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A symplectic extension map and a new Shubin class of pseudo-differential operators

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ABSTRACT

For an arbitrary pseudo-differential operator $A : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ with Weyl symbol $a \in \mathcal{S}'(\mathbb{R}^{2n})$, we consider the pseudo-differential operators $\tilde{A} : \mathcal{S}(\mathbb{R}^{n+k}) \rightarrow \mathcal{S}'(\mathbb{R}^{n+k})$ associated with the Weyl symbols $\tilde{a} = (a \otimes 1_{2k}) \circ s$, where $1_{2k}(x) = 1$ for all $x \in \mathbb{R}^{2k}$ and s is a linear symplectomorphism of $\mathbb{R}^{2(n+k)}$. We call the operators \tilde{A} symplectic dimensional extensions of A . In this paper we study the relation between A and \tilde{A} in detail, in particular their regularity, invertibility and spectral properties. We obtain an explicit formula allowing to express the eigenfunctions of \tilde{A} in terms of those of A . We use this formalism to construct new classes of pseudo-differential operators, which are extensions of the Shubin classes $HG_\rho^{m_1, m_0}$ of globally hypoelliptic operators. We show that the operators in the new classes share the invertibility and spectral properties of the operators in $HG_\rho^{m_1, m_0}$ but not the global hypoellipticity property. Finally, we study a few examples of operators that belong to the new classes and which are important in mathematical physics.

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1. Introduction

Every continuous linear operator $A : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ can be written as a pseudo-differential operator

$$A\psi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y)\cdot\xi} a_\tau((1-\tau)x + \tau y, \xi) \psi(y) dy d\xi \tag{1}$$

in terms of its τ -symbol $a_\tau \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ (Shubin [22]); the integral is convergent for $a_\tau \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$, and should otherwise be interpreted in the distributional sense. The usual left, right and Weyl pseudo-differential operators correspond to the cases $\tau = 0, 1$ and $1/2$, respectively.

Let $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ be the space of linear and continuous operators $A : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$. In this paper we will define and study a family of embedding maps $\mathbb{E}_s : \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n)) \rightarrow \mathcal{L}(\mathcal{S}(\mathbb{R}^{n+k}), \mathcal{S}'(\mathbb{R}^{n+k}))$ indexed by the elements s of the symplectic group of $\mathbb{R}^{2(n+k)}$, which we will call *symplectic dimensional extension maps*. In a nutshell, we are interested in these transformations because (i) they generate large classes of important operators $\tilde{A} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^{n+k}), \mathcal{S}'(\mathbb{R}^{n+k}))$ and (ii) many properties of the operators $\mathbb{E}_s[A]$, including the spectral properties, can be completely determined from those of A .

From now on we will write the Weyl symbols simply as $a = a_{1/2}$ and denote the Weyl correspondence by $A \xleftrightarrow{\text{Weyl}} a$ or $a \xleftrightarrow{\text{Weyl}} A$. The spaces \mathbb{R}^{2n} and \mathbb{R}^{2k} are equipped with the standard symplectic forms denoted by σ_n and σ_k , respectively. In $\mathbb{R}^{2(n+k)}$ we have the symplectic form $\sigma_{n+k} = \sigma_n \oplus \sigma_k$.

Since the operators $A \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ are in one-to-one correspondence with their Weyl symbols $a \in \mathcal{S}'(\mathbb{R}^{2n})$, an arbitrary map for operators can be defined in terms of the corresponding map for symbols. We then present our main definition

Definition 1. For arbitrary $k \in \mathbb{N}_0$ define the embedding map

$$E_s : \mathcal{S}'(\mathbb{R}^{2n}) \rightarrow \mathcal{S}'(\mathbb{R}^{2(n+k)}); \quad a \mapsto \tilde{a} = E_s[a] = (a \otimes 1_{2k}) \circ s \tag{2}$$

where $1_{2k} : \mathbb{R}^{2k} \rightarrow \mathbb{R}$ is the trivial function $1_{2k}(y, \eta) = 1$ and s is a linear symplectomorphism in $\mathbb{R}^{2(n+k)}$. A symplectic dimensional extension map \mathbb{E}_s is an embedding map

$$\begin{aligned} \mathbb{E}_s : \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n)) &\rightarrow \mathcal{L}(\mathcal{S}(\mathbb{R}^{n+k}), \mathcal{S}'(\mathbb{R}^{n+k})), \\ A &\mapsto \tilde{A} = \mathbb{E}_s[A] \end{aligned} \tag{3}$$

uniquely defined, for each E_s , by the following commutative diagram

$$\begin{array}{ccc}
 & \text{Weyl} & \\
 a < \text{-----} & & > A \\
 | & & | \\
 E_s & & \mathbb{E}_s \\
 \downarrow & & \downarrow \\
 \tilde{a} < \text{-----} & \text{Weyl} & > \tilde{A}.
 \end{array} \tag{4}$$

Remark 2. In order to write the symbol \tilde{a} (2) more explicitly let us make the identifications

$$\begin{aligned}
 \mathbb{R}^{2n} &= \mathbb{R}_x^n \times \mathbb{R}_\xi^n, & \mathbb{R}^{2k} &= \mathbb{R}_y^k \times \mathbb{R}_\eta^k, \\
 \mathbb{R}^{2(n+k)} &= \mathbb{R}_{x,y}^{(n+k)} \times \mathbb{R}_{\xi,\eta}^{(n+k)}.
 \end{aligned}$$

Then for $s = I$ we have in coordinates $\tilde{a}(x, y; \xi, \eta) = a(x, \xi)$. In the general case

$$s : \mathbb{R}^{2(n+k)} \longrightarrow \mathbb{R}^{2(n+k)}; \quad (x, y; \xi, \eta) \longrightarrow (x', y'; \xi', \eta') = s(x, y; \xi, \eta) \tag{5}$$

and

$$\tilde{a}(x, y; \xi, \eta) = a(x'(x, y; \xi, \eta), \xi'(x, y; \xi, \eta)). \tag{6}$$

The class of operators of the form $\tilde{A} = \mathbb{E}_s[A]$ is quite large. A few examples are:

- All partial differential operators $\tilde{A} = \partial_{x_i}$ on \mathbb{R}^m are symplectic dimensional extensions of the ordinary derivative operator $A = \partial_x$ on \mathbb{R} . In this case $n = 1, k = m - 1, s = I$ and $a(x, \xi) = i\xi$.
- The Landau Hamiltonian [20]

$$\tilde{H}_L = -(\partial_x^2 + \partial_y^2) + i(x\partial_y - y\partial_x) + \frac{1}{4}(x^2 + y^2) \tag{7}$$

which describes the motion of a test particle in the presence of a magnetic field, is a symplectic dimensional extension of the harmonic oscillator Hamiltonian $H_0 = -\partial_x^2 + x^2$ (Section 6.1).

- The Bopp pseudo-differential operators $\tilde{A}_B : \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$, formally $\tilde{A}_B = a \star$ where $a \in \mathcal{S}'(\mathbb{R}^{2n})$ and \star is the Moyal star product, are very important in the deformation quantization of Bayen et al. [3,4,18]. The operators $\tilde{A}_B = a \star$ are symplectic dimensional extensions of the Weyl operators $A \xleftrightarrow{\text{Weyl}} a$ (Section 6.2).

The first part of this paper is devoted to study the relation between the operators $A \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ and their dimensional extensions $\tilde{A} = \mathbb{E}_s[A]$. We prove several general results about the regularity, invertibility and spectral properties of \tilde{A} . In particular, we show that, in the general case, the eigenfunctions of \tilde{A} can be completely determined from those of A .

In the second part of the paper we define new classes of pseudo-differential operators, which are extensions of the Shubin classes of globally hypoelliptic operators. Using the results of the first part, we prove a complete set of results about the spectral, invertibility and hypoellipticity properties of the operators in the new classes. Finally, in Section 6 we present several examples of operators that belong to the new classes and have important applications in quantum mechanics and deformation quantization.

Notation. We will denote by $\mathcal{S}(\mathbb{R}^m)$ the Schwartz space of rapidly decreasing functions on \mathbb{R}^m ; its dual $\mathcal{S}'(\mathbb{R}^m)$ is the space of tempered distributions. The space of linear and continuous operators of the form $\mathcal{S}(\mathbb{R}^m) \rightarrow \mathcal{S}'(\mathbb{R}^m)$ is denoted by $\mathcal{L}(\mathcal{S}(\mathbb{R}^m), \mathcal{S}'(\mathbb{R}^m))$. The scalar product of two functions $f, g \in L^2(\mathbb{R}^m)$ is denoted by $(f|g)$ and the corresponding norm by $\|\cdot\|$. The distributional bracket is $\langle \cdot, \cdot \rangle$. The Euclidean product of two vectors x, y in \mathbb{R}^m is written $x \cdot y$ and the norm of $x \in \mathbb{R}^m$ is $|x|$. The Weyl correspondence is denoted by $A \xleftrightarrow{\text{Weyl}} a$ or $a \xleftrightarrow{\text{Weyl}} A$. The Weyl operators are also written $A = \text{Op}^w(a)$. The symplectic form on \mathbb{R}^{2n} is defined by $\sigma_n(z, z') = Jz \cdot z'$ where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. For $z = (x, \xi)$ and $z' = (x', \xi')$ we have explicitly $\sigma_n(z, z') = \xi \cdot x' - \xi' \cdot x$. The extension to higher dimensions is obviously $\sigma_{n+k} = \sigma_n \oplus \sigma_k$, that is

$$\sigma_{n+k}(z', u'; z'', u'') = \sigma_n(z', z'') + \sigma_k(u', u'')$$

for $(z', u'), (z'', u'') \in \mathbb{R}^{2n} \times \mathbb{R}^{2k}$.

