Z_\tau,\tau) + \int_0^t k^{\frac{1}{2}}(Z_\tau,\tau)\, d^+W^*_\tau \tag{85}$$

holds $\mathbb{P}_\mu$-a.s. for every $t \in [0,T]$, with b^* given by (60) and the forward stochastic integral defined with respect to $\mathcal{F}^+_{\tau\in[0,T]}$.

(b) There exists a d-dimensional Wiener process $W_{\tau\in[0,T]}$ such that the relation

$$Z_t = Z_T - \int_t^T d\tau b(Z_\tau,\tau) - \int_t^T k^{\frac{1}{2}}(Z_\tau,\tau)\, d^-W_\tau \tag{86}$$

holds $\mathbb{P}_\mu$-a.s. for every $t \in [0,T]$, with b given by (61) and the backward stochastic integral defined with respect to $\mathcal{F}^-_{\tau\in[0,T]}$.

Proof. We define $W^*_{\tau\in[0,T]}$ by

$$W^*_t := \int_0^t k^{-\frac{1}{2}}(Z_\tau,\tau)\, d^+Y^*_\tau \tag{87}$$

with respect to the continuous square-integrable martingale $Y^*_{\tau\in[0,T]}$ given by (67) and $\mathcal{F}^+_{\tau\in[0,T]}$. In order to show that (87) makes sense and indeed defines a Wiener process, we begin by proving that the quadratic variation of the martingale $y^*_{\tau\in[0,T]}$ is the absolutely continuous process

$$\langle y^* \rangle_t := \int_0^t d\tau \left(k(Z_\tau,\tau)\, q, q\right)_{\mathbb{R}^d} \tag{88}$$

where $q \in \mathbb{R}^d$. According to the Doob-Meyer decomposition, this amounts to showing that the process defined by

$$z_t^* := (y_t^*)^2 - \int_0^t d\tau \left(k\left(Z_\tau, \tau\right) q, q\right)_{\mathbb{R}^d} \tag{89}$$

is a martingale relative to $\mathcal{F}^+_{\tau\in[0,T]}$. In order to achieve this we proceed as we did in the proof of Lemma 4, by proving that the relation

$$\begin{aligned} &\lim_{r\to t_+} (r-t)^{-1} \mathbb{E}_\mu \left(z_r^* - z_t^* \left| \mathcal{F}_s^+ \right.\right) \\ &= \lim_{r\to t_+} (r-t)^{-1} \mathbb{E}_\mu \left(\mathbb{E}_\mu \left(z_r^* - z_t^* \left| \mathcal{F}_t^+ \right.\right) \left| \mathcal{F}_s^+ \right.\right) = 0 \end{aligned} \tag{90}$$

holds, where for $0 \leq t < r \leq T$ the inner conditional expectation is given by

$$\begin{aligned} &\mathbb{E}_\mu \left(z_r^* - z_t^* \left| \mathcal{F}_t^+ \right.\right) \\ &= \mathbb{E}_\mu \left((y_r^* - y_t^*)^2 \left| \mathcal{F}_t^+ \right.\right) - \int_t^r d\tau \mathbb{E}_\mu \left(\left(k\left(Z_\tau, \tau\right) q, q\right)_{\mathbb{R}^d} \left| \mathcal{F}_t^+ \right.\right) \end{aligned} \tag{91}$$

since $y^*_{\tau\in[0,T]}$ is a martingale relative to the filtration $\mathcal{F}^+_{\tau\in[0,T]}$.

We first show that

$$\lim_{r\to t_+} (r-t)^{-1} \mathbb{E}_\mu \left((y_r^* - y_t^*)^2 \left| \mathcal{F}_t^+ \right.\right) = \left(k\left(Z_t, t\right) q, q\right)_{\mathbb{R}^d} \tag{92}$$

$\mathbb{P}_\mu$-a.s. for every t. One the one hand, from (71) we have

$$\begin{aligned} &(y_r^* - y_t^*)^2 \\ &= (Z_r - Z_t, q)^2_{\mathbb{R}^d} + \left(\int_t^r d\tau \left(b^*(Z_\tau, \tau), q\right)_{\mathbb{R}^d}\right)^2 \\ &- 2\left(Z_r - Z_t, q\right)_{\mathbb{R}^d} \int_t^r d\tau \left(b^*(Z_\tau, \tau), q\right)_{\mathbb{R}^d} \end{aligned} \tag{93}$$

and we note that

$$\begin{aligned} &\lim_{r\to t_+} (r-t)^{-1} \mathbb{E}_\mu \left((Z_r - Z_t, q)^2_{\mathbb{R}^d} \left| \mathcal{F}_t^+ \right.\right) \\ &= \lim_{r\to t_+} (r-t)^{-1} \mathbb{E}_\mu \left((Z_r - Z_t, q)^2_{\mathbb{R}^d} \left| Z_t \right.\right) = \left(k\left(Z_t, t\right) q, q\right)_{\mathbb{R}^d} \end{aligned}$$

since $\tau \mapsto Z_\tau$ is not constant $\mathbb{P}_\mu$-a.s.. Indeed this follows immediately from (49), which implies the relation

$$\lim_{r\to t_+} (r-t)^{-1} \int_D dy m^*(Z_t, t; y, r) \left(y - Z_t, q\right)^2_{\mathbb{R}^d} = \left(k\left(Z_t, t\right) q, q\right)_{\mathbb{R}^d}$$

by switching to the quadratic form formulation. On the other hand, by dominated convergence the remaining terms on the right-hand side of (93) do not contribute to the conditional expectation (92), for

$$\lim_{r\to t_+} (r-t)^{-1} \left(\int_t^r d\tau \left(b^*(Z_\tau, \tau), q\right)_{\mathbb{R}^d}\right)^2 = 0$$

and

$$\lim_{r\to t_+}(r-t)^{-1}(Z_r-Z_t,q)_{\mathbb{R}^d}\int_t^r d\tau\,(b^*(Z_\tau,\tau),q)_{\mathbb{R}^d}$$
$$=(b^*(Z_t,t),q)_{\mathbb{R}^d}\lim_{r\to t_+}(Z_r-Z_t,q)_{\mathbb{R}^d}=0$$

$\mathbb{P}_\mu$-a.s. for every t, as a consequence of the boundedness and the continuity of b^* and $Z_{\tau\in[0,T]}$. Therefore (92) holds.

