PUZZLES WITH POLYHEDRA AND PERMUTATION GROUPS

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1. Introduction

Consider a polyhedron. For example, a platonic, an arquemidean, or a dual of an arquemidean polyhedron. Construct flat polygonal plates in the same number, shape and size as the faces of the referred polyhedron. Adjacent to each side of each plate draw a number like it is shown in figure 1. Some of the plates, or all, can have numbers on both faces. We call these plates, two-faced plates. In this case, they have the same number adjacent to the same side. In figure 1, c and d are the two faces of a pentagonal two-faced plate example.

Now the game is to put the plates over the polyhedron faces in such a way that the two numbers near each polyhedron edge are equal. If there is at least one solution for this puzzle one says that we have a polyhedron puzzle with numbers.

These puzzles are a tool in teaching and learning mathematics, and a source of examples and exercises. They may be used in various fields such as elementary group theory and computational geometry.

It is obvious that not all puzzles are interesting. But, amazingly or not, there are natural ways of constructing interesting puzzles.

From now on, assume that the numbers belong to the set $\{1, 2, \ldots, n\}$, and that all the numbers are used.

If we have plate faces which have the shape of a regular polygon with j sides, one can ask how many possible ways ν are there to draw the numbers $1, 2, \ldots, n$, without repeating them on each plate face. The answer is

$$\nu = (j-1)!\binom{n}{j} = \frac{n!}{(n-j)!j}.$$

For $n = 3$ and $j = 3$ (equilateral triangle), this gives $\nu = 2$.

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For $n = 4$ and $j = 3$, this gives $\nu = 8$. Note that 8 is precisely the number of the octahedron faces. We naturally call the related puzzle, the octahedron puzzle.

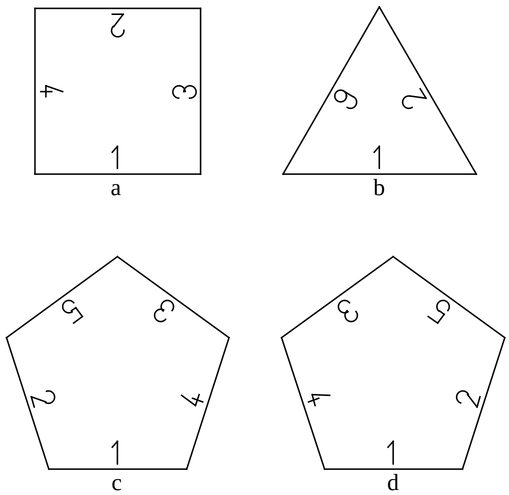

FIGURE 1.

For $n = 5$ and $j = 3$, this gives $\nu = 20$. Note that 20 is precisely the number of the icosahedron faces. We call the related puzzle, the icosahedron first puzzle (or icosahedron (1)).

For $n = 6$ and $j = 3$, this gives $\nu = 40$. Note that 40 is the double of the number of the icosahedron faces. Construct different plates with the numbers written on both faces. This gives 20 plates. We call the related puzzle, the icosahedron second puzzle (or icosahedron (2)).

Consider again $n = 6$ and $j = 3$. Construct different plates with the numbers written only on one face, but in such a way that the numbers grow if we read them, beginning with the minimum, counter clock-wise. See an example in figure 1*b*. This gives 20 plates. We call the related puzzle, the icosahedron third puzzle (or icosahedron (3)).

For $n = 4$ and $j = 4$ (square), this gives $\nu = 6$. Note that 6 is precisely the number of the cube faces. We naturally call the related puzzle, the cube puzzle.

For $n = 5$ and $j = 5$ (regular pentagon), this gives $\nu = 24$. Note that 24 is precisely the double of the number of the dodecahedron faces. Construct different plates with the numbers written on both faces. This gives 12 plates. In figure 1, c and d show the two faces of a plate of this type. We call the related puzzle, the dodecahedron first puzzle (or dodecahedron (1)).

Let again $n = 5$ and $j = 5$. Construct different plates with the numbers written only on one face, but in such a way that the numbers read counter clock-wise, $abcd5$, are such that $abcd$ form an even permutation. This gives 12 plates. In figure 1, c and d show two plates of this type. We call the related puzzle, the dodecahedron second puzzle (or dodecahedron (2)).

Consider $n = 4$. With $j = 3$, one has $\nu = 8$. With $j = 4$, one has $\nu = 6$. Note that 8 is precisely the number of the cuboctahedron triangular faces and 6 is precisely the number of its square faces. This is an example of an interesting puzzle using an arquemidean polyhedron.

Take now a deltoidal icositetrahedron. It has 24 deltoidal faces. If we have 24 plates which have the deltoidal shape the number of possible different ways to draw the numbers $1, 2, 3, 4$, without repeating them on each plate is precisely 24. This an example of an interesting puzzle using a dual of an arquemidean polyhedron.