2. Symplectic covariance of Weyl calculus: Review

For details and proofs we refer to Folland [13], de Gosson [15,17], or Wong [23].

2.1. Standard Weyl calculus

In view of Schwartz kernel theorem all operators $A \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ admit the representation

$$A\psi(x) = \langle K_A(x, \cdot), \psi(\cdot) \rangle \tag{8}$$

where $K_A \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ and $\langle \cdot, \cdot \rangle$ is the distributional bracket. The Weyl symbol of A is then

$$a(x, \xi_x) = \int_{\mathbb{R}^n} e^{-i\xi_x \cdot y} K_A\left(x + \frac{1}{2}y, x - \frac{1}{2}y\right) dy \tag{9}$$

with inverse

$$K_A(x, y) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} e^{i\xi \cdot (x-y)} a\left(\frac{x+y}{2}, \xi\right) d\xi. \tag{10}$$

These integrals are well defined for $a \in \mathcal{S}(\mathbb{R}^{2n}) \iff K_A \in \mathcal{S}(\mathbb{R}^{2n})$, and should otherwise be interpreted as generalized Fourier (or inverse Fourier) transforms (i.e. in the sense of distributions).

Substituting (10) in (8) and writing the distributional bracket as an integral we obtain the standard formula for Weyl pseudo-differential operators

$$A\psi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} a\left(\frac{1}{2}(x+y), \xi\right) \psi(y) dy d\xi. \tag{11}$$

2.2. *The metaplectic group*

The symplectic group $\text{Sp}(2n, \mathbb{R})$ of $\mathbb{R}^{2n} \equiv \mathbb{R}^n \times \mathbb{R}^n$ consists of all linear automorphisms s of \mathbb{R}^{2n} such that $\sigma_n(s(z), s(z')) = \sigma_n(z, z')$ for all $z, z' \in \mathbb{R}^{2n}$. The group $\text{Sp}(2n, \mathbb{R})$ is a connected classical Lie group; it has covering groups of all orders. Its double cover admits a faithful (but not irreducible) representation by a group of unitary operators on $L^2(\mathbb{R}^n)$, the metaplectic group $\text{Mp}(2n, \mathbb{R})$. Thus, to every $s \in \text{Sp}(2n, \mathbb{R})$ one can associate two unitary operators S and $-S \in \text{Mp}(2n, \mathbb{R})$. One shows (Leray [21], de Gosson [14,15]) that every element of $\text{Mp}(2n, \mathbb{R})$ is the product of exactly two Fourier integral operators of the type

$$S_{W,m}f(x) = \left(\frac{1}{2\pi i}\right)^{n/2} \Delta_m(W) \int_{\mathbb{R}^n} e^{iW(x,x')} f(x') dx' \tag{12}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$ where

$$W(x, x') = \frac{1}{2}Px \cdot x - Lx \cdot x' + \frac{1}{2}Qx' \cdot x'$$

is a quadratic form with $P = P^T$, $Q = Q^T$ and $\det L \neq 0$, and

$$\Delta_m(W) = i^m \sqrt{|\det L|} \tag{13}$$

where m is an integer (called the Maslov index) corresponding to a choice of $\arg \det L$. The operator $S_{W,m}$ belongs itself to $\text{Mp}(2n, \mathbb{R})$ and corresponds to the element $s \in \text{Sp}(2n, \mathbb{R})$ characterized by the condition

$$(x, \xi) = s(x', \xi') \iff \begin{cases} \xi = \partial_x W(x, x'), \\ \xi' = -\partial_{x'} W(x, x'). \end{cases} \tag{14}$$

It easily follows from the form of the generators (12) of $\text{Mp}(2n, \mathbb{R})$ that metaplectic operators are continuous mappings $\mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ which extend by duality to continuous mappings $\mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$. The inverse of the operator $S_{W,m}$ is given by $S_{W,m}^{-1} = S_{W,m}^* = S_{W^*,m^*}$ where $W^*(x, x') = -W(x', x)$ and $m^* = n - m$.

2.3. Symplectic covariance

The following important property is characteristic of Weyl pseudo-differential calculus:

Proposition 3. *Let $a \in \mathcal{S}'(\mathbb{R}^{2n})$ and let $A = \text{Op}^w(a)$ be the corresponding Weyl pseudo-differential operator. Let $s \in \text{Sp}(2n, \mathbb{R})$. We have*

$$\text{Op}^w(a \circ s) = S^{-1} \text{Op}^w(a)S \tag{15}$$

where S is anyone of the elements of $\text{Mp}(2n, \mathbb{R})$ corresponding to s .

This property has the following essential consequence for the symplectic dimensional extensions. We will denote by \tilde{S} the elements of $\text{Mp}(2(n+k), \mathbb{R})$ because these operators act on the “extended” space $L^2(\mathbb{R}^{n+k})$.

Proposition 4. *Let $s \in \text{Sp}(2(n+k), \mathbb{R})$ and $a \in \mathcal{S}'(\mathbb{R}^{2n})$. Let $\tilde{A}_I = \mathbb{E}_I[A]$ and $\tilde{A}_s = \mathbb{E}_s[A]$ be the corresponding symplectic dimensional extensions of $A = \text{Op}^w(a)$. We have*

$$\tilde{A}_s = \tilde{S}^{-1} \tilde{A}_I \tilde{S} \tag{16}$$

where $\tilde{S} \in \text{Mp}(2(n+k), \mathbb{R})$ is any of the two metaplectic operators that projects onto s .

Proof. In view of the definition of the dimensional extensions $\tilde{A}_I = \mathbb{E}_I[A]$ and $\tilde{A}_s = \mathbb{E}_s[A]$ formula (16) is equivalent to

$$\text{Op}^w[(a \otimes 1_{2k}) \circ s] = \tilde{S}^{-1} \text{Op}^w((a \otimes 1_{2k}))\tilde{S} \tag{17}$$

and the latter is a straightforward consequence of Proposition 3. \square

From this property it is easy to deduce some continuity properties for the operators \tilde{A}_s knowing those of \tilde{A}_I . Suppose for instance that \tilde{A}_I is bounded on $L^2(\mathbb{R}^{n+k})$ then so is \tilde{A}_s since the metaplectic operators $\tilde{S} \in \text{Mp}(2(n+k), \mathbb{R})$ are bounded on $L^2(\mathbb{R}^{n+k})$. It actually suffices in this case to assume that the operator A is bounded on $L^2(\mathbb{R}^n)$ as will follow from the intertwining results of the next section.

3. Properties of the dimensional extensions

3.1. A redefinition of \tilde{A}

We begin by making a few general observations. Assume that $a \in \mathcal{S}(\mathbb{R}^{2n})$, $A \xleftrightarrow{\text{Weyl}} a$ and let $\tilde{A} = \mathbb{E}_I[A]$. Then, for $\Psi \in \mathcal{S}(\mathbb{R}^{n+k})$ we have from (11)

$$\tilde{A}\Psi(x, y) = \left(\frac{1}{2\pi}\right)^{n+k} \int_{\mathbb{R}^{2(n+k)}} e^{i[(x-x')\xi+(y-y')\eta]} a\left(\frac{1}{2}(x+x'), \xi\right) \Psi(x', y') dx' dy' d\xi d\eta \tag{18}$$

where the integral in η is viewed as the inverse Fourier transform of 1; we thus have

$$\tilde{A}\Psi(x, y) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^{2n}} e^{i(x-x')\xi} a\left(\frac{1}{2}(x+x'), \xi\right) \Psi(x', y) dx' d\xi \tag{19}$$

which we can write in compact form as

$$\tilde{A}\Psi(\cdot, y) = A\psi_y \quad \text{with } \psi_y = \Psi(\cdot, y). \tag{20}$$

In particular, if $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^k)$ then

$$\tilde{A}(\psi \otimes \phi) = (A\psi) \otimes \phi. \tag{21}$$

These results can be extended to the more general case $a \in \mathcal{S}'(\mathbb{R}^{2n})$. In view of formula (8) we have

$$\tilde{A}\Psi(z) = \langle K_{\tilde{A}}(z, \cdot), \Psi(\cdot) \rangle \tag{22}$$

where $z = (x, y)$ and $K_{\tilde{A}} \in \mathcal{S}'(\mathbb{R}^{n+k} \times \mathbb{R}^{n+k})$ is given by

$$\begin{aligned} K_{\tilde{A}}(x, y; x', y') &= \left(\frac{1}{2\pi}\right)^{n+k} \int_{\mathbb{R}^{n+k}} e^{i\xi \cdot (x-x') + i\eta(y-y')} a\left(\frac{x+x'}{2}, \xi\right) d\xi d\eta \\ &= K_A(x; x') \delta(y - y') \end{aligned} \tag{23}$$

where the Fourier transform is interpreted in the sense of distributions and δ is the Dirac measure. It follows from (23) that

$$\begin{aligned} \tilde{A}\Psi(x, y) &= \langle K_A(x, \cdot), \Psi(\cdot, y) \rangle \\ &= \langle K_A(x, \cdot), \psi_y(\cdot) \rangle \\ &= A\psi_y(x) \end{aligned} \tag{24}$$

where, once again, $\psi_y = \Psi(\cdot, y)$. In particular, if $\Psi = \psi \otimes \phi$ with $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^k)$, we have again

$$\tilde{A}(\psi \otimes \phi) = (A\psi) \otimes \phi. \tag{25}$$

This intertwining relation will be extended to the general case $\tilde{A} = \mathbb{E}_s[A]$ in the next section.