Next, we observe that we also have

$$\lim_{r\to t_+}(r-t)^{-1}\int_t^r d\tau\mathbb{E}_\mu\left((k(Z_\tau,\tau)q,q)_{\mathbb{R}^d}\left|\mathcal{F}_t^+\right.\right)=(k(Z_t,t)q,q)_{\mathbb{R}^d} \tag{94}$$

$\mathbb{P}_\mu$-a.s. for every t or, equivalently, that

$$\lim_{r\to t_+}(r-t)^{-1}\int_t^r d\tau\int_D dym^*(Z_t,t;y,\tau)\left((k(y,\tau)-k(Z_t,t))q,q\right)_{\mathbb{R}^d}=0.$$

Indeed this relation follows from exactly the same arguments as those which led to (82), since the matrix elements of k are all jointly Hölder continuous relative to space-time variables according to Hypothesis (K). Relations (91), (92) and (94) then imply

$$\lim_{r\to t_+}(r-t)^{-1}\mathbb{E}_\mu\left(z_r^*-z_t^*\left|\mathcal{F}_t^+\right.\right)=0$$

$\mathbb{P}_\mu$-a.s. for every t, so that (90) follows from a dominated convergence argument similar to that given in the proof of Lemma 4. Thus $z^*_{\tau\in[0,T]}$ is a martingale relative to $\mathcal{F}^+_{\tau\in[0,T]}$, and consequently (88) is indeed the quadratic variation process of (71).

This implies (87) makes sense in that $k^{-\frac{1}{2}}(Z_\tau,\tau)$ is an admissible integrand there, and defines a continuous, square-integrable martingale whose quadratic variation is given by

$$\langle (W^*,q)_{\mathbb{R}^d}\rangle_t=t\left|q\right|^2$$

for each $t\in[0,T]$ and every $q\in\mathbb{R}^d$. This proves that $W^*_{\tau\in[0,T]}$ is a Wiener process, and furthermore that the combination of (87) with (67) leads to

$$\int_0^t k^{\frac{1}{2}}(Z_\tau,\tau)\,d^+W_\tau^*$$
$$=\int_0^t d^+Y_\tau^*=Z_t-Z_0-\int_0^t d\tau b^*(Z_\tau,\tau)$$

$\mathbb{P}_\mu$-a.s. for every $t\in[0,T]$, which is (85). The proof of (86) with

$$W_t:=-\int_t^T k^{-\frac{1}{2}}(Z_\tau,\tau)\,d^-Y_\tau \tag{95}$$

defined with respect to (68) is similar, with this time

$$z_t:=(y_t)^2-\int_t^T d\tau\,(k(Z_\tau,\tau)q,q)_{\mathbb{R}^d}$$

a martingale with respect to the decreasing filtration $\mathcal{F}^-_{\tau\in[0,T]}$. ■

We have already noted that (34) provides important information regarding $Z_{\tau\in[0,T]}$ in terms of the solutions to (1) and (2). It is, therefore, natural to ask whether those solutions can, in turn, be represented as suitable expectations of some functionals of $Z_{\tau\in[0,T]}$. This is indeed possible but not *a priori* evident since the elliptic operators on the right-hand side of (1) and (2) are *not* the generators of $Z_{\tau\in[0,T]}$. The problem lies, of course, in the presence of the logarithmic terms in (60), (61), and in the next result we show how to do do away with them by means of suitable Girsanov transformations.

Corollary. *Assume that the same hypotheses as in Theorem 3 are valid, and let $Z_{\tau\in[0,T]}$ be the reversible Markov diffusion of Theorem 2. Then, aside from $\mathbb{P}_\mu$ there exist two probability measures $\mathbb{P}^\pm_\mu$ on $(\Omega,\mathcal{F})$ such that the following statements hold:*

(a) There exists a d-dimensional Wiener process $\widetilde{W}^*_{\tau\in[0,T]}$ relative to $\mathcal{F}^+_{\tau\in[0,T]}$ such that the relation

$$Z_t = Z_0 + \int_0^t d\tau a^*\left(Z_\tau,\tau\right) + \int_0^t k^{\frac{1}{2}}\left(Z_\tau,\tau\right) d^+\widetilde{W}^*_\tau \tag{96}$$

holds $\mathbb{P}^+_\mu$-a.s. for every $t\in[0,T]$, with a^* given by (48).

(b) There exists a d-dimensional Wiener process $\widetilde{W}_{\tau\in[0,T]}$ relative to $\mathcal{F}^-_{\tau\in[0,T]}$ *such that the relation*

$$Z_t = Z_T - \int_t^T d\tau a\left(Z_\tau,\tau\right) - \int_t^T k^{\frac{1}{2}}\left(Z_\tau,\tau\right) d^-\widetilde{W}_\tau \tag{97}$$

holds $\mathbb{P}^-_\mu$-a.s. for every $t\in[0,T]$, with a given by (58).

Proof. Let us define

$$\widetilde{W}^*_t := \int_0^t d\tau X_\tau + W^*_t \tag{98}$$

for every $t\in[0,T]$, where

$$X_t := k^{\frac{1}{2}}\left(Z_t,t\right)\nabla_x \ln v_\psi(Z_t,t). \tag{99}$$

From the hypotheses regarding k and the properties of v_ψ and $Z_{\tau\in[0,T]}$, it is clear that (99) defines a continuous process adapted to the filtration $\mathcal{F}^+_{\tau\in[0,T]}$. It is also bounded $\mathbb{P}_\mu$-a.s. by a non-random positive constant, so that we have

$$\mathbb{E}_\mu \exp\left[\frac{1}{2}\int_0^T d\tau\left|X_\tau\right|^2\right] < +\infty$$

and thereby

$$\mathbb{E}_{\mu}\exp\left[-\int_0^T\left(X_\tau,d^+W_\tau^*\right)_{\mathbb{R}^d}-\frac{1}{2}\int_0^T d\tau\left|X_\tau\right|^2\right]=1$$

according to Theorem 12 in Section 1 of Chapter 3 of [14]. For every $E\in\mathcal{F}$ we then define

$$\mathbb{P}_\mu^+(E):=\int_E d\mathbb{P}_\mu\exp\left[-\int_0^T\left(X_\tau,d^+W_\tau^*\right)_{\mathbb{R}^d}-\frac{1}{2}\int_0^T d\tau\left|X_\tau\right|^2\right],$$

which indeed makes (98) a Wiener process on $(\Omega,\mathcal{F},\mathbb{P}_\mu^+)$ relative to $\mathcal{F}^+_{\tau\in[0,T]}$ according to Girsanov's standard construction (see, for instance, [14]). Furthermore we have

$$\begin{aligned}&\int_0^t k^{\frac{1}{2}}\left(Z_\tau,\tau\right)d^+\widetilde{W}_\tau^*\\&=\int_0^t d\tau k\left(Z_\tau,\tau\right)\nabla_x\ln v_\psi(Z_\tau,\tau)+\int_0^t k^{\frac{1}{2}}\left(Z_\tau,\tau\right)d^+W_\tau^*\end{aligned}$$