These are simple examples of polyhedron puzzles with numbers, which are enough in order to understand the following sections. There are, obviously, others. For more examples see Reference [2], which is a development of Reference [1]. Reference [3] is a collection of some of these puzzles paper models.

2. Polyhedron symmetries and permutation groups

Consider a puzzle with numbers $1, 2, \ldots, n$ drawn on the plates. From now on P denotes the set of its plate faces which have numbers drawn, and call it the plate set. S_n denotes the group of all permutations of $\{1, 2, \ldots, n\}$; $\sigma \in S_n$ means that σ is a one-to-one function $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. The identity is σ_0: $\sigma_0(1) = 1, \sigma_0(2) = 2, \ldots, \sigma_0(n) = n$. The alternating group, the S_n subgroup of the even permutations, is denoted by A_n.

We shall use also the group $\{-1, 1\} \times S_n$ denoted by $S_n^{\pm}$. If $\delta_1, \delta_2 \in \{-1, 1\}$ and $\sigma_1, \sigma_2 \in S_n$, then $(\delta_1, \sigma_1)(\delta_2, \sigma_2) = (\delta_1\delta_2, \sigma_1\sigma_2)$. We denote $S_n^+ = \{1\} \times S_n \equiv S_n$.

If Λ is a set, then $|\Lambda|$ denotes its cardinal. Hence, if G is group, $|G|$ denotes its order.

From now on E denotes the set of the polyhedron edges and F denotes the set of the polyhedron faces. A solution of the puzzle defines a function $\varepsilon : E \to \{1, 2, \ldots, n\}$. Denote $\mathcal{E}$ the set of these functions. One can say that $\mathcal{E}$ is the set of the puzzle solutions.

Denote Ω the group of the polyhedron symmetries. Every symmetry $\omega \in \Omega$ induces two bijections, that we shall also denote ω, whenever there is no confusion possible: $\omega : E \to E$ and $\omega : F \to F$. Denote also

$\Omega \equiv \{\omega : E \to E\} \equiv \{\omega : F \to F\}$, the two sets of these functions. One can say that each one of these two sets Ω is the set of the polyhedron symmetries. With the composition of functions each one of these two sets Ω forms a group that is isomorphic to the group of the polyhedron symmetries. If $\omega_1, \omega_2 \in \Omega$, we shall denote $\omega_1\omega_2 \equiv \omega_1 \circ \omega_2$.

If Ω_1 is a subgroup of Ω, then Ω_1 acts naturally on the face set, F: for $\omega \in \Omega_1$ and $\varphi \in F$, one defines the action $\omega\varphi = \omega(\varphi)$.

In the following Ω^+ denotes the subgroup of Ω of the symmetries with determinant 1.

We shall also consider the group $S_n \times \Omega$. If $(\sigma_1, \omega_1), (\sigma_2, \omega_2) \in S_n \times \Omega$, one defines the product $(\sigma_1, \omega_1)(\sigma_2, \omega_2) = (\sigma_1\sigma_2, \omega_2\omega_1)$.

On group theory we follow essentially references [4] and [5].

2.1. **The plate group.** Some S_n subgroups act naturally on P. Let $\pi \in P$ and $\sigma \in S_n$. Assume that $a, b, c, \ldots$ are drawn on π, by this order. Then $\sigma\pi$ is the plate face where the numbers $\sigma(a), \sigma(b), \sigma(c), \ldots$ are drawn replacing $a, b, c, \ldots$.

Let $s \in S_n^{\pm}$ and $\pi \in P$. If $s = (1, \sigma) \equiv \sigma$, then $s\pi = \sigma\pi$. If $s = (-1, \sigma) \equiv \sigma^-$, then $s\pi$ is a reflection of $\sigma\pi$. In this last case, if the numbers $a, b, c, \ldots$ are drawn on π, by this order, then $s\pi$ is a plate face where the numbers $\ldots, \sigma(c), \sigma(b), \sigma(a)$ are drawn by this order.

The plate group, G_P, is the greatest subgroup of $S_n^{\pm}$ that acts on P. If $s \in S_n^{\pm}$ and $s\pi \in P$, for every $\pi \in P$, then $s \in G_P$.