Another interesting formulation of $\tilde{A} = \mathbb{E}_I[A]$, also valid in the case $a \in \mathcal{S}'(\mathbb{R}^{2n})$, is as follows. For $\Psi, \Phi \in \mathcal{S}(\mathbb{R}^{n+k})$ the cross-Wigner transform is defined by [13,15,17]

$$W(\Psi, \Phi)(x, y; \xi, \eta) = \left(\frac{1}{2\pi}\right)^{n+k} \int_{\mathbb{R}^n \times \mathbb{R}^k} e^{-i(\xi \cdot x' + \eta \cdot y')} \times \Psi\left((x, y) + \frac{1}{2}(x', y')\right) \overline{\Phi\left((x, y) - \frac{1}{2}(x', y')\right)} dx' dy' \tag{26}$$

and formula (18) is equivalent to

$$(\tilde{A}\Psi|\Phi)_{L^2(\mathbb{R}^{n+k})} = \int_{\mathbb{R}^{2(n+k)}} a(x, \xi) W(\Psi, \Phi)(x, y; \xi, \eta) dx dy d\xi d\eta. \tag{27}$$

Note that since $W(\Psi, \Phi) \in \mathcal{S}(\mathbb{R}^{2(n+k)})$ the integral above makes sense not only when $a \in \mathcal{S}(\mathbb{R}^{2n})$, but also when a is measurable and does not increase too fast at infinity. In fact, viewing the integral as a distributional bracket, we can rewrite (27) as

$$\langle \tilde{A}\Psi, \bar{\Phi} \rangle = \langle \tilde{a}, W(\Psi, \Phi) \rangle \tag{28}$$

which makes sense for every $\tilde{a} = E_I[a] \in \mathcal{S}'(\mathbb{R}^{2(n+k)})$. More generally, we can redefine the operator $\tilde{A}_s = \tilde{S}^{-1} \tilde{A} \tilde{S} = \mathbb{E}_s[A]$ by the formula

$$\langle \tilde{A}_s \Psi, \bar{\Phi} \rangle = \langle \tilde{a} \circ s, W(\Psi, \Phi) \rangle \tag{29}$$

or, equivalently, by

$$\langle \tilde{A}_s \Psi, \bar{\Phi} \rangle = \langle \tilde{a}, W(\tilde{S}\Psi, \tilde{S}\Phi) \rangle. \tag{30}$$

3.2. Composition, invertibility and adjoints

We will need the following result:

Lemma 5. *If A is a continuous operator $\mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$, then $\tilde{A} = \mathbb{E}_s[A]$ is a continuous operator $\mathcal{S}(\mathbb{R}^{n+k}) \rightarrow \mathcal{S}(\mathbb{R}^{n+k})$.*

Proof. We first remark that it is sufficient to prove this result for $s = I$. This follows from the conjugation formula (16) in Proposition 4: if $\tilde{A} : \mathcal{S}(\mathbb{R}^{n+k}) \rightarrow \mathcal{S}(\mathbb{R}^{n+k})$ then $\tilde{S}^{-1} \tilde{A} \tilde{S} : \mathcal{S}(\mathbb{R}^{n+k}) \rightarrow \mathcal{S}(\mathbb{R}^{n+k})$ since metaplectic operators preserve the Schwartz classes; the continuity statement follows likewise since metaplectic operators are unitary. Assume that A is a continuous operator $\mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$, and let $\tilde{A} = \mathbb{E}_I[A]$ and $\Psi \in \mathcal{S}(\mathbb{R}^{n+k})$. Then, in view of formula (20), \tilde{A} is also continuous and $\tilde{A}\Psi \in \mathcal{S}(\mathbb{R}^{n+k})$. \square

Recall that if $A \overset{\text{Weyl}}{\longleftrightarrow} a$ then the formal adjoint of A corresponds to the complex conjugate of a , that is we have $A^* \overset{\text{Weyl}}{\longleftrightarrow} \bar{a}$. In particular A is (formally) self-adjoint if and only if its Weyl symbol is real. This property carries over to the case of symplectic dimensionally extended operators:

Proposition 6. *We have*

$$\mathbb{E}_s[A]^* = \mathbb{E}_s[A^*]. \tag{31}$$

Hence, in particular, $\tilde{A} = \mathbb{E}_s[A]$ is (formally) self-adjoint if and only if A is.

Proof. This property immediately follows from the definitions since we have $\mathbb{E}_s[A] \overset{\text{Weyl}}{\longleftrightarrow} (a \otimes 1_{2k}) \circ s$ and noting that the complex conjugate of $(a \otimes 1_{2k}) \circ s$ is $(\bar{a} \otimes 1_{2k}) \circ s$. \square

The proof of the following composition property is slightly more technical; it shows that the extension mapping \mathbb{E}_s preserves products of operators:

Proposition 7. *Let $A \overset{\text{Weyl}}{\longleftrightarrow} a$ and $B \overset{\text{Weyl}}{\longleftrightarrow} b$ and assume that $B : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$. In this case A and B can be composed. Then $\tilde{A} = \mathbb{E}_s[A]$ and $\tilde{B} = \mathbb{E}_s[B]$ can also be composed and we have*

$$\mathbb{E}_s[AB] = \mathbb{E}_s[A]\mathbb{E}_s[B]. \tag{32}$$

Proof. That $\tilde{A}\tilde{B}$ is well-defined follows from Lemma 5. Recall that the Weyl symbol of the product $C = AB$ is given by the formula

$$c(z) = \left(\frac{1}{4\pi}\right)^{2n} \int_{\mathbb{R}^{4n}} e^{\frac{i}{2}\sigma_n(z',z'')} a\left(z + \frac{1}{2}z'\right) b\left(z - \frac{1}{2}z''\right) dz' dz'' \tag{33}$$

(see for instance [17, p. 148]). Let us first assume that s is the identity; then

$$\begin{aligned} \tilde{a}\left(z + \frac{1}{2}z', u + \frac{1}{2}u'\right) &= a\left(z + \frac{1}{2}z'\right), \\ \tilde{b}\left(z - \frac{1}{2}z', u - \frac{1}{2}u'\right) &= b\left(z - \frac{1}{2}z'\right) \end{aligned}$$

and the Weyl symbol \tilde{c} of $\tilde{C} = \tilde{A}\tilde{B}$ is thus given by

$$\begin{aligned} \tilde{c}(z, u) &= \left(\frac{1}{4\pi}\right)^{2(n+k)} \int_{\mathbb{R}^{4(n+k)}} e^{\frac{i}{2}\sigma_{n+k}(z',u';z'',u'')} a\left(z + \frac{1}{2}z'\right) b\left(z - \frac{1}{2}z''\right) dz' dz'' du' du'' \\ &= \left(\frac{1}{4\pi}\right)^{2(n+k)} \int_{\mathbb{R}^{4k}} e^{\frac{i}{2}\sigma_k(u',u'')} du' du'' \end{aligned}$$

$$\begin{aligned} & \times \int_{\mathbb{R}^{4n}} e^{\frac{i}{2}\sigma_n(z',z'')} a\left(z + \frac{1}{2}z'\right) b\left(z - \frac{1}{2}z''\right) dz' dz'' \\ & = \left(\frac{1}{4\pi}\right)^{2n} \int_{\mathbb{R}^{4n}} e^{\frac{i}{2}\sigma_n(z',z'')} a\left(z + \frac{1}{2}z'\right) b\left(z - \frac{1}{2}z''\right) dz' dz'' \end{aligned}$$

where we have used the relation $\sigma_{n+k} = \sigma_n \oplus \sigma_k$ and the Fourier inversion formula to compute the integral in u', u'' ; thus $\tilde{c}(z, u) = c(z)$ which proves our claim in the case $s = I$. The general case follows from formula (16) of Proposition 4. \square

The next result is a corollary of the previous proposition.