$\mathbb{P}_\mu$-a.s. and $\mathbb{P}_\mu^+$-a.s. for every $t\in[0,T]$, so that (85) reduces to (96). The proof of (97) with

$$\widetilde{W}_t:=\int_t^T d\tau k^{\frac{1}{2}}\left(Z_\tau,\tau\right)\nabla_x\ln u_\varphi(Z_\tau,\tau)+W_t$$

is similar and therefore omitted. ■

Let us now write $\mathbb{E}_\mu^\pm$ for the expectation functional on $(\Omega,\mathcal{F},\mathbb{P}_\mu^\pm)$, and by $\mathbb{E}^\pm_{\mu,x,t}$ the conditional expectations corresponding to setting $Z_t=x$ for an arbitrary $(x,t)\in D\times[0,T]$. We then get the following Feynman-Kac representations for the solutions to (1) and (2) we alluded to above.

Theorem 4. *Assume that the same hypotheses as in the Corollary are valid, and let $Z_{\tau\in[0,T]}$ be the reversible Markov diffusion of that Corollary. Then the following statements hold:*

(a) The unique classical positive solution to (1) may be written as

$$u_\varphi(x,t)=\mathbb{E}^-_{\mu,x,t}\left(\exp\left[-\int_0^t d\sigma V\left(Z_\sigma,\sigma\right)\right]\varphi\left(Z_0\right)\right)\tag{100}$$

for all $(x,t)\in D\times[0,T]$.

(b) If in addition the vector-field l satisfies

$$\operatorname{div}_x l(x,t)=0\tag{101}$$

for each $t\in[0,T]$, then the unique classical positive solution to (2) may be written as

$$v_\psi(x,t)=\mathbb{E}^+_{\mu,x,t}\left(\exp\left[-\int_t^T d\sigma V\left(Z_\sigma,\sigma\right)\right]\psi\left(Z_T\right)\right)\tag{102}$$

for all $(x,t) \in D \times [0,T]$.

Proof. Regarding the proof of (a) we have

$$\frac{1}{2}\operatorname{div}_x\left(k(x,t)\nabla_x u_\varphi(x,t)\right) - \left(l(x,t), \nabla_x u_\varphi(x,t)\right)_{\mathbb{R}^d}$$
$$= \frac{1}{2}\left(k(x,t)\nabla_x, \nabla_x u_\varphi(x,t)\right)_{\mathbb{R}^d} - \left(a(x,t), \nabla_x u_\varphi(x,t)\right)_{\mathbb{R}^d} \tag{103}$$

by using (58), so that the sum of the first two terms on the right-hand side of the first line of (1) identifies with the generator of the backward diffusion (97). We then apply the corresponding *backward* Itô formula

$$\begin{aligned}
&F\left(Z_t,t\right)\\
&= F\left(Z_s,s\right) + \int_s^t d\tau \frac{\partial F}{\partial s}\left(Z_\tau,\tau\right) - \frac{1}{2}\int_s^t d\tau\left(k(Z_\tau,\tau)\nabla_x, \nabla_x F(Z_\tau,\tau)\right)_{\mathbb{R}^d}\\
&+ \int_s^t d\tau\left(a(Z_\tau,\tau), \nabla_x F(Z_\tau,\tau)\right)_{\mathbb{R}^d} + \int_s^t \left(\nabla_x F(Z_\tau,\tau), k^{\frac{1}{2}}\left(Z_\tau,\tau\right) d^- \widetilde{W}_\tau\right)_{\mathbb{R}^d}
\end{aligned} \tag{104}$$

to the function

$$F\left(x,s\right) := \exp\left[-\int_s^t d\sigma V\left(Z_\sigma,\sigma\right)\right] u_\varphi(x,s)$$

for each $x \in \overline{D}$ and all $s,t \in [0,T]$ satisfying $t \geq s$, by noticing in particular the crucial minus sign of the last term of the second line to (104). By substituting

$$\frac{\partial F}{\partial s}\left(x,s\right) = \exp\left[-\int_s^t d\sigma V\left(Z_\sigma,\sigma\right)\right]\left(\partial_s u_\varphi\left(x,s\right) + V(Z_s,s)u_\varphi(x,s)\right)$$

and

$$\nabla_x F(x,s) = \exp\left[-\int_s^t d\sigma V\left(Z_\sigma,\sigma\right)\right]\nabla_x u_\varphi(x,s)$$

into (104), and by using (103) together with the first relation in (1), we obtain

$$\begin{aligned}
&u_\varphi\left(Z_t,t\right)\\
&= \exp\left[-\int_s^t d\sigma V\left(Z_\sigma,\sigma\right)\right] u_\varphi\left(Z_s,s\right)\\
&+ \int_s^t \exp\left[-\int_\tau^t d\sigma V\left(Z_\sigma,\sigma\right)\right]\left(\nabla_x u_\varphi(Z_\tau,\tau), k^{\frac{1}{2}}\left(Z_\tau,\tau\right) d^- \widetilde{W}_\tau\right)_{\mathbb{R}^d}
\end{aligned} \tag{105}$$

where the stochastic integral in (105) is finite $\mathbb{P}_\mu^-$-a.s.. The desired result (100) then follows by setting $s = 0$ in the preceding relation, by using the initial condition in (1) and by taking the conditional expectation $\mathbb{E}_{\mu,x,t}^-$ of the resulting equality.