2.2. **The solution group.** Let $\varepsilon : E \to \{1, 2, \ldots, n\}$ be a solution of the puzzle. The group of this solution, G_ε, is a subgroup of $S_n \times \Omega$; $(\sigma, \omega) \in G_\varepsilon$ if and only if

$$\sigma \circ \varepsilon \circ \omega = \varepsilon.$$

Denote Ω_ε the following subgroup of Ω: $\omega \in \Omega_\varepsilon$ if and only if there exists $\sigma \in S_n$ such that $(\sigma, \omega) \in G_\varepsilon$. Note that if $\omega \in \Omega_\varepsilon$ there exists only one $\sigma \in S_n$ such that $(\sigma, \omega) \in G_\varepsilon$. From this one concludes that $\omega \mapsto (\sigma, \omega)$ defines an isomorphism between Ω_ε and G_ε. On the other hand $(\det \omega, \sigma) \in G_P$. This defines $g_\varepsilon : \Omega_\varepsilon \to G_P$, $g_\varepsilon(\omega) = (\det \omega, \sigma)$, which is an homomorphism of groups.

If all the plates are different, as is the case in the given examples and as we shall assume from now on, $(\det \omega, \sigma)$ defines completely ω. Hence, g_ε establishes an isomorphism between Ω_ε and $g_\varepsilon(\Omega_\varepsilon) \subset G_P$. Denote $G_{P\varepsilon} \equiv g_\varepsilon(\Omega_\varepsilon)$. Finally, G_ε and $G_{P\varepsilon}$ are isomorphic. We can identify (σ, ω) with $(\det \omega, \sigma)$, and G_ε with the subgroup $G_{P\varepsilon}$ of G_P.

One can define a function $u_\varepsilon : F \to P$, that associates to every face φ the plate face π that ε puts in φ. Define also $P_\varepsilon = u_\varepsilon(F)$. Note that

$|P_\varepsilon| = |F|$ and that $G_{P\varepsilon}$ acts on P_ε. Then one has

$$(\det \omega, \sigma)\,(u_\varepsilon \circ \omega\,(\varphi)) = u_\varepsilon\,(\varphi)\,,$$

for every $\varphi \in F$.

2.3. **Equivalent solutions.** Let $\varepsilon_1, \varepsilon_2 : E \to \{1, 2, \ldots, n\}$ be solutions of the puzzle. One says that these solutions are equivalent, $\varepsilon_1 \approx \varepsilon_2$, if there are $\omega_1 \in \Omega$ and $\sigma_1 \in S_n$, such that

$$\sigma_1 \circ \varepsilon_1 \circ \omega_1 = \varepsilon_2.$$

Note that $(\det \omega_1, \sigma_1) \in G_P$.

Then,

$$(\sigma, \omega) \mapsto \left(\sigma_1 \sigma \sigma_1^{-1}, \omega_1^{-1} \omega \omega_1\right) = (\sigma_1, \omega_1)\,(\sigma, \omega)\,(\sigma_1, \omega_1)^{-1}$$

defines an isomorphism between G_{ε_1} and G_{ε_2}. As $\det \left(\omega_1^{-1} \omega \omega_1\right) = \det \omega$, one has that

$$s \mapsto \sigma_1 s \sigma_1^{-1}$$

defines an isomorphism between $G_{P\varepsilon_1}$ and $G_{P\varepsilon_2}$.

If $\sigma_1 = \sigma_0$ and $\det \omega_1 = 1$, what distinguishes the solutions ε_1 and ε_2 is only a rotational symmetry. In this case $\varepsilon_1 \circ \omega_1 = \varepsilon_2$ expresses another equivalence relation, $\varepsilon_1 \sim \varepsilon_2$. When we make a puzzle, in practice, we do not recognize the difference between ε_1 and ε_2. We shall say that they represent the same natural solution, an equivalence class of the relation $\sim$.

Let $\varepsilon, \varepsilon_1, \varepsilon_2 \in \mathcal{E}$. As $\varepsilon_1 \sim \varepsilon_2$ and $\varepsilon_1 \approx \varepsilon$, implies $\varepsilon_2 \approx \varepsilon$, one can say that the natural solution represented by ε_1 is equivalent to ε. For $\varepsilon \in \mathcal{E}$, represent by $[\varepsilon]$ the set of natural solutions equivalent to ε.

Choose now $\omega_- \in \Omega$, such that $\det \omega_- = -1$. For $\varepsilon \in \mathcal{E}$ and $s = (\delta, \sigma) \in G_P$, denote $\varepsilon_s = \sigma \circ \varepsilon \circ \omega$, where ω is the identity if $\delta = 1$ and $\omega = \omega_-$ if $\delta = -1$. The set $\{\varepsilon_s : s \in G_P\}$ includes representatives of all natural solutions equivalent to ε. Then

$$|[\varepsilon]| = \frac{|G_P|}{|G_{P\varepsilon}|}.$$

The cardinal of all the natural solutions is then given by

$$\sum_{[\varepsilon]} \frac{|G_P|}{|G_{P\varepsilon}|},$$

where the sum is extended to all different equivalence classes $[\varepsilon]$.

