ON THE *PUZZLES WITH POLYHEDRA AND NUMBERS*

JORGE REZENDE

1. Introduction

The Portuguese Mathematical Society (SPM) published, in 2001, a set of didactical puzzles called *Puzzles com poliedros e números* (*Puzzles with polyhedra and numbers*) [2]. It is a set of paper models of eight puzzles with polyhedra that everyone with age and knowledge to manipulate them can construct even if the mathematical culture is poor.

For those that already have some mathematical knowledge, these puzzles are a source for many examples and exercises, that go from the elementary to complex ones, in combinatorics, group theory (including symmetry and permutation groups), programming, and so on. The object of this work is to point out some possibilities by giving simple examples. See also References [1] and [3].

2. Definitions of the puzzles using combinatorics

Consider, for example, a platonic polyhedron. Its faces are regular polygons. Construct plates with the same shape and size as the polyhedron faces. Near each side of these plates let us draw numbers $1, 2, \ldots, n$, as it is shown in figures 1,2 and 3. Assume that in each plate all the numbers are distinct. Let j be the number of the plate sides: $j = 3$ in the tetrahedron, octahedron and icosahedron cases; $j = 4$ in the cube case; $j = 5$ in the dodecahedron case.

How many different plates it is possible to construct in this way? If ν is the number of different plates the answer of this question is the following:

a) For $j = 3$ and $n = 3$, then $\nu = 2$ (see figure 1).
b) For $j = 3$ and $n = 4$, then $\nu = 8$ (see figure 2); 8 **is precisely the number of the octahedron faces**. With these 8 plates we make the octahedron puzzle.
c) For $j = 3$ and $n = 5$, then $\nu = 20$; 20 **is precisely the number of the icosahedron faces**. With these 20 plates we make the icosahedron (1) puzzle.

The Mathematical Physics Group is supported by the Portuguese Ministry for Science and Technology.

d) For $j = 3$ and $n = 6$, then $\nu = 40$; 40 **is precisely the double of the number of the icosahedron faces**.
e) For $j = 4$ and $n = 4$, then $\nu = 6$ (see figure 3); 6**is precisely the number of the cube faces**. With these 6 plates we make the cube puzzle.
f) For $j = 5$ and $n = 5$, then $\nu = 24$; 24 **is precisely the double of the number of the dodecahedron faces**.

The general formula is

$$\nu = (j-1)!\binom{n}{j} = \frac{n!}{(n-j)!\,j}\,.$$

From now on, we shall concentrate ourselves in the octahedron case. With $j = 3$ and $n = 4$, we have 8 distinct plates (figure 2), that we use in order to do the octahedron puzzle. One says that a solution of the puzzle is obtained if one has every plate over a face of the octahedron in such a way that the two numbers near the same edge are equal for every edge.

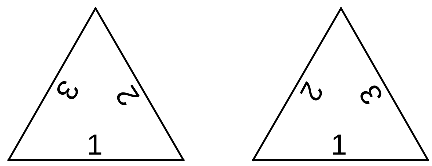

FIGURE 1.

3. THE OCTAHEDRON: NOTATIONS, PLANAR REPRESENTATIONS, SOLUTIONS

Figure 4 represents the octahedron centered at the origin. The set of the edges is denoted $E = \{e_1, e_2, \ldots, e_{12}\}$. The letter F denotes the face set.

Figures 5 and 6 represent two different octahedron planar representations where the edges are well identified.

Figure 7 has two different representations of the same solution of the octahedron puzzle. The first one is a planar representation of the puzzle as it is done, in practice. The second one is more schematic and shows that a puzzle solution associates to every edge a number of the set $\{1, 2, 3, 4\}$.

In fact, to give a puzzle solution implies to give a function $\varepsilon : E \to \{1, 2, 3, 4\}$ (the converse is not true). We shall say that this function ε

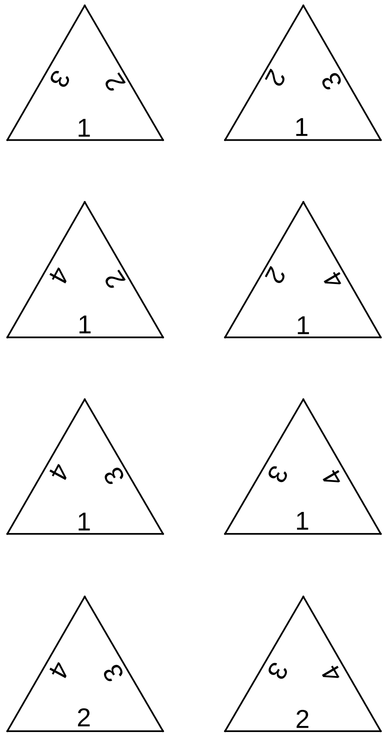

FIGURE 2.