As for the proof of (b), we first observe that (2) may be rewritten as

$$
\begin{aligned}
-\partial_t v(x,t) &= \frac{1}{2}\operatorname{div}_x\left(k(x,t)\nabla_x v(x,t)\right) + \left(l(x,t),\nabla_x v(x,t)\right)_{\mathbb{R}^d} - V(x,t)v(x,t), \\
&\quad (x,t)\in D\times[0,T), \\
v(x,T) &= \psi(x), \quad x\in D, \\
\frac{\partial v(x,t)}{\partial n_k(x,t)} &= 0, \quad (x,t)\in\partial D\times[0,T),
\end{aligned}
\tag{106}
$$

as a consequence of (101). Furthermore we have

$$
\begin{aligned}
&\frac{1}{2}\operatorname{div}_x\left(k(x,t)\nabla_x v_\varphi(x,t)\right) + \left(l(x,t),\nabla_x v_\varphi(x,t)\right)_{\mathbb{R}^d} \\
&= \frac{1}{2}\left(k(x,t)\nabla_x,\nabla_x v_\varphi(x,t)\right)_{\mathbb{R}^d} + \left(a^*(x,t),\nabla_x v_\varphi(x,t)\right)_{\mathbb{R}^d}
\end{aligned}
$$

by using (48), so that the sum of the first two terms on the right-hand side of the first line of (106) identifies this time with the generator of the forward diffusion (96). The remaining part of the argument then consists in applying the usual *forward* Itô formula

$$
\begin{aligned}
&F\left(Z_t,t\right) \\
&= F\left(Z_s,s\right) + \int_s^t d\tau\frac{\partial F}{\partial t}\left(Z_\tau,\tau\right) + \frac{1}{2}\int_s^t d\tau\left(k(Z_\tau,\tau)\nabla_x,\nabla_x F(Z_\tau,\tau)\right)_{\mathbb{R}^d} \\
&+ \int_s^t d\tau\left(a^*(Z_\tau,\tau),\nabla_x F(Z_\tau,\tau)\right)_{\mathbb{R}^d} + \int_s^t\left(\nabla_x F(Z_\tau,\tau),k^{\frac{1}{2}}\left(Z_\tau,\tau\right)d^+\widetilde{W}_t^*\right)_{\mathbb{R}^d}
\end{aligned}
\tag{107}
$$

to the function given by

$$
F\left(x,t\right) := \exp\left[-\int_s^t d\sigma V\left(Z_\sigma,\sigma\right)\right]v_\psi(x,t)
$$

for every $(x,t)\in\overline{D}\times[s,T]$, which eventually allows us to proceed as in the first part of the proof by using (106) to obtain (102). ■

Remarks. (1) The backward Itô formula (104) with the appropriate sign change has its origins in the theory developed in Section 13 of [23], particularly in Relation (2) of that section which in fact refers to a simpler situation. Its proof is similar to that of its forward counterpart (107) which is stated in many places in a more general form, for instance in Theorem 3.6 of Chapter 3 in [19] or in Theorem 5.1 of Chapter 2 in [17]. In this connection it is worth mentioning that we are not aware of any other *direct* proofs of Feynman-Kac representations such as (100) regarding the solutions to *forward* non-autonomous equations of the form (1). Such a proof is indeed made possible in our case thanks to the *backward* Itô equation (97), which allows us to carry out the proofs of (100) and

(102) quite independently as there are two Itô equations for the same process at our disposal.

(2) It is worth focusing again on two important aspects of our constructions, namely, on the one hand (34) which provides information on $Z_{\tau\in[0,T]}$ from the knowledge of (7) and (8) and, on the other hand, the representations (100) and (102) that make those two solutions emerge as functionals of $Z_{\tau\in[0,T]}$. In the latter case we remark that the comparison of (100) and (102) with (7) and (8) gives

$$\mathbb{E}^{-}_{\mu,x,t}\left(\exp\left[-\int_0^t d\sigma V\left(Z_\sigma,\sigma\right)\right]\varphi\left(Z_0\right)\right)=\int_D dy g_A(x,t;y,0)\varphi(y),$$
$$\mathbb{E}^{+}_{\mu,x,t}\left(\exp\left[-\int_t^T d\sigma V\left(Z_\sigma,\sigma\right)\right]\psi\left(Z_T\right)\right)=\int_D dy g^*_A(x,t;y,T)\psi(y),$$

respectively, that is, a concrete representation of the expectation functionals $\mathbb{E}^{\pm}_{\mu,x,t}$. In particular, for $V=0$ we obtain

$$\mathbb{E}^{-}_{\mu,x,t}\varphi\left(Z_0\right)=\int_D dy g_{V=0}(x,t;y,0)\varphi(y),$$
$$\mathbb{E}^{+}_{\mu,x,t}\psi\left(Z_T\right)=\int_D dy g^*_{V=0}(x,t;y,T)\psi(y),$$

respectively, with $g_{V=0}$ and $g^*_{V=0}$ the Green functions associated with (1) and (2) in this case.

In the next section we illustrate some of the preceding results.

3 Two examples

In the first example and for the sake of clarity we investigate the case of the simplest possible heat equation and its adjoint on a one-dimensional domain.

Example 1. Let us consider the initial-boundary value problem

$$\begin{aligned}\partial_t u(x,t)&=\frac{1}{2}\partial_{xx}u(x,t),\quad (x,t)\in(0,1)\times(0,T]\,,\\ u(x,0)&=\varphi(x),\quad x\in(0,1)\,,\\ \partial_x u(0,t)&=\partial_x u(1,t)=0,\quad t\in(0,T]\end{aligned}\tag{108}$$

and the corresponding final-boundary value problem for the adjoint equation

$$\begin{aligned}-\partial_t v(x,t)&=\frac{1}{2}\partial_{xx}v(x,t),\quad (x,t)\in(0,1)\times[0,T)\,,\\ v(x,T)&=\psi(x),\quad x\in(0,1)\,,\\ \partial_x v(0,t)&=\partial_x v(1,t)=0,\quad t\in[0,T)\,,\end{aligned}\tag{109}$$

where φ and ψ satisfy the hypotheses of the preceding section.