2.4. **Equivalent actions.** Consider two groups H_1 and H_2 that act on two sets K_1 and K_2. For $\rho_j \in H_j$ and $x_j \in K_j$, $\rho_j x_j \, (\in K_j)$ represents the action of ρ_j on x_j. One says that the action of H_1 on K_1 is equivalent to the action of H_2 on K_2 if there exists an isomorphism $\xi : H_1 \to H_2$ and a one-to-one function $\zeta : K_1 \to K_2$, such that for every $\rho \in H_1$ and every $x \in K_1$ one has

$$\xi (\rho) \zeta (x) = \zeta (\rho x) .$$

Note that if the action of H_1 on K_1 is equivalent to the action of H_2 on K_2, and H_3 is a subgroup of H_1, then the action of H_3 on K_1 is equivalent to the action of $H_4 = \xi (H_2)$ on K_2.

Assume that the action of H_1 on K_1 is equivalent to the action of H_2 on K_2. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be the sets of the equivalence classes of such actions. Then there exists a bijection $Z : \mathcal{K}_1 \to \mathcal{K}_2$, such that for every $\rho \in H_1$ and every $X \in \mathcal{K}_1$ one has

$$\xi (\rho) Z (X) = Z (\rho X)$$

and

$$|X| = |Z (X)| .$$

These two conditions are necessary in order to have equivalent actions.

Example 2.1. *If ε_1 and ε_2 are equivalent solutions of the polyhedron puzzle, the actions of $G_{P\varepsilon_1}$ and $G_{P\varepsilon_2}$ on P are equivalent. Let $\xi (s) = \sigma_1 s \sigma_1^{-1}$ and $\zeta (\pi) = (\det \omega_1, \sigma_1) \pi$. Then*

$$\xi (s) \zeta (\pi) = \sigma_1 s \sigma_1^{-1} (\det \omega_1, \sigma_1) \pi = (\det \omega_1, \sigma_1) s\pi = \zeta (s\pi) .$$

Example 2.2. *If ε_1 and ε_2 are equivalent solutions of the polyhedron puzzle, the actions of Ω_{ε_1} and Ω_{ε_2} on F are equivalent. Let $\xi (\omega) = \omega_1 \omega \omega_1^{-1}$ and $\zeta (\varphi) = \omega_1 \varphi$. Then*

$$\xi (\omega) \zeta (\varphi) = \omega_1 \omega \omega_1^{-1} \omega_1 \varphi = \omega_1 \omega \varphi = \zeta (\omega \varphi) .$$

Example 2.3. *If ε is a solution of the polyhedron puzzle, the actions of Ω_ε on F and of $G_{P\varepsilon}$ on $P_\varepsilon = u_\varepsilon (P)$ are equivalent. Let $\xi (\omega) = g_\varepsilon (\omega^{-1}) = (\det \omega, \sigma^{-1})$ and $\zeta (\varphi) = u_\varepsilon (\varphi)$. Then*

$$\xi (\omega) \zeta (\varphi) = \left(\det \omega, \sigma^{-1}\right) u_\varepsilon (\varphi) = (u_\varepsilon \circ \omega (\varphi)) = \zeta (\omega \varphi) .$$

Denote $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ and $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ be the sets of the equivalence classes of such actions. Then u_ε induces a one-to-one function $U : \mathcal{F} \to \mathcal{P}$, $U (F_j) = P_j$, $|F_j| = |P_j|$, and

$$\left(\det \omega, \sigma^{-1}\right) U (F_j) = U (\omega F_j) ,$$

for every $j = 1, 2, \ldots, k$, $\omega \in \Omega_\varepsilon$.

From now on we shall deal only with the action of a subgroup Ω_1 of Ω on F, or the action of a subgroup G_1 of G_P on P. Hence, when we say that two groups H_1 and H_2 are equivalent ($H_1 \approx H_2$), we are talking about these actions.

3. Examples

Some of the examples we give in this section can easily be studied directly. It is the case of the cube, the octahedron and the dodecahedron (2) puzzles.

We give also results for the dodecahedron (1), icosahedron (1) and (3) puzzles. These results were obtained mostly with a computer. Similar results for the icosahedron (2) puzzle are left to the reader.

There are good reasons for presenting results for these two icosahedron puzzles. The icosahedron (1) puzzle is mathematically rich, has a lot of natural solutions (over one million!). On the other hand, the icosahedron (3) puzzle is also very instructive because it has few natural solutions ("only" 2322), which means that it is difficult to do it without the help of a computer.

3.1. **The cube puzzle.** In this case there is only one equivalence class. $G_\varepsilon \approx \Omega^+ \approx S_4$. $|G_\varepsilon| = 24$. $|G_P| = 48$.