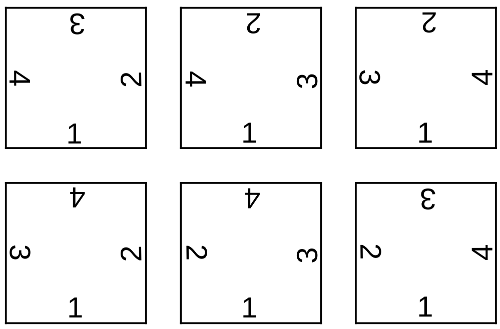

FIGURE 3.

is a solution of the puzzle. In the case of figure 7, ε is the following function:

$$\begin{array}{cccc} e_1 \mapsto 4 & e_2 \mapsto 1 & e_3 \mapsto 2 & e_4 \mapsto 3 \\ e_5 \mapsto 2 & e_6 \mapsto 3 & e_7 \mapsto 4 & e_8 \mapsto 1 \\ e_9 \mapsto 3 & e_{10} \mapsto 4 & e_{11} \mapsto 1 & e_{12} \mapsto 2 \end{array}.$$

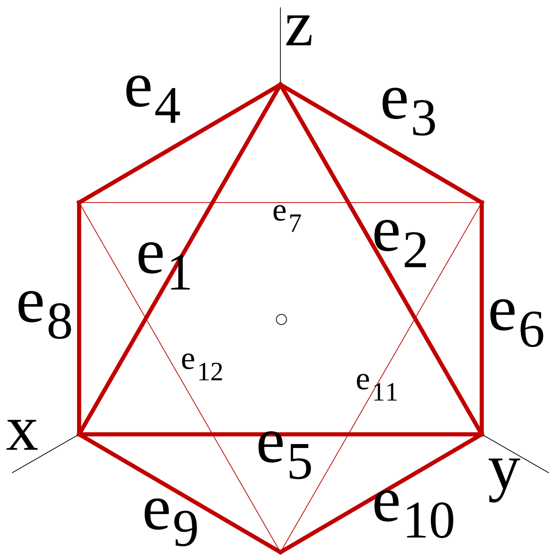

FIGURE 4.

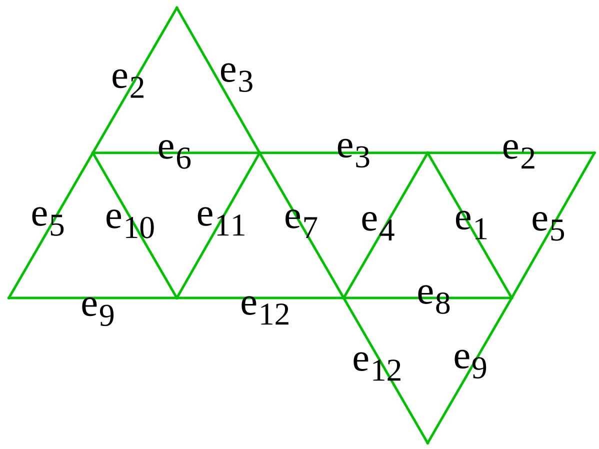

FIGURE 5.

4. THE PUZZLES AND THE ELEMENTARY GROUP THEORY

4.1. **Permutation groups.** Let $n \in \mathbb{N}$. The permutation group of $\{1, 2, \ldots, n\}$, S_n, is the set of the one-to-one functions $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, with the composition of functions as operation. We shall use the notation: $\sigma_1 \circ \sigma_2 \equiv \sigma_1\sigma_2$. Here the identity is denoted σ_0: $\sigma_0(1) = 1, \sigma_0(2) = 2, \ldots, \sigma_0(n) = n$.

We shall write $\sigma = (\alpha_1\alpha_2 \cdots \alpha_k) \cdots (\beta_1\beta_2 \cdots \beta_l)$, if

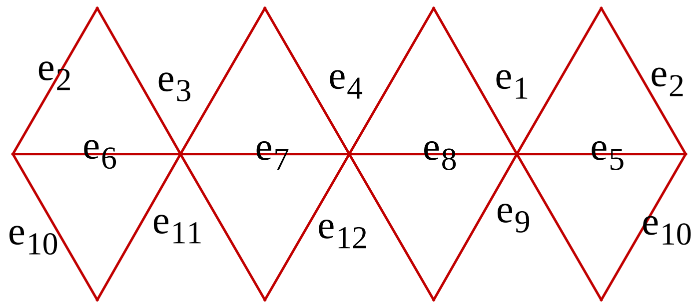

FIGURE 6.