As is well known, the solution to (108) may be written as

$$u_{\varphi}(x,t)=\sum_{n\in\mathbb{Z}}a_n\cos(\pi nx)\exp\left[-\frac{\pi^2n^2}{2}t\right] \tag{110}$$

with

$$a_n=\int_0^1 dx\varphi(x)\cos(\pi nx)$$

for every $n\in\mathbb{Z}$, while the solution to (109) reads

$$v_{\psi}(x,t)=\sum_{n\in\mathbb{Z}}b_n\cos(\pi nx)\exp\left[-\frac{\pi^2n^2}{2}(T-t)\right] \tag{111}$$

with

$$b_n=\int_0^1 dx\psi(x)\cos(\pi nx)\,,$$

both series (110) and (111) being absolutely and uniformly convergent. Furthermore (4) and (5) hold with the kernels

$$\begin{aligned}&g(x,t;y,s)\\&=g^*(x,s;y,t)=\sum_{n\in\mathbb{Z}}\cos(\pi nx)\cos(\pi ny)\exp\left[-\frac{\pi^2n^2}{2}(t-s)\right],\end{aligned} \tag{112}$$

and moreover all of the results of the preceding section are valid providing one chooses a joint probability distribution of the form (28) satisfying (29). Equations (85) and (86) then read

$$Z_t=Z_0+\int_0^t d\tau\partial_x\ln v_{\psi}(Z_\tau,\tau)+W_t^* \tag{113}$$

and

$$Z_t=Z_T+\int_t^T d\tau\partial_x\ln u_{\varphi}(Z_\tau,\tau)+W_t, \tag{114}$$

respectively.

Our goal now is to choose as simple a φ and ψ as possible in order to unveil further properties of the processes thus constructed, keeping in mind that we ought to disregard the trivial case

$$\varphi(x)=\psi(x)=1,$$

which does imply (29) but gives

$$u_{\varphi}(x,t)=v_{\psi}(x,t)=1$$

for all $(x,t) \in [0,1] \times [0,T]$ and thereby $b^* = b = 0$ identically according to (60) and (61). Let us choose instead

$$\varphi(x) = 1 + \frac{1}{2}\cos(\pi x) \tag{115}$$

and

$$\psi(x) = 1. \tag{116}$$

Then the normalization condition (29) holds and we have

$$u_{\varphi}(x,t) = 1 + \frac{1}{2}\cos(\pi x)\exp\left[-\frac{\pi^2}{2}t\right] \tag{117}$$

and

$$v_{\psi}(x,t) = 1 \tag{118}$$

for all $(x,t) \in [0,1] \times [0,T]$. We have the following result:

Proposition 4. *Let us consider (108) and (109) with the data (115) and (116), respectively. Then the following statements hold:*

(a) We have

$$\mathbb{P}_{\mu}(Z_t \in E) = |E| + \frac{1}{2}\exp\left[-\frac{\pi^2}{2}t\right]\int_E dx\cos(\pi x)$$

for every Borel subset $E \subseteq [0,1]$ of Lebesgue measure $|E|$ and every $t \in [0,T]$.

(b) When considered as a forward Markov diffusion the process $Z_{\tau\in[0,T]}$ is a Wiener process with zero drift which is instantaneously reflected at $x = 0$ and $x = 1$. Moreover, the Lebesgue measure on $[0,1]$ is an invariant measure for $Z_{\tau\in[0,T]}$.

(c) When considered as a backward Markov diffusion the process $Z_{\tau\in[0,T]}$ satisfies

$$Z_t = Z_T - \pi\int_t^T d\tau \frac{\sin(\pi Z_\tau)\exp\left[-\frac{\pi^2}{2}\tau\right]}{2+\cos(\pi Z_\tau)\exp\left[-\frac{\pi^2}{2}\tau\right]} + W_t$$

for every $t \in [0,T]$, which corresponds to the backward drift

$$b(x,t) = \frac{\pi\sin(\pi x)\exp\left[-\frac{\pi^2}{2}t\right]}{2+\cos(\pi x)\exp\left[-\frac{\pi^2}{2}t\right]} \tag{119}$$

defined for all $(x,t) \in [0,1] \times [0,T]$.

Proof. Statement (a) follows from (34), (117) and (118).

Let us now consider $Z_{\tau\in[0,T]}$ as a forward Markov diffusion. Then we have

$$b^*(x,t) = 0$$

for all $(x,t) \in [0,1] \times [0,T]$ according to (60), so that (113) reduces to

$$Z_t = Z_0 + W_t^*$$

which defines a Wiener process with zero drift. As for the instantaneous reflection at the boundaries of the domain, let us condition the process by setting $Z_0 = x$ for an arbitrary $x \in (0,1)$. Since $Z_t \in [0,1]$ we then have

$$Z_t = |x + W_t^*|$$

and by virtue of the identity

$$\begin{aligned} &\sum_{n\in\mathbb{Z}} \cos(\pi n x)\cos(\pi n y)\exp\left[-\frac{\pi^2 n^2}{2}t\right] \\ &= (2\pi t)^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} \left(\exp\left[-\frac{|x+y+2n|^2}{2t}\right] + \exp\left[-\frac{|x-y-2n|^2}{2t}\right]\right) \end{aligned}$$

valid for all $x, y \in (0,1)$ and every $t \in (0,T)$ (see, for instance, Appendix 1 to Chapter 6 in [4]), we have

$$\begin{aligned} &\mathbb{P}_\mu^x (Z_t \in E) \\ &= \int_E dy g^*(x,0;y,t) \\ &= (2\pi t)^{-\frac{1}{2}} \int_E dy \sum_{n\in\mathbb{Z}} \left(\exp\left[-\frac{|x+y+2n|^2}{2t}\right] + \exp\left[-\frac{|x-y-2n|^2}{2t}\right]\right) \end{aligned}$$

for each Borel subset $E \subseteq [0,1]$ and all $(x,t) \in (0,1) \times (0,T)$. Therefore, we may indeed identify $Z_{\tau\in[0,T]}$ with the doubly reflected Brownian motion as defined for instance in Section 2.8 C of [19]. Finally let us consider the function (23), which in this case takes the form

$$M^*(x,s;E,t) = \int_E dy g^*(x,s;y,t)$$

where g^* is given by (112). Since we have

$$\int_0^1 dx M^*(x,s;E,t) = |E|$$

for each Borel subset $E \subseteq [0,1]$ and all $s,t \in [0,T]$ satisfying $t > s$, the Lebesgue measure is indeed an invariant measure for $Z_{\tau\in[0,T]}$ on $[0,1]$. Consequently Statement (b) holds.