Corollary 8. *Let $A^{-1} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ be the inverse of $A : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$, i.e. $AA^{-1} = A^{-1}A = I_n$ where I_n is the identity operator in $\mathcal{S}(\mathbb{R}^n)$. Then $\mathbb{E}_s[A^{-1}]$ is the inverse of $\mathbb{E}_s[A]$, that is*

$$\mathbb{E}_s[A^{-1}] = (\mathbb{E}_s[A])^{-1}. \tag{34}$$

Proof. We first notice that $\mathbb{E}_s[I_n] = I_{n+k}$, which follows directly from (16) and (20). Then from (32)

$$\mathbb{E}_s[A]\mathbb{E}_s[A^{-1}] = \mathbb{E}_s[AA^{-1}] = \mathbb{E}_s[I_n] = I_{n+k} \tag{35}$$

and likewise $\mathbb{E}_s[A^{-1}]\mathbb{E}_s[A] = I_{n+k}$. Hence

$$\mathbb{E}_s[A^{-1}] = (\mathbb{E}_s[A])^{-1}. \quad \square$$

We notice for future reference that the composition and invertibility results extend by duality to $\mathcal{S}'(\mathbb{R}^{n+k})$.

Remark 9. If $A : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ is linear and continuous then it extends by duality to a continuous map that we shall denote by

$$A' : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n). \tag{36}$$

We have, of course, $A'|_{\mathcal{S}(\mathbb{R}^n)} = A$.

In this case, $\tilde{A} = \mathbb{E}_s[A]$ is also a linear and continuous map $\mathcal{S}(\mathbb{R}^{n+k}) \rightarrow \mathcal{S}(\mathbb{R}^{n+k})$ (Lemma 5) and so it also extends to a continuous map

$$\tilde{A}' : \mathcal{S}'(\mathbb{R}^{n+k}) \rightarrow \mathcal{S}'(\mathbb{R}^{n+k}).$$

Since both A' and \tilde{A}' are uniquely defined, we may extend the action of \mathbb{E} to the operators $A' : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ by setting

$$\mathbb{E}_s[A'] := \tilde{A}'. \tag{37}$$

Let B be also a continuous map $\mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$. It then follows trivially from the definition (37) that

$$\mathbb{E}_s[A'B'] = \mathbb{E}_s[A']\mathbb{E}_s[B'] \tag{38}$$

since $A'B' = (AB)'$; and that

$$\mathbb{E}_s[A'^{-1}] = (\mathbb{E}_s[A'])^{-1} \tag{39}$$

since $A'^{-1} = (A^{-1})'$.

We conclude that the three maps $'$, \mathbb{E}_s and $^{-1}$ commute with each other in the space of linear, continuous and invertible operators $\mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$.

3.3. Intertwiners

For $\chi \in \mathcal{S}(\mathbb{R}^k)$ and $\tilde{S} \in \text{Mp}(2(n+k), \mathbb{R})$ let us define the operator $T_{\tilde{S}, \chi} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^{n+k})$ by the formula

$$T_{\tilde{S}, \chi} \psi = \tilde{S}^{-1}(\psi \otimes \chi). \tag{40}$$

These operators $T_{\tilde{S}, \chi}$ have the following analytical properties:

Proposition 10. *The operator $T_{\tilde{S}, \chi}$ extends to a continuous map $\mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^{n+k})$ and, if $\|\chi\| = 1$, to a partial isometry $L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^{n+k})$.*

The adjoint $T_{\tilde{S}, \chi}^ : L^2(\mathbb{R}^{n+k}) \rightarrow L^2(\mathbb{R}^n)$ is given by the formula*

$$T_{\tilde{S}, \chi}^* \Phi(x) = \int_{\mathbb{R}^k} \overline{\chi(y)} \tilde{S} \Phi(x, y) dy \tag{41}$$

and also extends to the continuous operator $\mathcal{S}'(\mathbb{R}^{n+k}) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ defined by

$$\langle T_{\tilde{S}, \chi}^* \Phi, \psi \rangle = \langle \tilde{S} \Phi, \bar{\chi} \otimes \psi \rangle, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^n). \tag{42}$$

Proof. Let $\psi \in \mathcal{S}'(\mathbb{R}^n)$; then $\psi \otimes \chi \in \mathcal{S}'(\mathbb{R}^{n+k})$ and hence $\tilde{S}^{-1}(\psi \otimes \chi) \in \mathcal{S}'(\mathbb{R}^{n+k})$. The formula $T_{\tilde{S}, \chi} \psi = \tilde{S}^{-1}(\psi \otimes \chi)$ defines the desired extension. Assume now $\|\chi\| = 1$ and $\psi \in L^2(\mathbb{R}^n)$; dropping for notational simplicity the references to the spaces in scalar products and norms we have

$$\|T_{\tilde{S}, \chi} \psi\|^2 = \|\tilde{S}^{-1}(\psi \otimes \chi)\|^2 = \|\psi \otimes \chi\|^2$$

since \tilde{S} is unitary. Now $\|\psi \otimes \chi\|^2 = \|\psi\|^2 \|\chi\|^2$ hence $T_{\tilde{S}, \chi}$ is a partial isometry. Its adjoint is defined by $(T_{\tilde{S}, \chi}^* \Phi | \psi) = (\Phi | T_{\tilde{S}, \chi} \psi)$. Formula (41) follows since we have

$$\begin{aligned}
 (\Phi|T_{\tilde{S},\chi}\psi) &= (\Phi|\tilde{S}^{-1}(\psi \otimes \chi)) \\
 &= (\tilde{S}\Phi|\psi \otimes \chi) \\
 &= \int_{\mathbb{R}^n} \overline{\psi(x)} \left[\int_{\mathbb{R}^k} \overline{\chi(y)} \tilde{S}\Phi(x, y) dy \right] dx.
 \end{aligned}$$

Finally, using distributional brackets and the definition of the adjoint, the previous equation can be re-written as

$$\langle T_{\tilde{S},\chi}^* \Phi, \psi \rangle = \langle \tilde{S}\Phi, \bar{\chi} \otimes \psi \rangle, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^n)$$

which defines the distribution $T_{\tilde{S},\chi}^* \Phi \in \mathcal{S}'(\mathbb{R}^n)$ for every $\Phi \in \mathcal{S}'(\mathbb{R}^{n+k})$. \square

The following intertwining relations are essential for studying the spectral properties of the symplectic dimensional extensions:

Proposition 11. *Let $s \in \text{Sp}(2(n+k), \mathbb{R})$ and \tilde{S} be one of the two metaplectic operators that projects onto s . Then*

$$\tilde{A}_s T_{\tilde{S},\chi} = T_{\tilde{S},\chi} A, \quad T_{\tilde{S},\chi}^* \tilde{A}_s = A T_{\tilde{S},\chi}^* \tag{43}$$

for every $A = \text{Op}^w(a)$, $a \in \mathcal{S}'(\mathbb{R}^{2n})$ and $\tilde{A}_s = \mathbb{E}_s[A]$.

Proof. Let us first prove that $\tilde{A}_s T_{\tilde{S},\chi} = T_{\tilde{S},\chi} A$ for the case $\tilde{S} = I$. Let us set $T_\chi = T_{I,\chi}$ and let $\psi \in \mathcal{S}(\mathbb{R}^n)$. In view of formula (24) we have

$$\begin{aligned}
 \tilde{A}_I(T_\chi\psi)(x, y) &= A(T_\chi\psi)_y(x) \\
 &= A\psi(x)\chi(y) \\
 &= T_\chi(A\psi)(x, y).
 \end{aligned}$$

The general case follows using formula (16) since

$$\begin{aligned}
 \tilde{A}_s T_{\tilde{S},\chi} &= \tilde{S}^{-1} \tilde{A}_I \tilde{S} (\tilde{S}^{-1} T_\chi) \\
 &= \tilde{S}^{-1} \tilde{A}_I T_\chi \\
 &= \tilde{S}^{-1} T_\chi A \\
 &= T_{\tilde{S},\chi} A.
 \end{aligned}$$

The second formula (43) is deduced from the first: since $\tilde{A}_s^* T_{\tilde{S},\chi} = T_{\tilde{S},\chi}^* A^*$ (because of the equality (31)), we have $(\tilde{A}_s^* T_{\tilde{S},\chi})^* = (T_{\tilde{S},\chi}^* A^*)^*$ that is $T_{\tilde{S},\chi}^* \tilde{A}_s = A T_{\tilde{S},\chi}^*$. \square

4. Spectral results

4.1. Orthonormal bases and the partial isometries $T_{\tilde{S},\chi}$

In view of Proposition 10 the intertwining operators $T_{\tilde{S},\chi}$ are partial isometries of $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^{n+k})$. Hence the range $\mathcal{H}_{\tilde{S},\chi}$ of $T_{\tilde{S},\chi}$ is a closed subspace of $L^2(\mathbb{R}^{n+k})$. The following result shows how the operators $T_{\tilde{S},\chi}$ generate orthonormal bases of $L^2(\mathbb{R}^{n+k})$.