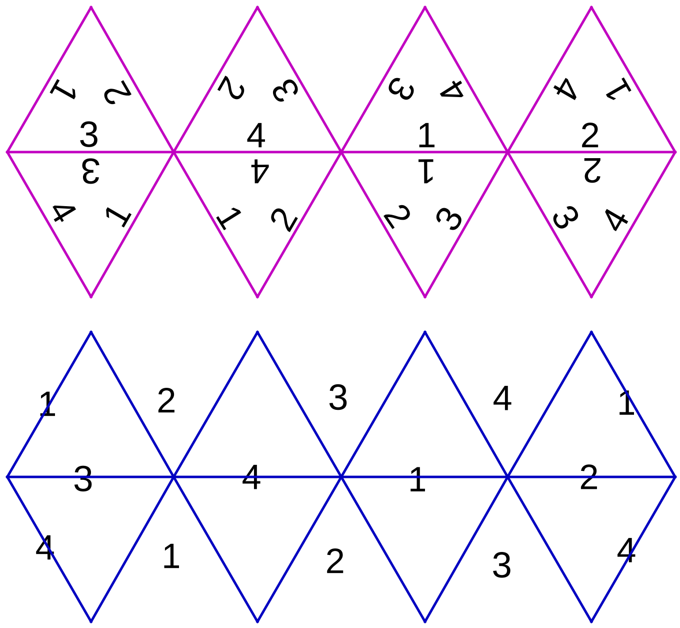

FIGURE 7.

$$\sigma(\alpha_1) = \alpha_2, \sigma(\alpha_2) = \alpha_3, \ldots, \sigma(\alpha_k) = \alpha_1, \ldots,$$
$$\sigma(\beta_1) = \beta_2, \sigma(\beta_2) = \beta_3, \ldots, \sigma(\beta_k) = \beta_1$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots, \beta_1, \beta_2, \ldots, \beta_l \in \{1, 2, \ldots, n\}$.

If $\gamma \in \{1, 2, \ldots, n\} \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots, \beta_1, \beta_2, \ldots, \beta_l\}$, then $\sigma(\gamma) = \gamma$.

In the octahedron case we are interested in the situation where $n = 4$. In what follows a, b, c, d are different elements of $\{1, 2, 3, 4\}$. S_4, a group of order 24 (the group cardinal), contains, obviously, σ_0, the 6 elements of the $(abcd)$ type, the 8 elements of the (abc) type, the 3 elements of the $(ab)(cd)$ type and the 6 elements of the (ab) type.

4.2. **The octahedron group.** The group of the octahedron symmetries, called the octahedron group, Ω, is the set of all isometries ω of $\mathbb{R}^3$, that send vertices to vertices, which implies that they send edges to edges, faces to faces. We shall denote: $\omega_1 \circ \omega_2 \equiv \omega_1\omega_2$. If $\omega \in \Omega$, then ω induces one-to-one functions $F \longrightarrow F$, $E \longrightarrow E$, that, we shall denote, when no confusion is possible, by the same letter ω. Remark that not all one-to-one functions $F \longrightarrow F$, $E \longrightarrow E$ are in Ω.

An element of Ω is, for example, the central symmetry $\omega(x, y, z) = -(x, y, z)$, that induces the function $\omega : E \longrightarrow E$

$$\begin{array}{cccc} \omega(e_1) = e_{11} & \omega(e_2) = e_{12} & \omega(e_3) = e_9 & \omega(e_4) = e_{10} \\ \omega(e_5) = e_7 & \omega(e_6) = e_8 & \omega(e_7) = e_5 & \omega(e_8) = e_6 \\ \omega(e_9) = e_3 & \omega(e_{10}) = e_4 & \omega(e_{11}) = e_1 & \omega(e_{12}) = e_2 \end{array} .$$

The central symmetry has determinant -1. The symmetries with determinant 1 (Ω^+), can be seen like this: one transports a chosen face in such a way that it goes to one of the eight faces of the octahedron; as one has three possibilities of making them coincide (they are equilateral triangles), there are 24 (3×8) symmetries with determinant 1. Figure 8 shows one of these symmetries. In this case the function $\omega : E \to E$, is the following:

$$\begin{array}{cccc} \omega(e_1) = e_4 & \omega(e_2) = e_7 & \omega(e_3) = e_{12} & \omega(e_4) = e_8 \\ \omega(e_5) = e_3 & \omega(e_6) = e_{11} & \omega(e_7) = e_9 & \omega(e_8) = e_1 \\ \omega(e_9) = e_2 & \omega(e_{10}) = e_6 & \omega(e_{11}) = e_{10} & \omega(e_{12}) = e_5 \end{array} .$$

The advantage of describing in this way the symmetries of Ω^+ is that it can be easily adapted to other polyhedra, and used in their computation in a computer program.

Another way of counting the symmetries of Ω^+ is the following: the identity (1); the rotations of 90°, 180° and 270° around the three axes defined by opposite vertices (9); the rotations of 180° around the six axes defined by the centers of opposite edges (6); the rotations of 120° and 240° around the four axes defined by the centers of opposite faces (8).

The symmetries with determinant -1 (Ω^-) are the compositions of the symmetries of Ω^+ with the central symmetry. The cardinal of Ω, the order of Ω, is, therefore, 48.