If one puts a plate on a face then one has 2 different possibilities for the other plates (2 natural solutions): 48/24.

These two different possibilities and the equivalences $G_\varepsilon \approx \Omega^+ \approx S_4$ can be used to translate the cube symmetries into permutations, in the same way as we shall do later with the icosahedron (1) puzzle and its canonical solution.

3.2. **The octahedron puzzle.** There three equivalence classes. One can distinguish them in the following way. Take a solution. On every vertex of the octahedron, note the numbers that correspond to its four edges. There are two possibilities for a vertex: a) four distinct numbers; b) three distinct numbers, with one of them repeated. In the solution, count the number of vertices where the situation a) happens. They can be 6, 2 or 0, that distinguish the three equivalence classes.

The first class group is of order 24. The second class group is of order 8. The third class group is of order 6. As $|G_P| = 48$, one has that if one puts a plate on a face then one has 16 different possibilities for the other plates (16 natural solutions): $\frac{48}{24} + \frac{48}{8} + \frac{48}{6}$.

Although the cube and the octahedron are dual polyhedra, puzzles are not. Solutions can be dual. The first class of the octahedron puzzle is dual of the cube puzzle class.

3.3. **The dodecahedron (1) puzzle.** This puzzle has three equivalence classes that we can distinguish in the following way. Consider a solution. On every dodecahedron vertex note the numbers that are on the edges around the vertex. There are 20 possibilities, but not all of them belong to the solution. There are some repetitions: 7 or 3. The solutions that have 7 repetitions on the vertices are equivalent. The solutions that have 3 repetitions on the vertices belong to 2 different equivalence classes. One of these classes has the repetitions on opposite vertices. The other has the repetitions on vertices that belong to the same edges.

The first class group is of order 8. The second class group is of order 24. The third class group is of order 12. As $|G_P| = 240$, one has that if one puts a plate on a face then one has 60 different possibilities for the other plates (120 natural solutions): $\frac{1}{2}\left(\frac{240}{8} + \frac{240}{24} + \frac{240}{12}\right)$.

3.4. **The dodecahedron (2) puzzle.** This puzzle has 2 equivalence classes that can be distinguished in the following form. Consider two opposite dodecahedron edges. There are other four that are orthogonal to these two. The six edges are over the faces of a virtual cube where the dodecahedron is inscribed. There are five such cubes. The first equivalence class has the same number associated to the edges that belong to the faces of each cube. The second equivalence class has the same number associated to the edges that belong to the faces of one of the fives cubes.

The first class group is of order 120 and the second class group is of order 24. As $|G_P| = 120$, one has that if one puts a plate on a face then one has 6 different possibilities for the other plates (6 natural solutions): $\frac{120}{120} + \frac{120}{24}$.

3.5. **The icosahedron (1) puzzle.** The icosahedron (1) puzzle has 5592 equivalence classes: 5366 have groups of order 1, 165 have groups of order 2, 36 have groups of order 3, 1 has a group of order 4, 4 have groups of order 5, 10 have groups of order 6, 1 has a group of order 8, 4 have groups of order 10, 2 have groups of order 12, 2 have groups of order 24, 1 has a group of order 120. As $|G_P| = 240$, one has that once one puts a plate over a face there are 1311360 different possibilities (1311360 natural solutions):

$$240\left(5366 + \frac{165}{2} + \frac{36}{3} + \frac{1}{4} + \frac{4}{5} + \frac{10}{6} + \frac{1}{8} + \frac{4}{10} + \frac{2}{12} + \frac{2}{24} + \frac{1}{120}\right).$$

There are two possible natural solutions with a group of order 120. One of them, that we call canonical solution, is dual of the dodecahedron (2) solution with a group of order 120. The group of these two solutions