Proposition 12. *Let $(\phi_j)_j$ be an orthonormal basis of $L^2(\mathbb{R}^n)$ and $(\chi_l)_l$ an orthonormal basis of $L^2(\mathbb{R}^k)$. Then $(T_{\tilde{S},\chi_l} \phi_j)_{j,l}$ is an orthonormal basis of $L^2(\mathbb{R}^{n+k})$ and $L^2(\mathbb{R}^{n+k}) = \bigoplus_l \mathcal{H}_{\tilde{S},\chi_l}$.*

Proof. Let us set $\Phi_{j,l} = T_{\tilde{S},\chi_l} \phi_j$. We have

$$\begin{aligned} (\Phi_{j,l} | \Phi_{j',l'}) &= (\tilde{S}^{-1}(\phi_j \otimes \chi_l) | \tilde{S}^{-1}(\phi_{j'} \otimes \chi_{l'})) \\ &= (\phi_j \otimes \chi_l | \phi_{j'} \otimes \chi_{l'}) \\ &= (\phi_j | \phi_{j'}) (\chi_l | \chi_{l'}) \end{aligned}$$

hence the vectors $\Phi_{j,l}$ form an orthonormal system in $L^2(\mathbb{R}^{n+k})$. Let us prove that this system is complete in $L^2(\mathbb{R}^{n+k})$; it suffices for that to show that if $\Psi \in L^2(\mathbb{R}^{n+k})$ is such that $(\Psi | \Phi_{j,l}) = 0$ for all indices j, l then $\Psi = 0$. Now,

$$(\Psi | \Phi_{j,l}) = (\Psi | \tilde{S}^{-1}(\phi_j \otimes \chi_l)) = (\tilde{S}\Psi | \phi_j \otimes \chi_l)$$

and $(\Psi | \Phi_{j,l}) = 0$ for all j, l implies $\tilde{S}\Psi = 0$ because the tensor product of the two orthonormal bases is an orthonormal basis of $L^2(\mathbb{R}^{n+k})$; it follows that $\Psi = 0$ as claimed. Moreover, for each l the set $(\Phi_{j,l})_j$ is an orthonormal basis of $\mathcal{H}_{\tilde{S},\chi_l}$. Since for $l \neq l'$ we have $\mathcal{H}_{\tilde{S},\chi_l} \cap \mathcal{H}_{\tilde{S},\chi_{l'}} = \{0\}$ it follows that $L^2(\mathbb{R}^{n+k}) = \bigoplus_l \mathcal{H}_{\tilde{S},\chi_l}$. \square

The observant Reader will have noticed that in the first part of the proof we established the equality

$$(T_{\tilde{S},\chi} \phi | T_{\tilde{S},\chi'} \phi')_{L^2(\mathbb{R}^{n+k})} = (\phi | \phi')_{L^2(\mathbb{R}^n)} (\chi | \chi')_{L^2(\mathbb{R}^k)} \tag{44}$$

which is valid for all pairs of functions (ϕ, ϕ') in $L^2(\mathbb{R}^n)$ and (χ, χ') in $L^2(\mathbb{R}^k)$.

4.2. Spectral result

Let us prove the main result of this section:

Proposition 13. *Let $s \in \text{Sp}(2(n+k), \mathbb{R})$, let $A : \mathcal{S}(\mathbb{R}^n) \mapsto \mathcal{S}'(\mathbb{R}^n)$ be a linear continuous operator and let $\tilde{A}_s = \mathbb{E}_s[A]$. Let \tilde{S} be one of the metaplectic operators associated with s . Then:*

- (i) The eigenvalues of A and \tilde{A}_s are the same.
- (ii) If ψ is an eigenfunction of A corresponding to the eigenvalue λ then $\Psi = T_{\tilde{S},\chi}\psi$ is an eigenfunction of \tilde{A}_s corresponding to λ , for every $\chi \in \mathcal{S}(\mathbb{R}^k) \setminus \{0\}$, and we have $\Psi \in \mathcal{S}(\mathbb{R}^{n+k})$.
- (iii) Conversely, if Ψ is an eigenfunction of \tilde{A}_s and $\psi = T_{\tilde{S},\chi}^* \Psi$ is different from zero then ψ is an eigenfunction of A and corresponds to the same eigenvalue.
- (iv) If $(\psi_j)_j$ is an orthonormal basis of eigenfunctions of A and $(\chi_l \in \mathcal{S}(\mathbb{R}^k))_l$ is an orthonormal basis of $L^2(\mathbb{R}^k)$ then $(T_{\tilde{S},\chi_l}\psi_j)_{j,l}$ is a complete set of eigenfunctions of \tilde{A}_s and forms an orthonormal basis of $L^2(\mathbb{R}^{n+k})$.

Proof. That every eigenvalue of A also is an eigenvalue of \tilde{A}_s is clear: if $A\psi = \lambda\psi$ for some $\psi \neq 0$ then

$$\tilde{A}_s(T_{\tilde{S},\chi}\psi) = T_{\tilde{S},\chi}A\psi = \lambda(T_{\tilde{S},\chi}\psi)$$

and $T_{\tilde{S},\chi}\psi \neq 0$ because $T_{\tilde{S},\chi}$ is injective; this shows at the same time that $\Psi = T_{\tilde{S},\chi}\psi$ is an eigenfunction of \tilde{A}_s . Since $T_{\tilde{S},\chi} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^{n+k})$ it is clear that $\Psi \in \mathcal{S}(\mathbb{R}^{n+k})$ which concludes the proof of (ii).

Assume conversely that $\tilde{A}_s\Psi = \lambda\Psi$ and $\Psi \neq 0$. For every χ we have, using the second intertwining relation (43),

$$AT_{\tilde{S},\chi}^*\Psi = T_{\tilde{S},\chi}^*\tilde{A}_s\Psi = \lambda T_{\tilde{S},\chi}^*\Psi$$

hence λ is an eigenvalue of A and $T_{\tilde{S},\chi}^*\Psi$ will be an eigenfunction of A if it is different from zero. This proves (iii).

To conclude the proof of (i) we still have to show that if Ψ is an eigenfunction of \tilde{A}_s then $T_{\tilde{S},\chi}^*\Psi \neq 0$ for some $\chi \in \mathcal{S}(\mathbb{R}^k)$. We note that $T_{\tilde{S},\chi}T_{\tilde{S},\chi}^* = P_{\tilde{S},\chi}$ is the orthogonal projection onto the range $\mathcal{H}_{\tilde{S},\chi}$ of $T_{\tilde{S},\chi}$. Assume that $T_{\tilde{S},\chi}^*\Psi = 0$ for all $\chi \in \mathcal{S}(\mathbb{R}^k)$; then $P_{\tilde{S},\chi}\Psi = 0$ for every $\chi \in \mathcal{S}(\mathbb{R}^k)$, and hence $\Psi = 0$ in view of Proposition 12; but this is not possible since Ψ is an eigenfunction.

Finally, the statement (iv) follows from (ii) and Proposition 12. \square

4.3. Generalized spectral theorem

If A is a continuous linear operator $\mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ then it can be extended to a continuous operator $\mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$. In view of Lemma 5, $\tilde{A}_s = \mathbb{E}_s[A]$ is also a continuous operator $\mathcal{S}(\mathbb{R}^{n+k}) \rightarrow \mathcal{S}(\mathbb{R}^{n+k})$ and thus can also be extended to a continuous operator $\mathcal{S}'(\mathbb{R}^{n+k}) \rightarrow \mathcal{S}'(\mathbb{R}^{n+k})$. In this case Proposition 13 can be extended to a generalized spectral result. Let $(\psi, \lambda) \in \mathcal{S}'(\mathbb{R}^n) \times \mathbb{C}$. We will say that ψ is a generalized eigenvector of A , corresponding to the generalized eigenvalue λ if $\psi \neq 0$ and

$$\langle \psi, \overline{A^*\phi} \rangle = \lambda \langle \psi, \bar{\phi} \rangle \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

We will write this equality formally as $(\psi|A^*\phi) = \lambda(\psi|\phi)$ since both sides coincide with the usual scalar products when ψ is a square integrable function.

Proposition 14. *Assume that A is a continuous linear operator of the form $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ and let $\tilde{A}_s = \mathbb{E}_s[A]$. Then:*

- (i) *The generalized eigenvalues of the operators A and \tilde{A}_s are the same.*
- (ii) *Let ψ be a generalized eigenvector of A . Then, for every $\chi \in \mathcal{S}(\mathbb{R}^k) \setminus \{0\}$ the vector $\Psi = T_{\tilde{S},\chi} \psi$ is a generalized eigenvector of \tilde{A}_s and corresponds to the same generalized eigenvalue.*
- (iii) *Conversely, if Ψ is a generalized eigenvector of \tilde{A}_s and $\psi = T_{\tilde{S},\chi}^* \Psi \neq 0$ then ψ is a generalized eigenvector of A corresponding to the same generalized eigenvalue.*

Proof. It is similar, *mutatis mutandis*, to the proof of Theorem 445, Chapter 19, in [17] (also see [16]). For instance, to prove (ii) one notes that if $(\psi|A^*\phi) = \lambda(\psi|\phi)$ for every $\phi \in \mathcal{S}(\mathbb{R}^n)$ then also $(T_{\tilde{S},\chi} \psi|\tilde{A}_s^* \Phi) = \lambda(T_{\tilde{S},\chi} \psi|\Phi)$ for every $\Phi \in \mathcal{S}(\mathbb{R}^{n+k})$. In fact, using the intertwining property (43),

$$\begin{aligned} (T_{\tilde{S},\chi} \psi|\tilde{A}_s^* \Phi) &= (\psi|T_{\tilde{S},\chi}^* \tilde{A}_s^* \Phi) \\ &= (\psi|(\tilde{A}_s T_{\tilde{S},\chi})^* \Phi) \\ &= (\psi|(T_{\tilde{S},\chi} A)^* \Phi) \\ &= (\psi|A^* T_{\tilde{S},\chi}^* \Phi) \\ &= \lambda(\psi|T_{\tilde{S},\chi}^* \Phi) \end{aligned}$$

hence the claim. \square

5. A new class of pseudo-differential operators

In this section we define a new class of pseudo-differential operators which is an extension of the well-known Shubin class $HG_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ of globally hypoelliptic operators. Moreover, we prove several general results about the spectral, invertibility and hypoellipticity properties of the operators in the new class.