Statement (c) follows directly from (61), (114) and (117). ■

Remark. The Bernstein diffusion $Z_{\tau\in[0,T]}$ of the preceding proposition is not reversible in the sense of [8] or [20] since its probability density is time-dependent according to Statement (a). This is a simple consequence of the third remark following the proof of Theorem 2.

Example 2. We now consider the initial-boundary value problem

$$\partial_t u(x,t) = \frac{1}{2}\triangle_x u(x,t), \quad (x,t) \in \mathbb{D} \times (0,T],$$
$$u(x,0) = \varphi(x), \quad x \in \mathbb{D},$$
$$\frac{\partial u(x,t)}{\partial n(x)} = 0, \quad (x,t) \in \partial\mathbb{D}\times (0,T] \tag{120}$$

and the corresponding final-boundary value problem for the adjoint equation

$$-\partial_t v(x,t) = \frac{1}{2}\triangle_x v(x,t), \quad (x,t) \in \mathbb{D} \times [0,T),$$
$$v(x,T) = \psi(x), \quad x \in \mathbb{D},$$
$$\frac{\partial v(x,t)}{\partial n(x)} = 0, \quad (x,t) \in \partial\mathbb{D}\times [0,T), \tag{121}$$

where

$$\mathbb{D} = \left\{x \in \mathbb{R}^2 : |x| < 1\right\}$$

is the two-dimensional open unit disk centered at the origin, and where φ and ψ satisfy the hypotheses of the preceding section. Let us restrict ourselves to radially symmetric solutions to (120) and (121); switching to polar coordinates and abusing the notation a bit we then rewrite (120) as

$$\partial_t u(r,t) = \frac{1}{2}\left(\partial_{rr} + r^{-1}\partial_r\right) u(r,t), \quad (r,t) \in (0,1] \times (0,T],$$
$$u(r,0) = \varphi(r), \quad r \in [0,1],$$
$$\partial_r u(1,t) = 0, \quad t \in (0,T], \tag{122}$$

whose solution reads

$$u_\varphi(r,t) = \sum_{n=1}^{+\infty} a_{n,0} J_0\left(\sqrt{\mu_{n,0}}r\right) \exp\left[-\frac{\mu_{n,0}}{2}t\right] \tag{123}$$

where J_0 stands for the Bessel function of zeroth order of the first kind (see, for instance, [31] or [32]). In (123) we have

$$a_{n,0} = 2J_0^{-2}\left(\sqrt{\mu_{n,0}}\right)\int_0^1 rdr\varphi\left(r\right) J_0\left(\sqrt{\mu_{n,0}}r\right)$$

where the $\mu_{n,0}$'s are ordered in such a way that

$$0 = \mu_{1,0} < \mu_{2,0} < \mu_{3,0} < ..... \tag{124}$$

and satisfy

$$J_0'\left(\sqrt{\mu_{n,0}}\right) = 0, \tag{125}$$

the preceding relation being Neumann's boundary condition in this case. In a similar way we have

$$\begin{aligned} -\partial_t v(r,t) &= \frac{1}{2}\left(\partial_{rr} + r^{-1}\partial_r\right) v(r,t), \quad (r,t) \in (0,1] \times [0,T)\,, \\ v(r,T) &= \psi(r), \quad r \in [0,1]\,, \\ \partial_r v(1,t) &= 0, \quad t \in [0,T)\,, \end{aligned} \tag{126}$$

for (121), whose solution is

$$v_\psi(r,t) = \sum_{n=1}^{+\infty} b_{n,0} J_0\left(\sqrt{\mu_{n,0}}r\right) \exp\left[-\frac{\mu_{n,0}}{2}(T-t)\right] \tag{127}$$

where

$$b_{n,0} = 2J_0^{-2}\left(\sqrt{\mu_{n,0}}\right)\int_0^1 rdr\psi\left(r\right) J_0\left(\sqrt{\mu_{n,0}}r\right),$$

both series (123) and (127) being again absolutely and uniformly convergent. Consequently we may write

$$u_\varphi(r,t) = \int_0^1 r'dr' g\left(r,t;r',0\right)\varphi\left(r'\right)$$

and

$$v_\psi(r,t) = \int_0^1 r'dr' g^*\left(r,t;r',T\right)\psi\left(r'\right)$$

where the radial Green functions are

$$\begin{aligned} &g\left(r,t;r',s\right) \\ &= g^*\left(r,s;r',t\right) \\ &= 2\sum_{n=1}^{+\infty} J_0^{-2}\left(\sqrt{\mu_{n,0}}\right) J_0\left(\sqrt{\mu_{n,0}}r\right) J_0\left(\sqrt{\mu_{n,0}}r'\right) \exp\left[-\frac{\mu_{n,0}}{2}(t-s)\right], \end{aligned}$$

so that all of the results of the preceding section remain valid in this case provided one chooses again a probability distribution of the form (28) satisfying (29).

Let us take for instance

$$\varphi(r) = \frac{1}{\pi}\left(1 + J_0\left(\sqrt{\mu_{2,0}}r\right)\right) \tag{128}$$

and

$$\psi\left(r\right) = 1 \tag{129}$$

for every $r \in [0,1]$, where $\sqrt{\mu_{2,0}}$ is the first positive root of J_0' according to (124) and (125). It then follows from the standard properties of J_0 that φ satisfies all the required hypotheses of our theory, including positivity and (29);

furthermore, the orthogonality properties of J_0 with respect ot the $\mu_{n,0}$'s imply that the corresponding solutions are

$$u_\varphi(r,t) = \frac{1}{\pi}\left(1 + J_0\left(\sqrt{\mu_{2,0}}r\right)\exp\left[-\frac{\mu_{2,0}}{2}t\right]\right) \tag{130}$$

and

$$v_\psi(r,t) = 1 \tag{131}$$

for all $(r,t) \in [0,1] \times [0,T]$. Let us denote again by $Z_{\tau\in[0,T]}$ the Bernstein diffusion associated with (122), (126) and (130), (131), respectively, and by $J_1 = -J_0'$ the Bessel function of first order. We have the following result, similar to that of Proposition 4:

Proposition 5. *Let us consider (122) and (126) with the data (128) and (129), respectively. Then the following statements hold:*

(a) We have

$$\mathbb{P}_\mu\left(Z_t \in E\right) = \frac{1}{\pi}\left(|E| + \exp\left[-\frac{\mu_{2,0}}{2}t\right]\int_E dx J_0\left(\sqrt{\mu_{2,0}}\,|x|\right)\right)$$

for every Borel subset $E \subseteq \mathbb{D}$ of Lebesgue measure $|E|$ and every $t \in [0,T]$.