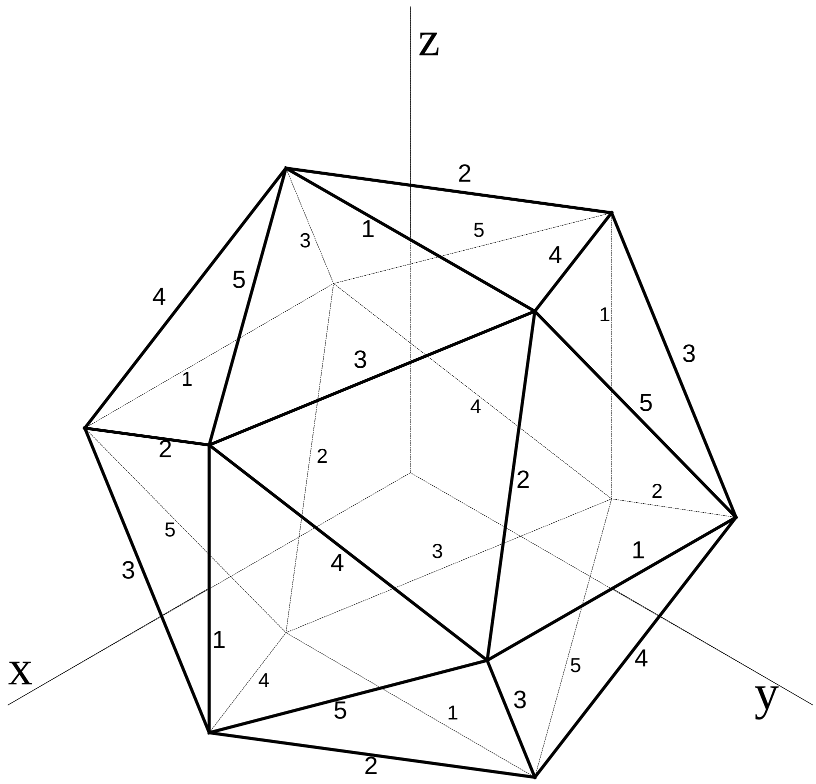

FIGURE 2.

is equivalent to the icosahedron group: $G_\varepsilon \approx \Omega \approx \{-1,1\} \times A_5$. The canonical solution is represented in figure 2.

3.6. **The icosahedron (3) puzzle.** This puzzle has 197 equivalence classes: 190 have groups of order 1, and 7 have groups of order 2. As $|G_P| = 12$, one has that, once one puts a plate over a face there are 2322 different possibilities (2322 natural solutions): $190 \times 12 + 7 \times \frac{12}{2}$.

Note that the actions of all these 7 groups of order 2 are equivalent. Figure 3 shows representatives of all these 7 equivalence classes.

4. More on icosahedron puzzles

As we have already seen there is a natural solution of the icosahedron (1) puzzle which is dual of the dodecahedron (2) natural solution with a group of order 120 (see figure 2). This group is equivalent to the icosahedron group Ω: $G_\varepsilon \approx \Omega \approx \{-1,1\} \times A_5$. As this solution is very easy to construct, one can use it in order to translate in terms of

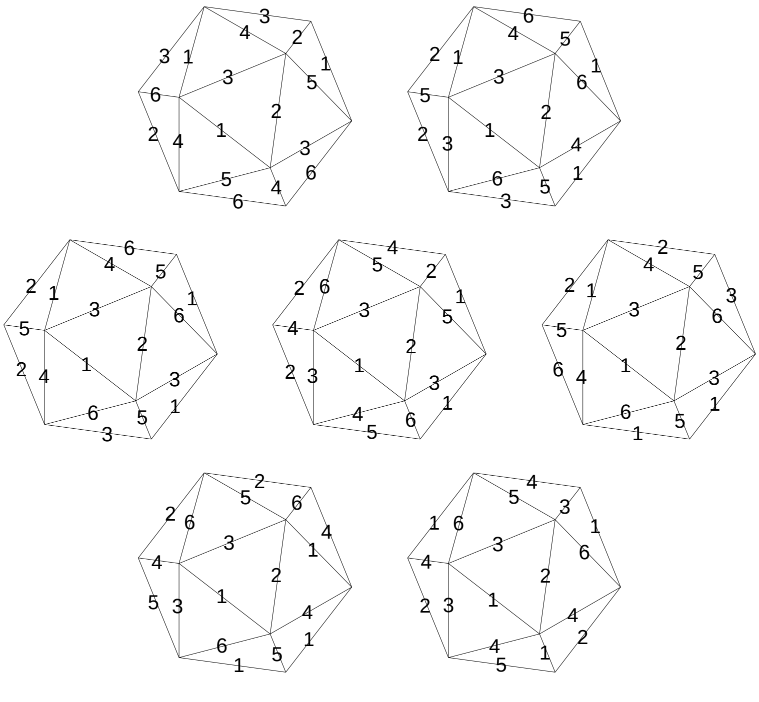

FIGURE 3.

$\{-1,1\} \times A_5$ everything that happens in Ω. For example, all subgroups of Ω are equivalent to the corresponding subgroups of $\{-1,1\} \times A_5$.

4.1. **Subgroups of the icosahedron group.** The icosahedron group Ω has 22 different equivalence classes of subgroups: 1 of order 1 ($\{\sigma_0\}$); 3 of order 2; 1 of order 3; 3 of order 4; 1 of order 5; 3 of order 6; 1 of order 8; 3 of order 10; 2 of order 12; 1 of order 20; 1 of order 24; 1 of order 60; 1 of order 120 ($\Omega \approx \{-1,1\} \times A_5$). In the following $a, b, c, d, e \in \{1,2,3,4,5\}$ are different numbers, and in angle brackets we give the number of equivalence classes of solutions in the icosahedron (1) puzzle.

