Yang-Mills equations on quantum spheres

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The idea of this talk is

- to present the quantum group $\mathrm{SU}_q(2)$ and the standard Podleś sphere S^2_q as its quantum homogeneous space, and to equip them with a suitable class of differential calculi;

- to introduce Laplacian operators on them through the definition of Hodge dualities on both $\Omega(\mathrm{SU}_q(2))$ and $\Omega(\mathrm{S}^2_q)$;

- to describe their gauge coupling corresponding to a class of Hopf fibration.

- to present the class of monopole solutions.

The classical setting: a classical Hopf bundle

- SU(2) is a matrix Lie group

$$g \;=\; \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}, \qquad (u,v) \in \mathbb{C}^2 :\; \bar{u}u + \bar{v}v = 1 \;\sim\; S^3;$$

with a Lie algebra $\mathfrak{su}(2) \sim \mathbb{R}^3$

$$[L_a, L_b] = \epsilon_{abc} L_c, \qquad\qquad [R_a, R_b] = -\epsilon_{abc} R_c$$

- The group manifold is parallelizable. Left and right vector fields give a *global* basis for the differential calculus.

$$\mathrm{d}\phi = \sum_a (L_a \cdot \phi)\omega^a = \sum_b (R_b \cdot \phi)\eta^b \qquad\qquad \text{on } \phi \in \mathcal{A}(S^3)$$

- The exterior algebra

$$\Omega(S^3) = (\oplus_{k=1}^{N} \Omega^k(S^3), \wedge, \mathrm{d} : \Omega^k(S^3) \to \Omega^{k+1}(S^3), \mathrm{d}^2 = 0)$$

is given by free $\mathcal{A}(S^3)$-bimodules on the basis of left (right) – invariant $\{\omega^a\}$ ($\{\eta^b\}$) 1-forms.

- SU(2) contains a U(1) subgroup

$$\mathrm{U}(1) \ni h = \exp\left[\frac{is}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \begin{pmatrix} e^{is/2} & 0 \\ 0 & e^{-is/2} \end{pmatrix};$$

the quotient of its right principal action $r_h(g) = g \cdot h$ is $\mathrm{SU}(2)/\,\mathrm{U}(1) \sim S^2$.

- The U(1)-action generator is the vertical field L_z of the fibration, a connection gives the horizontal subspaces.

- The exterior algebra over the 2-dim sphere is:

$$\Omega^r(S^2) = \{\phi \in \Omega^r(S^3) \,:\, i_{L_z}\phi = 0; \quad \mathrm{r}_k^*(\phi) = \phi\}$$

- A connection can be given by a suitable projection on $\Omega^1(S^3)$:

$$\Pi(\omega_\pm) = 0 \qquad\qquad \Pi(\omega_z) = \omega_z + A$$

with $A \in \Omega^1(S^2)$.

- $\forall\, n \in \mathbb{Z}$ there is an IRREP of the gauge group U(1),

$$\rho_{(n)} : \mathrm{U}(1) \longrightarrow \mathbb{C}^*, \qquad \rho_{(n)}(e^{i\alpha}) = e^{in\alpha}$$

- Horizontal and ρ_n-equivariant r-forms on S^3 give the line bundles

$$\mathcal{L}_n^{(r)} = \Big\{\phi \in \Omega^r(S^3) : i_{L_z}\phi = 0, \qquad \mathrm{r}_k^*(\phi) = \rho_{(n)}^{-1}(k)\phi \quad \Leftrightarrow \quad L_z(\phi) = -\frac{in}{2}\phi\Big\}.$$

or equivalently sections of the associated bundles (matter fields).

- Covariant derivatives $\nabla : \mathcal{L}_n^{(r)} \longrightarrow \mathcal{L}_n^{(r+1)}$ act as

$$\nabla\phi = \mathrm{d}\phi + \omega \wedge \phi$$

with $\omega = (i\, n/2)(\omega_z + A)$ the connection 1-form.
The monopole connection corresponds to $A = 0$.

- On the Lie algebra $\mathfrak{su}(2) \sim \mathbb{R}^3$ the Cartan-Killing metric is *round* ($S^3 \subset \mathbb{R}^4$):

$$g = L_x \otimes L_x + L_y \otimes L_y + L_z \otimes L_z$$

- The Laplacian obtained via its corresponding Hodge duality operator

$$\star : \Omega^k(S^3) \to \Omega^{3-k}(S^3) \qquad\qquad \star^2 = (-1)^{k(3-k)}(\mathrm{sgn}\, g)$$

is the Casimir operator

$$\Box\, x = \star \mathrm{d} \star \mathrm{d}\, x = (L_x^2 + L_y^2 + L_z^2)\, x,$$

- The induced metric on S^2 is $g = 1/2\,(\omega_+ \otimes \omega_- + \omega_- \otimes \omega_+)$, the corresponding Hodge duality defines the Laplacian

$$\Box f = \frac{1}{2}\,(L_- L_+ + L_+ L_-)\, f$$

- The second order covariant derivative

$$\nabla^2 \phi = F \wedge \phi$$

gives the curvature of the connection; on each associated bundle it is

$$\Omega^2(S^2) \ni F = n\,\omega_- \wedge \omega_+ + \frac{i\,n}{2}\,\mathrm{d}A$$

$$F = F' \qquad \Leftrightarrow \qquad A = A' + i\,\mathrm{d}f$$

(gauge transformations, $f \in \mathcal{A}(S^2)$)

- The Bianchi identity holds

$$D\,F = 0;$$

- The Yang-Mills field equations (the stationary points of the action functional) are

$$D(\star F) = \star J$$

- On the Hopf bundle ($J = 0$)

$$\mathrm{d} \star (\mathrm{d}A) = 0$$

has the non perturbative (Dirac) monopole solution $A = 0$.