5.1. The Shubin classes of global hypoelliptic operators

A linear operator $A : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is “globally hypoelliptic” if it satisfies

$$\psi \in \mathcal{S}'(\mathbb{R}^n) \quad \text{and} \quad A\psi \in \mathcal{S}(\mathbb{R}^n) \implies \psi \in \mathcal{S}(\mathbb{R}^n). \tag{45}$$

This notion is different with respect to the ordinary C^∞ hypoellipticity (which is a local property) because it incorporates the decay at infinity of the involved functions or distributions.

The Shubin class of globally hypoelliptic operators $HG_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ is a subclass of the Shubin class $G_\rho^{m_1}(\mathbb{R}^{2n})$. It is defined (Shubin [22], de Gosson [17]) as follows:

Definition 15. Let m_0, m_1 , and ρ be real numbers such that $m_0 \leq m_1$ and $0 < \rho \leq 1$. The symbol class $H\Gamma_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ consists of all functions $a \in C^\infty(\mathbb{R}^{2n})$ such that for $|z|$ sufficiently large the following estimates hold:

$$C_0|z|^{m_0} \leq |a(z)| \leq C_1|z|^{m_1} \tag{46}$$

for some $C_0, C_1 > 0$ and, for every $\alpha \in \mathbb{N}^{2n}$ there exists $C_\alpha \geq 0$ such that

$$|\partial_z^\alpha a(z)| \leq C_\alpha |a(z)| |z|^{-\rho|\alpha|}. \tag{47}$$

The Shubin class $HG_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ consists of all operators $A : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ with Weyl symbols $a \in H\Gamma_\rho^{m_1, m_0}(\mathbb{R}^{2n})$.

The main appeal of the classes $HG_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ is the set of general properties which is satisfied by their operators. Moreover, they contain many operators which are important in mathematical physics (a trivial example is the harmonic oscillator Hamiltonian). Let us then state the main properties of the operators in $HG_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ (for proofs see [22, Chapter IV]).

Since $H\Gamma_\rho^{m_1, m_0}(\mathbb{R}^{2n}) \subset \Gamma_\rho^{m_1}(\mathbb{R}^{2n})$, every operator $A \in HG_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ is a continuous operator $\mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ which extends by duality to a continuous operator $\mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$. Moreover,

Proposition 16 (Shubin). Let $A \in HG_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ with $m_0 > 0$. If A is formally self-adjoint, that is if $(\widehat{A}\psi|\phi) = (\psi|\widehat{A}\phi)$ for all test functions $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$, then A is essentially self-adjoint and has discrete spectrum in $L^2(\mathbb{R}^n)$. Moreover there exists an orthonormal basis of eigenfunctions $\phi_j \in \mathcal{S}(\mathbb{R}^n)$ ($j = 1, 2, \dots$) with eigenvalues $\lambda_j \in \mathbb{R}$ such that $\lim_{j \rightarrow \infty} |\lambda_j| = \infty$.

In addition, every $A \in HG_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ is globally hypoelliptic and, if $\text{Ker } A = \text{Ker } A^* = \{0\}$, it is also invertible with inverse $A^{-1} \in HG_\rho^{-m_0, -m_1}(\mathbb{R}^{2n})$.

5.2. The extended Shubin classes $\widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{2n})$

We now define a new class of pseudo-differential operators:

Definition 17. The symbol class $\widetilde{H\Gamma}_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ consists of all functions $\tilde{a} \in C^\infty(\mathbb{R}^{2n})$ for which there is $a \in H\Gamma_\rho^{m_1, m_0}(\mathbb{R}^{2(n-k)})$ (for some $k \in \{0, 1, \dots, n-1\}$) and a symplectic extension map $E_s : \mathcal{S}'(\mathbb{R}^{2(n-k)}) \rightarrow \mathcal{S}'(\mathbb{R}^{2n})$ such that $\tilde{a} = E_s[a]$. The set of pseudo-differential operators with Weyl symbols $\tilde{a} \in \widetilde{H\Gamma}_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ is called an “extended Shubin class” and denoted by $\widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{2n})$.

We notice that, by construction, the symbols in the classes $\widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ satisfy the property

$$\tilde{a} \in \widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{2n}) \iff \tilde{a} \circ s \in \widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{2n}), \quad \forall s \in \text{Sp}(2n, \mathbb{R}).$$

Apart from the trivial case $k = 0$ or the case where $a(z)$ is a polynomial, the symbol $E_s[a]$ does not belong any more to the Shubin classes. Hence the Shubin results concerning hypoellipticity and spectral theory do not apply to the operators in the extended Shubin classes of [Definition 17](#).

In the next section we will present several examples of operators in the classes $\widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{2n})$, which are important for applications in quantum mechanics and deformation quantization. First, let us study the general properties of the operators in $\widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{2n})$.

Lemma 18. *Let $\tilde{A} \in \widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ and let $A \in HG_\rho^{m_1, m_0}(\mathbb{R}^{2(n-k)})$ be the associated operator such that $\tilde{A} = \mathbb{E}_s[A]$. Then $\text{Ker } \tilde{A} = \{0\}$ iff $\text{Ker } A = \{0\}$. (To be clear, both operators are considered as continuous maps of the form $\mathcal{S}(\mathbb{R}^m) \rightarrow \mathcal{S}(\mathbb{R}^m)$, where $m = n$ and $m = n - k$ for \tilde{A} and A , respectively.)*

Proof. Assume that $\text{Ker } A \neq \{0\}$ and let $\psi \in \mathcal{S}(\mathbb{R}^{n-k}) \setminus \{0\}$ be such that $A\psi = 0$. Let $\Psi = T_{\tilde{S}, \chi} \psi = \tilde{S}^{-1} \psi \otimes \chi$, where \tilde{S} is (one of the two) metaplectic operators associated with s , and $\chi \in \mathcal{S}(\mathbb{R}^k) \setminus \{0\}$. Then $\Psi \neq 0$ and in view of [Proposition 11](#)

$$\tilde{A}\Psi = \tilde{A}T_{\tilde{S}, \chi} \psi = T_{\tilde{S}, \chi} A\psi = 0.$$

We conclude that $\text{Ker } \tilde{A} \neq \{0\}$.

Conversely, assume that $\text{Ker } \tilde{A} \neq \{0\}$ and let $\Psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ be such that $\tilde{A}\Psi = 0$. Once again, in view of [Proposition 11](#) we have for $\psi = T_{\tilde{S}, \chi}^* \Psi$

$$A\psi = AT_{\tilde{S}, \chi}^* \Psi = T_{\tilde{S}, \chi}^* \tilde{A}\Psi = 0.$$

It remains to prove that there is always some $\chi \in \mathcal{S}(\mathbb{R}^k)$ such that $\psi = T_{\tilde{S}, \chi}^* \Psi \neq 0$. From [Proposition 12](#) we know that $(T_{\tilde{S}, \chi_l} \phi_j)_{jl}$ is an orthogonal basis of $L^2(\mathbb{R}^n)$ provided $(\phi_j)_j$ and $(\chi_l)_l$ are orthogonal bases of $L^2(\mathbb{R}^{n-k})$ and $L^2(\mathbb{R}^k)$, respectively. In addition, we may choose $\phi_j \in \mathcal{S}(\mathbb{R}^{n-k})$ and $\chi_l \in \mathcal{S}(\mathbb{R}^k)$. It follows that there is always some l, j such that

$$(\Psi | T_{\tilde{S}, \chi_l} \phi_j) = (T_{\tilde{S}, \chi_l}^* \Psi | \phi_j) \neq 0$$

and so $\psi = T_{\tilde{S}, \chi_l}^* \Psi \neq 0$, which concludes the proof. \square

Remark 19. Every operator $A \in HG_\rho^{m_1, m_0}(\mathbb{R}^{2(n-k)})$ extends to a continuous operator of the form $A' : \mathcal{S}'(\mathbb{R}^{n-k}) \rightarrow \mathcal{S}'(\mathbb{R}^{n-k})$. Its kernel, however, satisfies [Shubin [22, p. 187]]

$$\text{Ker}(A'|_{\mathcal{S}'(\mathbb{R}^{n-k})}) = \text{Ker}(A|_{\mathcal{S}(\mathbb{R}^{n-k})}).$$

This property is not shared by the continuous extension $\tilde{A}' : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ of $\tilde{A} = \mathbb{E}_s[A]$. If $\text{Ker } \tilde{A}' \neq \{0\}$ we always have

$$(\text{Ker } \tilde{A}') \cap (\mathcal{S}'(\mathbb{R}^n) \setminus \mathcal{S}(\mathbb{R}^n)) \neq \emptyset.$$

We will prove this relation in Proposition 23 where we will also show that the operators $\tilde{A} \in \widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ are not, in general, globally hypoelliptic.