Let us see two examples:

a) Consider, in the figures 9 and 10, the rotations of 0°, 90°, 180° and 270° around the z-axis. They form a subgroup of Ω^+ of order four. The action of these rotations on the octahedron faces, sends every green face, by order of succession, to the place of all other green faces and every magenta face, by order of succession, to the place of all other magenta faces. The set of the green faces and the set of the magenta faces are the

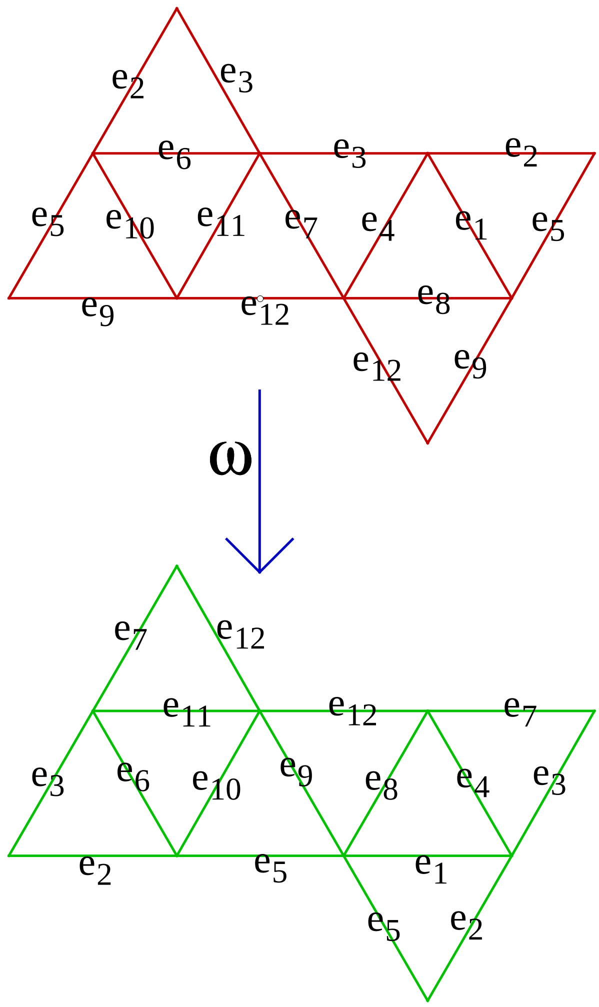

FIGURE 8.

orbits of this action (see figure 10). Hence, there are two orbits, each one of them with four elements. We say that the action of this subgroup is of the $(1, 2 \times 4)$ type. The first component, the 1, is the determinant of the generator.

b) Consider, in figures 9 and 11, the rotations of 0°, 120° and 240° around the w-axis. They form a subgroup of Ω^+ of order three. The action of these rotations on the octahedron faces keeps the green face fixed, as well the yellow one (which is the green face opposite), and sends every blue face, by order of succession, to the place of all other blue faces and every magenta face, by order of succession, to the place of all other magenta faces. The set with the green face, the set with the yellow face,

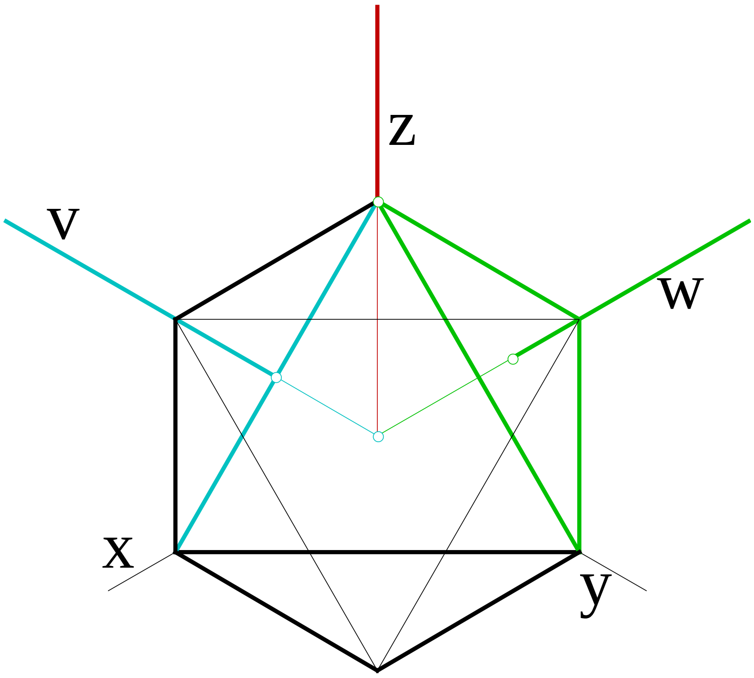

FIGURE 9.

the set of the blue faces and the set of the magenta faces are the orbits of this action (see figure 11). Therefore, there are four orbits, two with three elements each, and two with only one element each. We say that the action of this subgroup is of the $(1, 2\times 1 + 2\times 3)$ type. The first component, the 1, is the determinant of the generator.