(b) When considered as a forward Markov diffusion the process $Z_{\tau\in[0,T]}$ is a Wiener process with zero drift and the Lebesgue measure on $\mathbb{D}$ is an invariant measure for $Z_{\tau\in[0,T]}$.

(c) When considered as a backward Markov diffusion the process $Z_{\tau\in[0,T]}$ satisfies

$$Z_t = Z_T - \int_t^T d\tau b\left(Z_\tau,\tau\right) + W_t$$

for every $t \in [0,T]$, where the backward drift is given by

$$b(x,t) = \frac{\sqrt{\mu_{2,0}}J_1\left(\sqrt{\mu_{2,0}}\,|x|\right)\exp\left[-\frac{\mu_{2,0}}{2}t\right]}{1 + J_0\left(\sqrt{\mu_{2,0}}\,|x|\right)\exp\left[-\frac{\mu_{2,0}}{2}t\right]} \times \frac{x}{|x|} \tag{132}$$

for all $(x,t) \in \mathbb{D} \times [0,T]$ with $x \neq 0$, and by

$$b(0,t) = \lim_{|x|\to 0} b(x,t) = 0 \tag{133}$$

for every $t \in [0,T]$.

Proof. Statement (a) follows from (34), this time with (130) and (131).

Let us now consider $Z_{\tau\in[0,T]}$ as a forward Markov diffusion. Then we have

$$Z_t = Z_0 + W_t^*$$

since

$$b^*(x,t) = 0$$

for all $(x,t) \in \mathbb{D} \times [0,T]$ according to (60) and (131), which indeed defines a Wiener process with zero drift, and the fact that the Lebesgue measure on $\mathbb{D}$ is an invariant measure for $Z_{\tau\in[0,T]}$ follows from an argument similar to that given in Proposition 4. Consequently Statement (b) holds.

Statement (c) is a simple consequence of (61), (86) and (130), with (133) following from the fact that $J_0(0) = 1$ and

$$J_1\left(\sqrt{\mu_{2,0}}\,|x|\right) \sim \frac{1}{2}\sqrt{\mu_{2,0}}\,|x|$$

as $|x| \to 0$. $\blacksquare$

Remarks. (1) Relations (132) and (133) reveal two interesting features regarding the drift of the backward diffusion $Z_{\tau\in[0,T]}$. On the one hand, $b(x,t)$ points to the radial direction and its norm $|b(x,t)|$ is rotationally invariant, which is no surprise since (130) is radially symmetric to start with. On the other hand, $|b(x,t)|$ vanishes at the center of the disk and on its boundary since $J_1\left(\sqrt{\mu_{2,0}}\right) = 0$, and thereby reaches a maximal value at some $|x^*| = r^* \in (0,1)$ depending on t since $J_1 > 0$ on the interval $\left(0, \sqrt{\mu_{2,0}}\right)$. We note that such a feature is already present in the one-dimensional backward drift (119).

(2) The differential operator on the right-hand side of (122) and (126) is, of course, formally the generator of a Bessel process of order zero. Along the same line we can also consider radially symmetric solutions to Problems (120) and (121) in the d-dimensional ball centered at the origin of $\mathbb{R}^d$, where the radial part of half of the Laplacian identifies with the generator of a Bessel process of order $\nu := \frac{d}{2} - 1$. In such cases we can naturally expect the existence of nontrivial connections between Bernstein diffusions and Bessel processes of order ν. Moreover, because of (37) and (39) it is tempting to conjecture that all Bernstein diffusions constructed in this article are in fact reflected diffusions, as is the case in Statement (b) of Proposition 4. We defer a detailed analysis of these questions to a separate publication, including one of the boundary local time

$$L_Z(t) := \int_0^t ds \mathbb{I}_{\partial\mathbb{D}}(Z_s) = \lim_{\varepsilon\to 0_+} \frac{1}{\varepsilon}\int_0^t ds \mathbb{I}_{\mathbb{D}_\varepsilon}(Z_s)$$

in light of the results of [22], [25] and [30], where $\mathbb{I}_{\partial\mathbb{D}}$ is the indicator function of the boundary $\partial\mathbb{D}$ and $\mathbb{I}_{\mathbb{D}_\varepsilon}$ that of the annulus

$$\mathbb{D}_\varepsilon := \left\{x \in \mathbb{R}^d : 0 < 1 - \varepsilon \leq |x| \leq 1\right\}.$$

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4 Appendix. A Variational Construction of Weak Solutions in $L^2(D)$.

The solutions u_φ and v_ψ we used throughout this article are generated by two evolution systems $U_A(t,s)_{0\leq s\leq t\leq T}$ and $U_A^*(t,s)_{0\leq s\leq t\leq T}$ on $L^2(D)$. We show here how to construct these evolution systems by applying the standard methods of [29], under the following hypotheses regarding the coefficients k, l and V in (1) and (2):

(K′) The function $k : D\times[0,T]\mapsto\mathbb{R}^{d^2}$ is matrix-valued and for every $i,j\in\{1,...,d\}$ we have $k_{i,j}=k_{j,i}\in L^\infty(D\times(0,T))$; moreover, there exists a finite constant $\underline{k}>0$ such that the inequality

$$(k(x,t)q,q)_{\mathbb{R}^d}\geq\underline{k}\,|q|^2 \tag{134}$$

holds uniformly in $(x,t)\in D\times[0,T]$ for all $q\in\mathbb{R}^d$. Finally, there exist finite constants $c_*>0$, $\beta\in\left(\frac{1}{2},1\right]$ such that the Hölder continuity estimate

$$\max_{i,j\in\{1,\ldots,d\}}|k_{i,j}(x,t)-k_{i,j}(x,s)|\leq c_*\,|t-s|^\beta$$

is valid for every $x\in D$ and all $s,t\in[0,T]$.