4.1.1. *Groups of order* 2. The 3 equivalence classes are generated by $(ab)(cd)$ $\langle 148 \rangle$, $(-1,(ab)(cd))$ $\langle 5 \rangle$ and $(-1,\sigma_0)$ $\langle 12 \rangle$. Look at figure 2. The rotation around the z-axis by an angle of $180°$, corresponds to the permutation $(23)(45)$. This rotation belongs to the first equivalence class. The reflection using the xy-plane (orthogonal to the z-axis) as a mirror, corresponds to $(-1,(23)(45))$. This reflection belongs to the second equivalence class. The third equivalence class corresponds to the central symmetry $(x,y,z) \mapsto -(x,y,z)$.

4.1.2. *Groups of order* 3. The unique equivalence class is generated by (abc) $\langle 36 \rangle$. In figure 2, the rotations around the axis $x = y = z$ by angles multiple of 120° correspond to group generated by (234).

4.1.3. *Groups of order* 4. Let $\sigma_1 = (ab)\,(cd)$ and $\sigma_2 = (ac)\,(bd)$. Note that $\sigma_1\sigma_2 = \sigma_2\sigma_1 = (ad)\,(bc)$. The 3 equivalence classes are generated by σ_1 and σ_2 $\langle 0 \rangle$, σ_1 and $(-1, \sigma_0)$ $\langle 1 \rangle$, σ_1 and $(-1, \sigma_2)$ $\langle 0 \rangle$, respectively.

4.1.4. *Groups of order* 5. The unique equivalence class is generated by $(abcde)$ $\langle 4 \rangle$. In figure 2, the rotations around the axis $x = 0$, $z = -\frac{1+\sqrt{5}}{2}y$ (this is the z'-axis in figure 4), by angles multiple of 72°, c orrespond to group generated by (12345).

4.1.5. *Groups of order* 6. Let $\sigma_1 = (ab)\,(cd)$ and $\sigma_2 = (ae)\,(cd)$. The 3 equivalence classes are generated by σ_1 and σ_2 $(\equiv S_3)$ $\langle 3 \rangle$, $(-1, \sigma_1)$ and $(-1, \sigma_2)$ $\langle 0 \rangle$, (abc) and $(-1, \sigma_0)$ $\langle 7 \rangle$, respectively.

4.1.6. *Groups of order* 8. Let $\sigma_1 = (ab)\,(cd)$ and $\sigma_2 = (ac)\,(bd)$. As before, note that $\sigma_1\sigma_2 = \sigma_2\sigma_1 = (ad)\,(bc)$. The unique equivalence class is generated by σ_1 and σ_2 and $(-1, \sigma_0)$ $\langle 1 \rangle$.

4.1.7. *Groups of order* 10. Let $\sigma_1 = (ab)\,(cd)$ and $\sigma_2 = (ac)\,(be)$. The 3 equivalence classes are generated by σ_1 and σ_2 ($\equiv D_5$, the dihedral group of order 10) $\langle 2 \rangle$, $(-1, \sigma_1)$ and $(-1, \sigma_2)$ $\langle 0 \rangle$, $(abcde)$ and $(-1, \sigma_0)$ $\langle 2 \rangle$, respectively.

4.1.8. *Groups of order* 12. Let $\sigma_1 = (ab)\,(cd)$ and $\sigma_2 = (acd)$. The 2 equivalence classes are generated by σ_1 and σ_2 $(\equiv A_4)$ $\langle 2 \rangle$, S_3 and $(-1, \sigma_0)$ $(\equiv \{-1, 1\} \times S_3)$ $\langle 0 \rangle$, respectively.

4.1.9. *Groups of order* 20. The unique equivalence class is generated by D_5 and $(-1, \sigma_0)$ $(\equiv \{-1, 1\} \times D_5)$ $\langle 0 \rangle$.

4.1.10. *Groups of order* 24. The unique equivalence class is generated by A_4 and $(-1, \sigma_0)$ $(\equiv \{-1, 1\} \times A_4)$ $\langle 2 \rangle$.

4.1.11. *Groups of order* 60. The unique equivalence class is A_5 $\langle 0 \rangle$.

4.1.12. *Groups of order* 120. The unique equivalence class is $\{-1, 1\} \times A_5 \approx \Omega$ $\langle 1 \rangle$.

4.2. **Subgroups of plate groups.** All the subgroups of the icosahedron group have cyclic groups of order 2, 3 or 5 as generators. Let us look at the actions on the faces of such cyclic groups.