4.3. **The plate group.** The octahedron plate group is $S_4^{\pm} \equiv \{-1, 1\} \times S_4$. If $(\delta_1, \sigma_1), (\delta_2, \sigma_2) \in S_4^{\pm}$, then $(\delta_1, \sigma_1)(\delta_2, \sigma_2) = (\delta_1\delta_2, \sigma_1\sigma_2)$. This group acts on the puzzle plates in the way that figure 12 describes, with $s_1 = (1, \sigma)$, $s_2 = (-1, \sigma)$, $a_1 = \sigma(a)$, $b_1 = \sigma(b)$, $c_1 = \sigma(c)$.

Let us look at two examples:

a) If $s = (1, (1234))$, see figure 13.

In this case $s \equiv \sigma = (1234)$, and $A_1 = \sigma(A_0)$, $A_2 = \sigma(A_1)$, $A_3 = \sigma(A_2)$, $A_0 = \sigma(A_3)$, $B_1 = \sigma(B_0)$, $B_2 = \sigma(B_1)$, $B_3 = \sigma(B_2)$, $B_0 = \sigma(B_3)$. The sets $\{A_0, A_1, A_2, A_3\}$, $\{B_0, B_1, B_2, B_3\}$, are the action orbits of the subgroup, of order four, generated by $s \equiv \sigma = (1234)$, on the plate set. We say that the action of the subgroup generated by s is of the $(1, 2\times 4)$ type. The first component, the 1, is the first component of s.

b If $s = (1, (123))$, see figure 14.

In this case $s \equiv \sigma = (123)$, and $A = \sigma(A)$, $B = \sigma(B)$, $C_1 = \sigma(C_0)$, $C_2 = \sigma(C_1)$, $C_0 = \sigma(C_2)$, $D_1 = \sigma(D_0)$, $D_2 = \sigma(D_1)$, $D_0 = \sigma(D_2)$. The

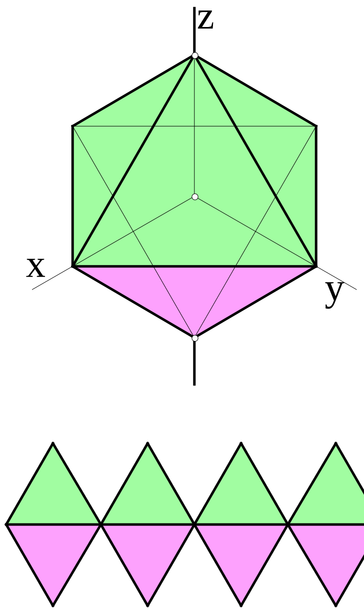

FIGURE 10.

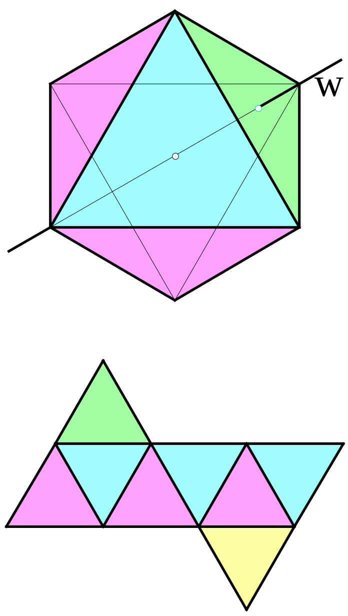

FIGURE 11.

sets $\{A\}$, $\{B\}$, $\{C_0, C_1, C_2\}$, $\{D_0, D_1, D_2\}$, are the action orbits of the subgroup, of order three, generated by $s \equiv \sigma = (123)$, on the plate set. We say that the action of the subgroup generated by s is of the

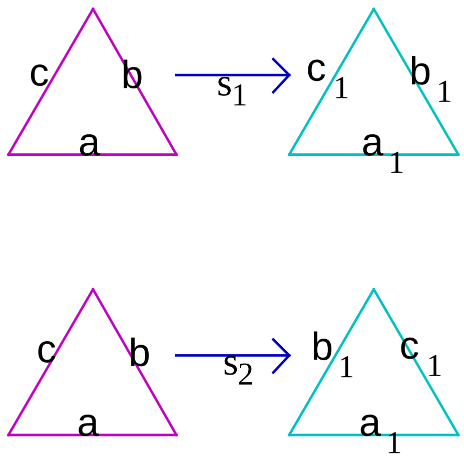

FIGURE 12.