The spectral properties of the operators in the classes $\widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{2n})$, for $m_0 > 0$, follow from Propositions 13 and 16.

Proposition 20. *Let $\tilde{A} \in \widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ and let $m_0 > 0$. Assume in addition that \tilde{A} is formally self-adjoint. Then:*

- (i) \tilde{A} has discrete spectrum $(\lambda_j)_{j \in \mathbb{N}}$ and $\lim_{j \rightarrow \infty} |\lambda_j| = \infty$;
- (ii) A complete set of eigenfunctions of \tilde{A} is given by $\tilde{\Phi}_{j,l} = T_{\tilde{S}, \chi_l} \phi_j$ where $(\phi_j)_j$ is a complete set of eigenfunctions of the associated operator $A \in HG_\rho^{m_1, m_0}(\mathbb{R}^{2(n-k)})$ (such that $\tilde{A} = \mathbb{E}_s[A]$) and $(\chi_l \in \mathcal{S}(\mathbb{R}^k))_l$ is an orthonormal basis of $L^2(\mathbb{R}^k)$;
- (iii) We have $\tilde{\Phi}_{j,l} \in \mathcal{S}(\mathbb{R}^n)$ and $(\tilde{\Phi}_{j,l})_{j,l}$ form an orthonormal basis of $L^2(\mathbb{R}^n)$.

Proof. From Definition 17, $\tilde{A} \in \widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ iff there is some operator $A \in HG_\rho^{m_1, m_0}(\mathbb{R}^{2(n-k)})$ such that $\tilde{A} = \mathbb{E}_s[A]$ for some extension map \mathbb{E}_s . Recall, also from Proposition 6, that \tilde{A} is formally self-adjoint iff A is also formally self-adjoint. The results of the present theorem are then corollaries of the spectral Proposition 13 after taking into account the spectral properties of the Shubin classes listed in Proposition 16 above. \square

We also have

Proposition 21. *Let $\tilde{A} \in \widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{2n})$. If $\text{Ker } \tilde{A} = \text{Ker } \tilde{A}^* = \{0\}$ then \tilde{A} is invertible with inverse $\tilde{A}^{-1} \in \widetilde{HG}_\rho^{-m_0, -m_1}(\mathbb{R}^{2n})$.*

Proof. If $\tilde{A} \in \widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ then also $\tilde{A}^* \in \widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ and there are some $A, A^* \in HG_\rho^{m_1, m_0}(\mathbb{R}^{2(n-k)})$ such that $\tilde{A} = \mathbb{E}_s[A]$ and $\tilde{A}^* = \mathbb{E}_s[A^*]$. If $\text{Ker } \tilde{A} = \text{Ker } \tilde{A}^* = \{0\}$ then, in view of Lemma 18, we also have $\text{Ker } A = \text{Ker } A^* = \{0\}$. It follows from Shubin [22] that A is invertible with inverse $A^{-1} \in HG_\rho^{-m_0, -m_1}(\mathbb{R}^{2(n-k)})$. From Corollary 8 we conclude that $\mathbb{E}_s[A^{-1}] \in \widetilde{HG}_\rho^{-m_0, -m_1}(\mathbb{R}^{2n})$ is the inverse of \tilde{A} . \square

Contrary to what happens in $HG_\rho^{m_1, m_0}(\mathbb{R}^{2n})$, the operators $\tilde{A} \in \widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ are not in general globally hypoelliptic. A set of sufficient conditions for hypoellipticity in $\widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ is given by

Proposition 22. *If $\tilde{A} \in \widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ and $\text{Ker } \tilde{A} = \text{Ker } \tilde{A}^* = \{0\}$ then the extension $\tilde{A}' : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ is globally hypoelliptic.*

Proof. In view of the previous proposition, $\tilde{A} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ is invertible. Then its extension by duality $\tilde{A}' : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ is also invertible (Remark 9) and satisfies

$$(\tilde{A}')^{-1} = (\tilde{A}^{-1})' \implies (\tilde{A}')^{-1}|_{\mathcal{S}(\mathbb{R}^n)} = \tilde{A}^{-1}.$$

Hence $(\tilde{A}')^{-1}\phi \in \mathcal{S}(\mathbb{R}^n)$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$. It follows that $\tilde{A}' : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ is globally hypoelliptic. In fact,

$$\tilde{A}'\psi = \phi \in \mathcal{S}(\mathbb{R}^n) \implies \psi = (\tilde{A}')^{-1}\phi \in \mathcal{S}(\mathbb{R}^n). \quad \square$$

However,

Proposition 23. *Let $\tilde{A} \in \widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{2n})$. If $\text{Ker } \tilde{A}' \neq \{0\}$ then \tilde{A}' is not globally hypoelliptic.*

Proof. If $(\text{Ker } \tilde{A}') \cap (\mathcal{S}'(\mathbb{R}^n) \setminus \mathcal{S}(\mathbb{R}^n)) \neq \emptyset$ then \tilde{A}' is obviously not global hypoelliptic.

Assume that $\text{Ker } \tilde{A}' \subset \mathcal{S}(\mathbb{R}^n)$. Let $A \in HG_\rho^{m_1, m_0}(\mathbb{R}^{2(n-k)})$ be such that $\tilde{A} = \mathbb{E}_s[A]$. Since $\text{Ker } \tilde{A} = \text{Ker } \tilde{A}' \neq \{0\}$, it follows from Lemma 18 that $\text{Ker } A \neq \{0\}$. Let then $\phi \in \mathcal{S}(\mathbb{R}^{n-k}) \setminus \{0\}$ be such that $A\phi = 0$ and let $\chi \in \mathcal{S}'(\mathbb{R}^k) \setminus \mathcal{S}(\mathbb{R}^k)$. Then $\Phi = T_{\tilde{S}, \chi}\phi \notin \mathcal{S}(\mathbb{R}^n)$ and, using a trivial extension of the first intertwining relation (43) to the case $\chi \notin \mathcal{S}(\mathbb{R}^n)$, we get

$$\tilde{A}'\Phi = \tilde{A}'T_{\tilde{S}, \chi}\phi = T_{\tilde{S}, \chi}A'\phi = 0 \in \mathcal{S}(\mathbb{R}^n).$$

Hence, $\text{Ker } \tilde{A}' \not\subset \mathcal{S}(\mathbb{R}^n)$ and \tilde{A}' is not globally hypoelliptic. \square

6. Examples

In this section we present a few examples of operators that belong to the new classes $\widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ and which play an important role in mathematical physics. We also determine the intertwining operators that generate the complete sets of eigenfunctions for each of these operators. Our results recover the spectral properties that were obtained in some previous studies [16,18,19].

6.1. The Landau Hamiltonian

The Landau Hamiltonian [20]

$$\tilde{H}_L = -(\partial_x^2 + \partial_y^2) + i(x\partial_y - y\partial_x) + \frac{1}{4}(x^2 + y^2)$$

describes the motion of a test particle in the presence of a magnetic field. The spectral properties of this operator have been studied in detail in [16,19]. Here we show that $\tilde{H}_L \in \widetilde{HG}_1^{2,2}(\mathbb{R}^4)$ and so not only its spectral properties, but also its invertibility and hypoellipticity properties, follow directly from Propositions 20, 21 and 22.

Proposition 24. *We have $\tilde{H}_L \in \widetilde{HG}_1^{2,2}(\mathbb{R}^4)$.*

Proof. Recall that the harmonic oscillator Hamiltonian

$$H_0 = -\partial_x^2 + x^2$$

is a Weyl operator with symbol $h_0(x, \xi) = \xi^2 + x^2$. We have, of course, $h_0 \in H\Gamma_1^{2,2}(\mathbb{R}^2)$. The Weyl symbol of the Landau Hamiltonian is

$$\tilde{h}_L(x, y; \xi, \eta) = \xi^2 + \eta^2 - x\eta + y\xi + \frac{1}{4}(x^2 + y^2)$$

and one can easily check that $\tilde{h}_L = E_s[h_0]$ where $E_s : \mathcal{S}'(\mathbb{R}^2) \rightarrow \mathcal{S}'(\mathbb{R}^4)$ is the symplectic dimensional extension map associated with $s \in \text{Sp}(4, \mathbb{R})$

$$s = \begin{pmatrix} \frac{1}{2}I & -D \\ \frac{1}{2}D & I \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{48}$$

Hence, $\tilde{h}_L \in \widetilde{H\Gamma}_1^{2,2}(\mathbb{R}^4)$ and so $\tilde{H}_L \in \widetilde{HG}_1^{2,2}(\mathbb{R}^4)$. \square

Since $\text{Ker } H_0 = \{0\}$, we conclude from Lemma 18 that also $\text{Ker } \tilde{H}_L = \{0\}$. Since \tilde{H}_L is formally self-adjoint, it is also globally hypoelliptic and invertible with inverse $\tilde{H}_L^{-1} \in \widetilde{HG}_1^{-2,-2}(\mathbb{R}^4)$ (Propositions 21 and 22).