The first equivalence class of the order 2 groups has 10 orbits with 2 elements and the determinant of the generator is 1: $(1, 10 \times 2)$. The second equivalence class has 8 orbits with 2 elements, 4 orbits with 1 element and the determinant of the generator is -1: $(-1, 8 \times 2 + 4 \times 1)$. The third equivalence class has 10 orbits with 2 elements and the determinant of the generator is -1: $(-1, 10 \times 2)$.

The equivalence class of the order 3 groups has 6 orbits with 3 elements, 2 orbits with 1 element and the determinant of the generator is 1: $(1, 6 \times 3 + 2 \times 1)$.

The equivalence class of the order 5 groups has 4 orbits with 5 elements and the determinant of the generator is 1: $(1, 4 \times 5)$.

4.2.1. *The icosahedron (1) puzzle.* The plate group is $G_P = \{-1, 1\} \times S_5$. The only cyclic groups of G_P whose actions on the plates have orbits of the type $(1, 10 \times 2)$, $(-1, 8 \times 2 + 4 \times 1)$, $(-1, 10 \times 2)$, $(1, 6 \times 3 + 2 \times 1)$ and $(1, 4 \times 5)$ are those generated, precisely, by $(1, (ab)(cd))$, $(-1, (ab)(cd))$, $(-1, \sigma_0)$, $(1, (abc))$ and $(1, (abcde))$.

4.2.2. *The icosahedron (3) puzzle.* Let $\sigma_1 = (123456)$ and $\sigma_2 = (16)(25)(34)$. The plate group, G_P, is generated by $(1, \sigma_1)$ and $(-1, \sigma_2)$. The only cyclic groups of G_P whose actions on the plates have orbits of the type $(1, 10 \times 2)$, $(-1, 8 \times 2 + 4 \times 1)$, $(-1, 10 \times 2)$ and $(1, 6 \times 3 + 2 \times 1)$ are generated by $(1, \sigma_1^3)$, $\left(-1, \sigma_1^j \sigma_2\right)$, $\left(-1, \sigma_1^k \sigma_2\right)$, $(1, \sigma_1^2)$, $j = 1, 3, 5$, $k = 0, 2, 4$. There are no subgroups of order 5. It is easy to see directly that there are no solutions corresponding to the subgroups of order 3. The solutions corresponding to subgroups of order 2, are only those related to the generators $\left(-1, \sigma_1^k \sigma_2\right)$, $k = 0, 2, 4$.

4.3. **Finding solutions of a puzzle with a prescribed group.** A way to find a solution for a puzzle is to use the group relations between the polyhedron and the plate groups. Take, for example, the icosahedron (1) puzzle and the third equivalence class for groups of order 10.

This class is generated by an element σ of order 5 and $(-1, \sigma_0)$. The numbers on the edges must respect the central symmetry and a rotational symmetry group of order 5. Let $\sigma = (12345)$. In figure 4, that illustrates this example, $a, b \in \{1, 2, 3, 4, 5\}$, and $a_j = \sigma^j(a)$, $b_j = \sigma^j(b)$, $j = 1, 2, 3, 4$. The numbers a and b must be chosen so that $a \neq b$, $a \neq \sigma^3(b) = b_3$, $b \neq \sigma^2(a) = a_2$. These three conditions give 9 possibilities, but only 5 of them correspond to solutions of the puzzle: $(a, b) = (2, 1), (2, 5), (3, 4), (4, 2), (4, 3)$. The third one is the canonical

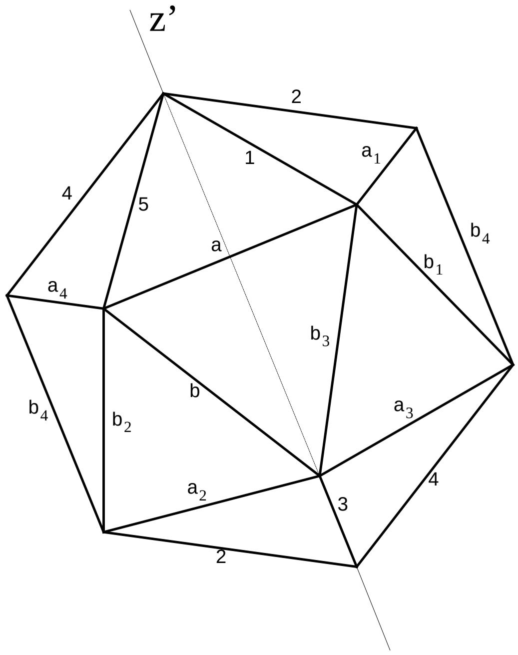

Figure 4.

solution. The first and the fourth ones belong to the same equivalence class. The same happens with the second and the fifth ones. These last two classes are the ones already listed.

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