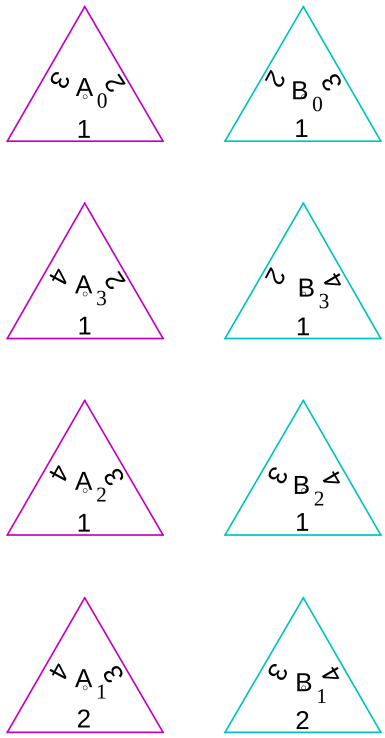

FIGURE 13.

$(1, 2\times 1 + 2\times 3)$ type. The first component, the 1, is the first component of s.

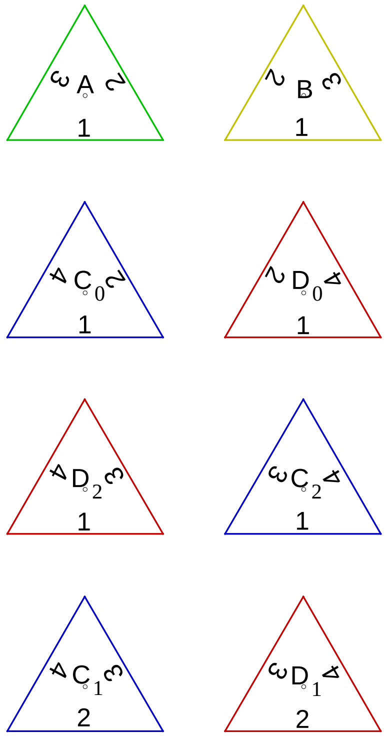

FIGURE 14.

5. SOLUTIONS

As we have already seen, to a puzzle solution corresponds a function $\varepsilon : E \longrightarrow \{1, 2, 3, 4\}$. We shall now study in more detail the octahedron puzzle solutions.

5.1. **Natural solutions.** Consider $\varepsilon_1, \varepsilon_2 : E \longrightarrow \{1, 2, 3, 4\}$ two solutions of the octahedron puzzle. One says that they represent the same **natural solution**, if there is $\omega : E \longrightarrow E$, $\omega \in \Omega^+$, such that

$$\varepsilon_1 \circ \omega = \varepsilon_2.$$

This equation involving ε_1 and ε_2 defines an equivalence relation, and a natural solution is an equivalence class of this relation. Remark that if $\varepsilon_1 = \varepsilon_2$, then ω is the identity.

In practice, we do not distinguish two representatives of the same natural solution, because there are no Cartesian axes associated to the polyhedron as we see in figure 4.

Figure 15 shows two solutions that, although they are not equal, represent the same natural solution. The octahedron puzzle has sixteen different natural solutions.

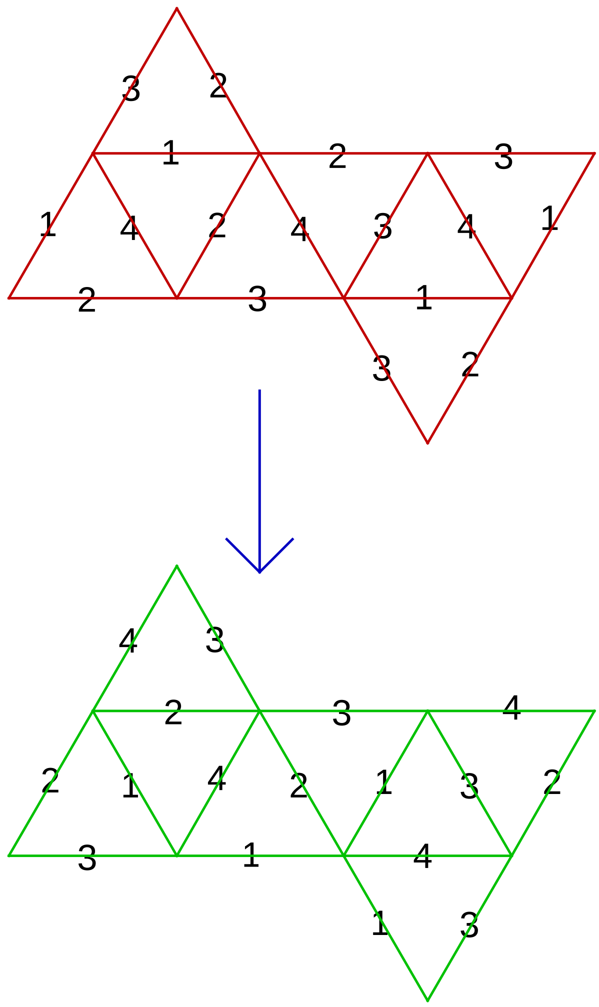

FIGURE 15.

