PERIOD AND ENERGY
IN ONE DEGREE OF FREEDOM SYSTEMS

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ABSTRACT. For one degree of freedom systems there exists a very well known formula for the period ($T$) of periodic solutions. In this note we give a detailed description of the behavior of $T$ in function of the total energy ($E$), near a stable equilibrium position of energy $E_0$. A formula for $\frac{dT}{dE}(E_0)$ is established. We illustrate these formulae with two examples. The second one is a new proof of a Bertrand’s theorem.

1. INTRODUCTION

As it is very well known, a system with one degree of freedom is a differential equation of the form

$$m \ddot{x} = -U'(x),$$

where $m > 0$, $U$, the potential, is a smooth function of $x$, defined in a real interval, and $\ddot{x} \equiv \frac{d^2x}{dt^2}$, $U'' \equiv \frac{dU}{dx}$. Equation (1) is nothing more than a Newton’s equation of motion in one dimension. We follow essentially [1].

If $t \mapsto x(t)$ is a solution of (1), then

$$E \equiv E(x(t), \dot{x}(t)) = \frac{m}{2} \dot{x}(t)^2 + U(x(t)),$$

the energy, is constant.

Let $\xi$ be such that $U'(\xi) = 0$. Then, $\xi$ is called an equilibrium position; $t \mapsto x(t) \equiv \xi$ is a solution of (1). We shall assume that $U''(\xi) > 0$, which implies that $\xi$ is a stable equilibrium position. Denote $E_0 = E(\xi, 0)$.

Under the above assumption, for $E$ near and $> E_0$, there are periodic movements $t \mapsto x(t)$ around $\xi$ and of energy $E$. The period $T$ of such movements

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is given by

\[(2) \quad T(E) = 2 \int_{x_i}^{x_2} \left[ \frac{2}{m} \right]^{\frac{1}{2}} \frac{E - U(x)}{x} dx, \]

where \(x_i \equiv x_i(E)\) are the turning points, \(U(x_i(E)) = E, i = 1, 2.\)

In the limit \(E \to E_0\) one obtains the well known formula

\[T(E_0) \equiv \lim_{E \to E_0} T(E) = 2\pi \left[ \frac{1}{m} U''(\xi) \right]^{\frac{1}{2}}.\]

The object of this note is to study the behavior of \(T(E)\) near an equilibrium position

Assume that \(U\) is of class \(C^n, n \geq 2,\) and that \(U'' > 0\) in \([x_1, x_2]\). Define the function \(x \mapsto X = \varphi(x),\) for \(x \in [x_1, x_2],\) such that

\[\varphi(x) = \pm \sqrt{\frac{2(U(x) - E_0)}{U''(\xi)}}.\]

where + stands for \(x \geq \xi\) and − stands for \(x \leq \xi;\) \(\varphi\) is of class \(C^{n-1}\) and \(\varphi' = U''(\xi)(U''(\xi) U''(\xi))^{-\frac{1}{2}} > 0,\) with \(\xi_1\) and \(\xi_2\) between \(\xi\) and \(x.\)

Denote \([-X_0, X_0] = \varphi([x_1, x_2])\) and consider the function \(f : [-X_0, X_0] \to \mathbb{R},\) such that

\[f(X) = \frac{1}{\varphi'(-1 \varphi(X))}.\]

The function \(f\) is of class \(C^{n-2}.\) Denote \(f' = \frac{df}{dX}, f'' = \frac{d^2f}{dX^2},\) and so on. For \(n \geq 4,\) and as \(U''(\xi) X = U'(\varphi^{-1}(\xi)) f(X),\) one has

\[f(0) = 1, f'(0) = -\frac{U'''(\xi)}{3U''(\xi)}, f''(0) = \frac{5U''''(\xi)^2 - 3U''(\xi) U''''(\xi)}{12U''(\xi)^2}.\]

Formula (2) can be written

\[T(E) = 2 \int_{-X_0}^{X_0} f(X) \left[ \frac{1}{m} \right]^{\frac{1}{2}} \frac{2(E - E_0) - U''(\xi) X^2}{X^2} dX.\]

Writing \(f(X) = 1 + f'(0) X + \frac{1}{2} f''(\theta) X^2,\) for some \(\theta \in [-X_0, X_0],\) and as

\[\int_{-X_0}^{X_0} X^2 \left[ \frac{1}{m} \right]^{\frac{1}{2}} \frac{2(E - E_0) - U''(\xi) X^2}{X^2} dX = \frac{\pi \sqrt{m}}{U''(\xi)} (E - E_0),\]
one has

\[ T (E) - T (E_0) = \sigma \frac{\pi \sqrt{m}}{U'' (\xi)} (E - E_0), \]

for some \( \sigma \in f'' ([-X_0, X_0]) \).