Moreover, \tilde{H}_L displays a discrete spectrum (Proposition 20). The intertwining operators (40) that generate the eigenfunctions of \tilde{H}_L from those of H_0 are given by

Proposition 25. *The intertwiners for the Landau Hamiltonian are explicitly*

$$T_{\tilde{S},x} \phi(x, y) = -i \left(\frac{\pi}{2}\right)^{1/2} W(\phi, \tilde{\chi}) \left(\frac{x}{2}, \frac{y}{2}\right) \tag{49}$$

where W is the cross Wigner function

$$W(\phi, \psi)(x, y) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} e^{-iy \cdot x'} \phi\left(x + \frac{1}{2}x'\right) \overline{\psi\left(x - \frac{1}{2}x'\right)} dx' \tag{50}$$

for $n = 1$ and $\hat{\chi}$ is the Fourier transform of χ .

Proof. We have from (40) that

$$T_{\tilde{S}, \chi} \phi = \tilde{S}^{-1}(\phi \otimes \chi)$$

where \tilde{S}^{-1} is one of the two metaplectic operators that projects into the symplectomorphism which is the inverse of $s \in \text{Sp}(4, \mathbb{R})$ given by (48)

$$s^{-1} = \begin{pmatrix} I & D \\ -\frac{1}{2}D & \frac{1}{2}I \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The quadratic form $W(x, y; x', y')$ that generates s^{-1} satisfies (cf. formula (14))

$$(x, y; \xi, \eta) = s^{-1}(x', y'; \xi', \eta') \iff \begin{cases} (\xi, \eta) = (\partial_x W, \partial_y W), \\ (\xi', \eta') = -(\partial_{x'} W, \partial_{y'} W). \end{cases} \tag{51}$$

We have

$$W(x, y; x', y') = -(y \cdot x' + x \cdot y') + x' \cdot y' + \frac{1}{2}x \cdot y.$$

Choosing the Maslov index $m = 0$ in (13) and noticing that $|\det L| = 1$ we get from (12):

$$\tilde{S}^{-1}\Psi(x, y) = \left(\frac{1}{2\pi i}\right) \int_{\mathbb{R}^2} e^{iW(x, y; x', y')} \Psi(x', y') dx' dy'. \tag{52}$$

Setting $\Psi(x, y) = \phi(x) \overline{\hat{\chi}(y)}$ and using

$$\overline{\hat{\chi}(y)} = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{iz \cdot y} \overline{\chi(z)} dz \tag{53}$$

with $n = 1$, we get

$$T_{\tilde{S}, \bar{\chi}} \phi(x, y) = -i \left(\frac{1}{2\pi}\right)^{3/2} \int_{\mathbb{R}^3} e^{iW(x, y; x', y')} e^{iz y'} \phi(x') \overline{\chi(z)} dz dx' dy'.$$

A trivial calculation then shows that

$$\tilde{S}^{-1} \phi \otimes \bar{\chi}(x, y) = \left(\frac{-i}{2}\right) \left(\frac{1}{2\pi}\right)^{1/2} \int_{\mathbb{R}} e^{-i\frac{y}{2} \cdot x'} \phi\left(\frac{1}{2}(x + x')\right) \overline{\chi\left(\frac{1}{2}(x - x')\right)} dx'$$

and so

$$T_{\tilde{S}, \bar{\chi}}\phi(x, y) = -i\left(\frac{\pi}{2}\right)^{1/2} W(\phi, \chi)\left(\frac{x}{2}, \frac{y}{2}\right).$$

Taking into account that $\varphi = \bar{\chi} \implies \chi = \bar{\varphi}$ we easily obtain (49). \square

This result coincides exactly with the formula for the intertwiners that was obtained in [16,19].

6.2. Bopp pseudo-differential operators

Let $a \in \mathcal{S}'(\mathbb{R}^{2n})$ and consider the operator defined by

$$\tilde{A}_B : \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n}); \quad \Psi \longrightarrow \tilde{A}_B\Psi = a \star \Psi \tag{54}$$

where \star is the Moyal star-product

$$a \star b(z) = \left(\frac{1}{4\pi}\right)^{2n} \int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} e^{-\frac{i}{2}\sigma_n(v,u)} a\left(z + \frac{1}{2}u\right) b\left(z - \frac{1}{2}v\right) du dv \tag{55}$$

familiar from deformation quantization [3,4,11,12]. The operators $\tilde{A}_B = a\star$ are usually called “Bopp pseudo-differential operators” [7] (or “Moyal–Weyl pseudo-differential operators” [10]) and are very important in deformation quantization where they were used, for instance, to obtain a precise formulation of the spectral problem [18].

Moreover, the map $a \longrightarrow \tilde{A}_B = a\star$ yields a phase space representation of quantum mechanics [10], called the “Moyal representation”, that finds interesting applications in semiclassical dynamics [6], hybrid dynamics [5] and non-commutative quantum mechanics [1,2,8,9].

In this section we show that for every $a \in H\Gamma_\rho^{m_1, m_0}(\mathbb{R}^{2n})$, the operator $\tilde{A}_B = a\star$ belongs to $\widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{4n})$. Hence its spectral, invertibility and hypoellipticity properties are given by our Propositions 20, 21, 22 and 23.

We start by recalling that:

Lemma 26. *Let $a \in \mathcal{S}'(\mathbb{R}^{2n})$. Then the operator formally $\tilde{A}_B = a\star$ is a Weyl operator $\mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$ with symbol*

$$\tilde{a}_B(x, y; \xi, \eta) = a\left(x - \frac{1}{2}\eta, y + \frac{1}{2}\xi\right). \tag{56}$$

For a proof see [18]. We then have

Proposition 27. *Let $a \in H\Gamma_\rho^{m_1, m_0}(\mathbb{R}^{2n})$. Then $\tilde{A}_B \in \widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{4n})$.*

Proof. The Weyl symbol \tilde{a}_B (56) is given by $\tilde{a}_B = E_s[a]$ where $E_s : \mathcal{S}'(\mathbb{R}^{2n}) \rightarrow \mathcal{S}'(\mathbb{R}^{4n})$ is the symplectic dimensional extension map associated with the symplectomorphism $s \in \text{Sp}(4n, \mathbb{R})$

$$s = \begin{pmatrix} I & -\frac{1}{2}D \\ D & \frac{1}{2}I \end{pmatrix}, \quad D = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{57}$$

Hence, for $a \in H\Gamma_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ we have

$$\tilde{a}_B \in \widetilde{H\Gamma}_\rho^{m_1, m_0}(\mathbb{R}^{4n}) \quad \text{and so} \quad \tilde{A}_B \in \widetilde{HG}_\rho^{m_1, m_0}(\mathbb{R}^{4n}). \quad \square$$

Let us then determine the explicit form of the intertwining operators that generate the eigenfunctions of $\tilde{A}_B = a\star$ from those of $A = \text{Op}^w(a)$.

Proposition 28. *The intertwiners for the Bopp operators are given by*

$$T_{\tilde{S}, \chi} \phi = (2\pi)^{n/2} i^{-n} W(\phi, \bar{\chi}) \tag{58}$$

where W is the cross-Wigner function (50).

Proof. The intertwining operators are given by $T_{\tilde{S}, \chi} \phi = \tilde{S}^{-1} \phi \otimes \chi$ where \tilde{S}^{-1} is (one of the two) metaplectic operators that project into the symplectomorphism $s^{-1} \in \text{Sp}(4n, \mathbb{R})$, inverse of (57).

$$s^{-1} = \begin{pmatrix} \frac{1}{2}I & \frac{1}{2}D \\ -D & I \end{pmatrix}, \quad D = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

From (51) we find the quadratic form that generates s^{-1}

$$W(x, y; x', y') = -2(y \cdot x' + x \cdot y') + x' \cdot y' + 2x \cdot y.$$

Choosing the Maslov index $m = 0$ in (13) and taking into account that $|\det L| = 2^{2n}$ we get from (12)

$$\tilde{S}^{-1} \Psi(x, y) = \left(\frac{1}{\pi i}\right)^n \int_{\mathbb{R}^{2n}} e^{iW(x, y; x', y')} \Psi(x', y') dx' dy'. \tag{59}$$

Setting $\Psi = \phi \otimes \bar{\chi}$ and using (53) we get

$$\begin{aligned} T_{\tilde{S}, \bar{\chi}} \phi(x, y) &= \left(\frac{1}{2\pi}\right)^{n/2} \left(\frac{1}{i\pi}\right)^n \int_{\mathbb{R}^{3n}} e^{iW(x, y; x', y')} e^{iz \cdot y'} \phi(x') \overline{\chi(z)} dz dx' dy' \\ &= \left(\frac{2}{\pi}\right)^{n/2} i^{-n} \int_{\mathbb{R}^n} e^{-iy \cdot (2x' - 2x)} \phi(x') \overline{\chi(2x - x')} dx'. \end{aligned}$$

Making the change of variables $x'' = 2x' - 2x$ and taking into account that $\varphi = \bar{\chi} \implies \chi = \bar{\varphi}$ we easily obtain (58). \square

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