5.2. **The solution group.** The group $S_4 \times \Omega$ is defined with the following product: if $(\sigma_1, \omega_1), (\sigma_2, \omega_2) \in S_4 \times \Omega$, then $(\sigma_1, \omega_1)(\sigma_2, \omega_2) = (\sigma_1\sigma_2, \omega_2\omega_1)$.

Let $\varepsilon : E \longrightarrow \{1, 2, 3, 4\}$ be a puzzle solution. The group of this solution, G_ε, is the $S_4 \times \Omega$ subgroup, such that $(\sigma, \omega) \in G_\varepsilon$ if

$$\sigma \circ \varepsilon \circ \omega = \varepsilon,$$

or, equivalently,

$$\varepsilon \circ \omega = \sigma^{-1} \circ \varepsilon.$$

This relation implies that ω order equals σ order, which means that ω^k is the identity if and only if $\sigma^k = \sigma_0$. The generalization of this property can be seen in [3]. In the two following examples we shall look for solutions with groups which have prescribed cyclic subgroups.

a) Figure 16 shows the different possibilities when ω is a rotations of $90°$ around the z-axis (figure 9) and $\sigma = (1234)$. The equivalence class A of figure 13 must be either over all the green faces or over all the magenta faces of figure 10. If A is over the green faces, then B is over the magenta faces. If A is over the magenta faces, then B is over the green faces.

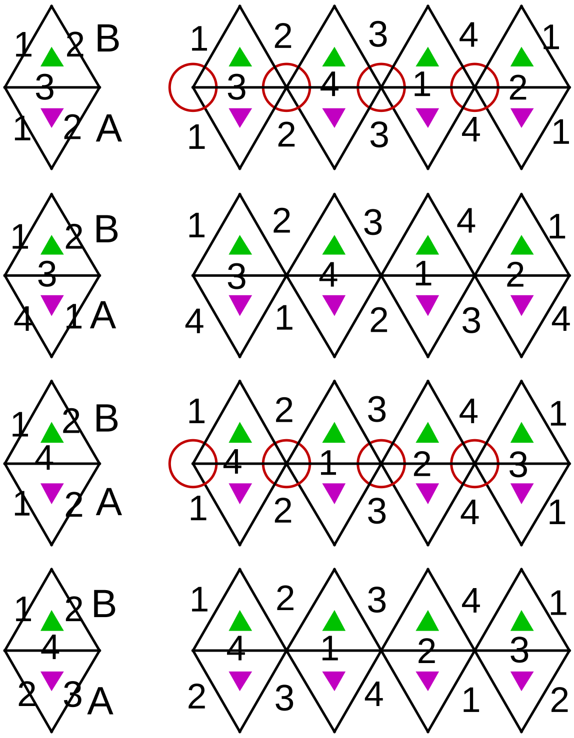

FIGURE 16.

b) Figure 17 shows the different possibilities when ω is the rotation of $120°$ around the w-axis (figure 9) and $\sigma = (123)$. We put the plate A, of figure 14, over the green face. Of course that B, of figure 14, must be over the yellow face. The equivalence class C of figure 14 must be either over all the blue faces or over all the magenta faces. If C is over the blue faces, then D is over the magenta faces. If C is over the magenta faces, then D is over the blue faces.

5.3. **Equivalent solutions.** Consider $\varepsilon_1, \varepsilon_2 : E \to \{1,2,3,4\}$ two solutions of the octahedron puzzle. We say that they are equivalent, if there are $\omega \in \Omega$, $\sigma \in S_4$ such that

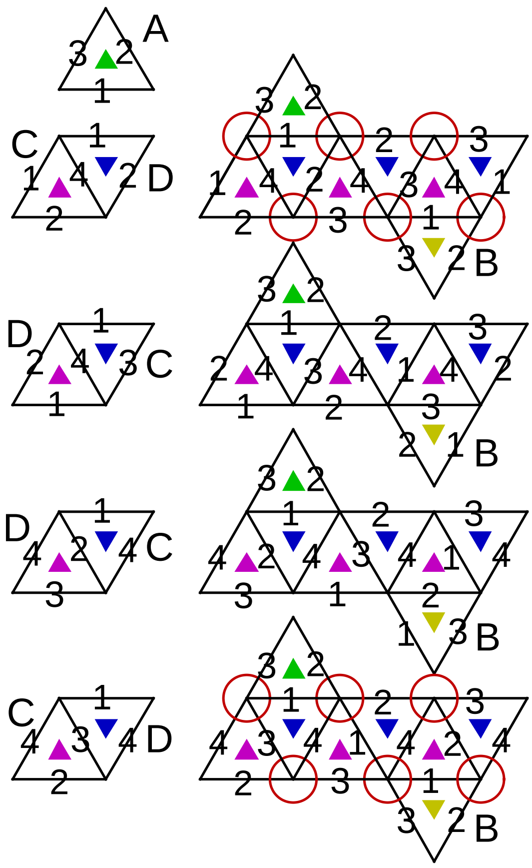

Figure 17.