Hence

\[ T' (E_0) \equiv \frac{dT}{dE} (E_0) = f'' (0) \frac{\pi \sqrt{m}}{U'' (\xi)} = \pi \sqrt{m} \left( \frac{5U'''' (\xi)^2 - 3U'''' (\xi) U'' (\xi)}{12U'''' (\xi)^2} \right). \]

**Example 1.** Consider a mathematical pendulum of length \( l \) under the free fall acceleration \( g \). The potential is \( U (x) = -\frac{g}{l} \cos x \), where \( x \) is the angle of deviation of the pendulum from the vertical.

In this case \( \varphi (x) = 2 \sin \frac{x}{2} \) and \( f (X) = 2 \left( 4 - X^2 \right)^{-\frac{1}{2}} \). Simple calculations show that formula (3) becomes

\[ T (E) - T (E_0) = 2\sigma \pi \sqrt{\frac{l}{8}} \left( \sin \frac{x_0}{2} \right)^2, \]

for some \( \sigma \in \left[ \frac{1}{4}, \frac{1 + 2 \left( \sin \frac{x_0}{2} \right)^2}{4 \left( \cos \frac{x_0}{2} \right)^5} \right] \),

where \( x_0 \) is the maximum deviation angle.

**Example 2.** The following example is another proof of a Bertrand’s theorem. We follow partially [1], Chapter 2, § 8.D. See also [2], page 51 and [4], page 90. The original proof is in [3].

Let \( U \) be the potential of a central force, \( \mu \) the angular momentum and \( V \) the effective potential, \( V (r) = U (r) + \frac{\mu^2}{2mr^2} \), where \( r \mapsto r (t) \) is the radial movement. Assume that there are periodic radial movements and that \( r_0 \) is a minimum for \( V \). Then \( r \mapsto r_0 \) is a stable circular orbit. Let \( r \mapsto r (t) \) be a periodic radial movement, \( \Phi \) the angle between a pericenter and an apocenter which are adjacent and let \( r_1 \) and \( r_2 \) be the distances from the pericenters and the apocenters to the center of the field, \( r_1 < r_0 < r_2 \). Making the change \( x = \frac{1}{r}, x_1 = \frac{1}{r_1}, x_2 = \frac{1}{r_2}, x_0 = \frac{1}{r_0} \), \( W (x) = U \left( \frac{1}{x} \right) + \frac{\mu^2}{2m} x^2 \), one has \( \Phi = \frac{\mu}{2 \sqrt{m}} T (E) \),

where \( T (E) \) is as in (2) with \( U \) replaced by \( W \) and \( m \) by 1. As in [1] one easily obtains

\[ \lim_{r_1, r_2 \to r_0} \Phi = \Phi_{\text{circ}} = \lim_{E \to E_0} \frac{\mu}{2 \sqrt{m}} T (E) \]

\[ = \pi \mu \left[ MW'' (x_0) \right]^{-\frac{1}{2}} = \pi \left( \frac{U' (r_0)}{3U'' (r_0) + r_0 U''' (r_0)} \right)^{\frac{1}{2}}. \]

As in [1] we consider the differential equation \( U' (r) = C (3U'' (r) + rU''' (r)) \), for \( r \) in some interval and \( C > 0 \). The solutions are \( U (r) = ar^e \) and \( U (r) =
$b \log r$, with $\varepsilon, a, b \neq 0$ and $\varepsilon > -2$. The second case ($b \log r$) can easily be excluded.

As $\Phi$ does not depend on $E$, $T(E)$ does not depend on $E$. Hence $\frac{dT}{dE}(E_0) = 0$. From (4), one has

$$5W'''(x_0)^2 - 3W^{(4)}(x_0)W''(x_0) = 0.$$ 

As $W(x) = ax^{-\varepsilon} + \frac{\mu^2}{2m}x^2$, we have that $x_0 = \left(\frac{\text{sum}}{\mu^2}\right)^{\frac{1}{\varepsilon+2}}$ which, together with (5), implies $\varepsilon = -1$ or $\varepsilon = 2$. Remember that, in order to have bounded movements, for $\varepsilon = -1$, $a < 0$, and for $\varepsilon = 2$, $a > 0$.

References