As for the lower-order differential operators we assume that the following hypotheses are valid, where we assume without restricting the generality that the constants c_* and β are the same as in Hypothesis (K′):

(L′) Each component of the vector-field $l : D\times[0,T]\mapsto\mathbb{R}^d$ satisfies $l_i\in L^\infty(D\times(0,T))$. Moreover, the Hölder continuity estimate

$$\max_{i\in\{1,\ldots,d\}}|l_i(x,t)-l_i(x,s)|\leq c_*\,|t-s|^\beta$$

holds for every $x\in D$ and all $s,t\in[0,T]$.

(V′) The function $V : D\times(0,T)\mapsto\mathbb{R}$ is such that $V\in L^\infty(D\times(0,T))$ and satisfies

$$|V(x,t)-V(x,s)|\leq c_*\,|t-s|^\beta$$

for every $x\in D$ and all $s,t\in[0,T]$.

Moreover, both the initial condition φ and the final condition ψ are real-valued and the following hypothesis holds:

(IF′) We have $\varphi,\psi\in L^2(D)$.

Remark. In the variational theory we are reviewing here we observe that the Hölder continuity requirement relative to the time variable in Hypotheses

(K′), (L′) and (V′) is stronger than that of Hypotheses (K), (L) and (V), since $\beta \in \left(\frac{1}{2}, 1\right]$ whereas $\frac{\alpha}{2} \in \left(0, \frac{1}{2}\right)$. However, it is easy to show by uniqueness arguments that the evolution operators $U_A(t,s)_{0\leq s\leq t\leq T}$ and $U_A^*(t,s)_{0\leq s\leq t\leq T}$ introduced in Section 2 are identical to those constructed below. The reason why $\beta \in \left(\frac{1}{2}, 1\right]$ is required here is intimately tied up with the variational structure of the problem, and is thoroughly discussed in [29].

Under the preceding three conditions, it is easily verified that the quadratic form $a : [0,T] \times H^1(D) \times H^1(D) \mapsto \mathbb{C}$ defined by

$$\begin{aligned} a\left(t,f,h\right) &:= \frac{1}{2}\int_D dx \left(k(x,t)\nabla_x f(x), \nabla_x h(x)\right)_{\mathbb{C}^d} \\ &\quad + \int_D dx \left(l(x,t), \nabla_x f(x)\right)_{\mathbb{C}^d} \overline{h}(x) \\ &\quad + \int_D dx V(x,t) f(x)\overline{h}(x) \end{aligned}$$

satisfies the estimates

$$\left|a\left(t,f,h\right)\right| \leq c \left\|f\right\|_{1,2} \left\|h\right\|_{1,2}, \tag{135}$$

$$\operatorname{Re} a\left(t,f,f\right) \geq \underline{k} \left\|f\right\|_{1,2}^2 - c\left\|f\right\|_2^2, \tag{136}$$

$$\left|a\left(t,f,h\right) - a\left(s,f,h\right)\right| \leq c\left|t-s\right|^{\beta} \left\|f\right\|_{1,2} \left\|h\right\|_{1,2} \tag{137}$$

for all $s,t \in [0,T]$ and all $f,h \in H^1(D)$, where $\|.\|_2$ and $\|.\|_{1,2}$ stand for the usual norms in $L^2(D)$ and $H^1(D)$, respectively, and where $(.,.)_{\mathbb{C}^d}$ denotes the Hermitian inner product in $\mathbb{C}^d$. Consequently, the formal elliptic operator

$$A(t) := -\frac{1}{2}\operatorname{div}\left(k(.,t)\nabla\right) + \left(l(.,t), \nabla\right)_{\mathbb{C}^d} + V(.,t)$$

corresponding to the right-hand side of (1) can be realized as a regularly accretive operator defined on some time-dependent and dense domain $\mathcal{D}(A(t)) \subset L^2(D)$, and as such generates an evolution system $U_A(t,s)_{0\leq s\leq t\leq T}$ in $L^2(D)$ given by

$$U_A(t,s)f(x) = \begin{cases} f(x) & \text{if } t = s, \\ \int_D dy g_A(x,t;y,s)f(y) & \text{if } t > s \end{cases} \tag{138}$$

for every $f \in L^2(D)$, where g_A denotes the parabolic Green function associated with (1). Indeed all these assertions follow directly from estimates (135)-(137) and the general theory developed in Section 5.4 of [29], together with Schwartz's kernel theorem which guarantees the existence of g_A (see [27] for a summary of the many possible applications of that theorem).

In a similar way, the Hermitian conjugate form

$$a^*\left(t,f,h\right) := \overline{a\left(t,h,f\right)}$$

is associated with the linear operator $A^*(t)$ adjoint to $A(t)$, which in turn generates the adjoint evolution system

$$U_A^*(t,s)f(x) = \begin{cases} f(x) & \text{if } t = s, \\ \int_D dy g_A^*(x,s;y,t)f(y) & \text{if } t > s, \end{cases} \tag{139}$$

where G_A^* is the parabolic Green function associated with (2) that satisfies the relation

$$g_A^*(x,s;y,t) = g_A(y,t;x,s)$$

for all $s,t \in [0,T]$ with $t > s$.

The important features of (138) and (139) are that they provide the real-valued functions defined by

$$u_\varphi(x,t) := U_A(t,0)\varphi(x) = \int_D dy g_A(x,t;y,0)\varphi(y), \quad t \in (0,T] \tag{140}$$

and

$$v_\psi(x,t) := U_A^*(T,t)\psi(x) = \int_D dy g_A^*(x,t;y,T)\psi(y), \quad t \in [0,T)\,, \tag{141}$$

which satisfy

$$\left(\frac{\partial}{\partial t}u_\varphi(.,t),h\right)_2 + a(t,u_\varphi(.,t),h) = 0, \quad t \in (0,T]$$

and

$$-\left(\frac{\partial}{\partial t}v_\psi(.,t),h\right)_2 + a^*(t,v_\psi(.,t),h) = 0, \quad t \in [0,T)$$

for every $h \in H^1(D)$, respectively, where $(.,.)_2$ stands for the usual inner product in $L^2(D)$. Moreover we have $u_\varphi, v_\psi \in L^2(D \times (0,T))$, so that (140) and (141) provide *weak* solutions to (1) and (2), respectively (see, for instance, Section 5.5 in [29]).

These solutions are those which ultimately possess the properties listed in Lemma 1 of Section 2, according to the above remark regarding the Hölder regularity in time.

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