$$\sigma \circ \varepsilon_1 \circ \omega = \varepsilon_2.$$

As the name says, this relation between ε_1 and ε_2 is an equivalence relation. Two representatives of the same natural solution are, of course, equivalent. The octahedron puzzle has three equivalence classes that one can identify looking at the vertices. Take a solution, a vertex and note the numbers in the four edges that meet in this vertex. One can have two different cases: a) the numbers are all different, or b) there is one number that is repeated. The vertices of the b) case are marked in figures 16 and 17 with a circle. In the solution either there are 0 vertices of the b) type, or there are 4 vertices of the b) type, or there are 6 vertices of the b) type. Those are the three equivalence classes, that, by the way, one can see in figures 16 and 17.

If a solution has 0 vertices of b) type, one says that it is one of the canonical natural solutions (there are two). Its group has order 24 and it helps to identify Ω^+ with S_4. Look at figure 18.

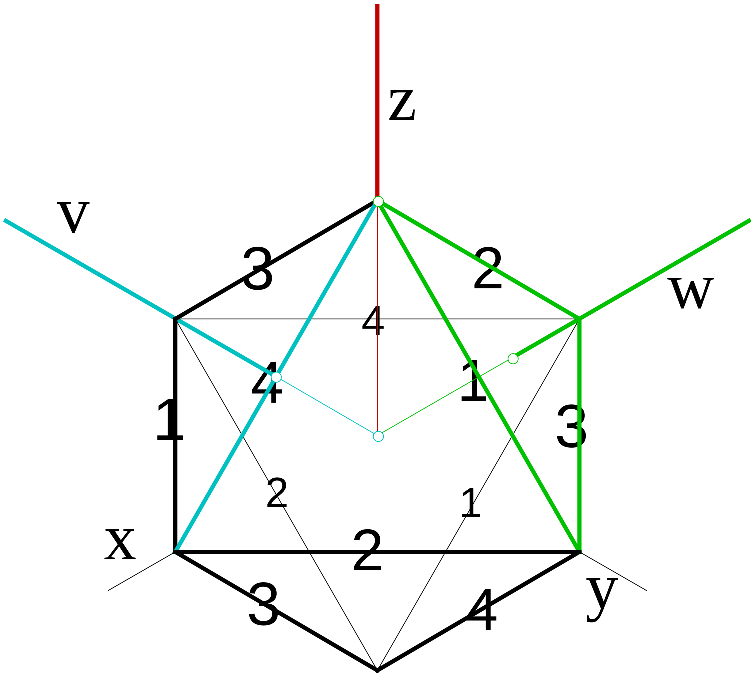

FIGURE 18.

The four rotations around the z-axis, that belong to Ω, can easily be associated with the four elements of the group generated by (1234). The three rotations around the w-axis, that belong to Ω, can easily be associated with the three elements of the group generated by (123). The two rotations around the v-axis, that belong to Ω, can easily be associated with the two elements of the group generated by (23). And so on.

If a solution has 4 vertices of b) type, its group contains also (σ_0, ω) where ω is the reflection (determinant -1) that uses as mirror the equator plane, the plane xy. Its group has order 8. There are $48/8$ such natural solutions, being 48 the order of the plate group.

Remark that these two first are the maximal equivalence classes of the octahedron puzzle. The last one is not maximal.

If a solution has 6 vertices of b) type, its group contains also three elements $((ab), \omega)$, where ω is a rotation of $180°$ around an axis which is orthogonal to the w-axis, and contains the centers of opposite edges. Its group has order 6. There are $48/6$ such natural solutions.

Hence, the number of natural solutions is 16:

$$16 = 48\left(\frac{1}{24} + \frac{1}{8} + \frac{1}{6}\right).$$

For more details see reference [3].

Acknowledgments
The author acknowledges the support of the SPM (the Portuguese Mathematical Society) and, specially, the support of Professor Suzana Metello de Nápoles.

References

[1] Jorge Rezende: *Jogos com poliedros e permutações*. Boletim da SPM, nº 43, 105-124 (2000). `http://gfm.cii.fc.ul.pt/Members/jr_poliedros.pdf`
[2] Jorge Rezende: *Puzzles com poliedros e números*. Edição da SPM, Lisboa (2001). http://www.spm.pt/SPM/lojaSPM.html
[3] Jorge Rezende: *Puzzles with polyhedra and permutation groups*. Pre-print, 2001.

Grupo de Física-Matemática da Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal, e Departamento de Matemática, Faculdade de Ciências da Universidade de Lisboa
E-mail address: `rezende@cii.fc.ul.pt`