Yang-Mills field equations require a rich geometrical setting.

The aim of the talk is then:

- to describe how Yang Mills equations can be introduced in an algebraic setting where groups are *quantum*

- to describe which kind of non perturbative solutions they have in the case of the lowest dimensional Hopf bundle.

- The unifying idea of NCG is to extend some concepts (measure theory, topology, differential geometry) to spaces which are no point sets, and are then studied in terms of algebras and states

- The prototype of this analysis is the G.N.S. theorem.

- If A is a commutative C^*-algebra, then a locally compact Hausdorff space X exists (the Gelfand spectrum of A) with an isometric $*$-isomorphism

$$A \sim C_0(X)$$

- If A is a non commutative C^*-algebra, then a separable Hilbert space $\mathcal{H}$ exists, with again an isometric $*$-isomorphism

$$A \sim B(\mathcal{H})$$

The quantum group $\mathrm{SU}_q(2)$

- In the spirit of Gelfand duality, (following [Wo]) a compact quantum group $G = (A, \Delta)$ is separable unital C^*-algebra with a (dense) coproduct Δ.

- The set of all linear combinations of matrix elements *all finite dimensional unitary IRREPs of* G is a dense Hopf $*$-algebra $\mathcal{A} \subset A$.

- As quantum group $\mathrm{SU}_q(2)$ consider the Hopf $(S, \varepsilon, \Delta, *)$, polynomial *unital* $*$-algebra (with $q \in \mathbb{R}$) generated by

$$U = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}, \qquad \begin{array}{l} ac = qca \quad ac^* = qc^*a \quad cc^* = c^*c \\ a^*a + c^*c = aa^* + q^2cc^* = 1. \end{array}$$

- Given $\mathrm{U}(1) := \mathbb{C}[z, z^*]\big/ < zz^* - 1 >$. The map

$$\pi : \mathrm{SU}_q(2) \longrightarrow \mathrm{U}(1), \qquad \pi \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} := \begin{pmatrix} z & 0 \\ 0 & z^* \end{pmatrix}$$

is a surjective Hopf $*$-algebra homomorphism.

- $\mathrm{U}(1)$ is a quantum subgroup of $\mathrm{SU}_q(2)$ with right coaction:

$$\delta_R := (\mathrm{id} \otimes \pi) \circ \Delta \ : \ \mathrm{SU}_q(2) \longrightarrow \mathrm{SU}_q(2) \otimes \mathrm{U}(1)$$

- The algebra S^2_q of the standard Podleś sphere is given by the *coinvariants* for this coaction

$$\mathrm{S}^2_q = \{ b \in \mathrm{SU}_q(2) : \delta_R(b) = b \otimes 1 \}$$

- This coaction allows to define the line bundles ($n \in \mathbb{Z}$)

$$\mathcal{L}_n := \{ x \in \mathrm{SU}_q(2) \quad : \quad \delta_R(x) = x \otimes z^{-n} \}$$

- This talk will not be about the spectral geometry

$$(\mathcal{A}, \mathcal{H}, D)$$

of these quantum spaces ($\mathcal{A} = \mathrm{SU}_q(2), \mathrm{S}^2_q$) [G,DLSVvS, CP,NT], since their corresponding spectral differential calculus is not manageable.

- Connes' formulation gives a top-down approach to an algebraic formulation of the geometry. A Dirac operator encodes the differential calculus and the metric aspects of a non commutative space. Our approach is bottom-up: the idea is to construct the various aspects of a complete algebraic formalism.

- In order to have consistent differential operators, we shall equip $\mathrm{SU}_q(2)$ (and then S^2_q) with a class of Woronowicz type differential calculi, build the exterior algebras, and then introduce Hodge duality operators.

Differential calculi over $\mathrm{SU}_q(2)$ [Wo]

- The dually paired Hopf universal envelopping algebra to $\mathrm{SU}_q(2)$ is $\mathcal{U}_q(\mathfrak{su}(2)) = \{K^{\pm}, E, F = E^*\}$

$$K^{\pm}E = q^{\pm}EK^{\pm} \qquad K^{\pm}F = q^{\mp}FK^{\pm} \qquad [E, F] = \frac{K^2 - K^{-2}}{q - q^{-1}}$$

- A family of left-covariant 3D $*$-calculi (i.e. $\mathrm{d}(x^*) = (\mathrm{d}x)^*$)

$$\mathrm{d}x = \sum_a (X_a \triangleright x)\omega_a, \qquad X_a \in \mathcal{U}_q(\mathfrak{su}(2)), \quad (a = \pm, z),$$

- A bicovariant $4D_+$ $*$-calculus with $R_a = -S^{-1}(L_a) \in \mathcal{U}_q(\mathfrak{su}(2))$ $(a = \pm, 0, z)$

$$\mathrm{d}x = \sum_a (L_a \triangleright x)\omega_a = \sum_a \omega_a (R_a \triangleright x) = \sum_a \eta_a (x \triangleleft R_a)$$

- Any of these calculi has a braiding $\sigma : (\Omega^1(\mathrm{SU}_q(2)))^{\otimes 2} \circlearrowleft$

$$(\sigma \otimes 1) \circ (1 \otimes \sigma) \circ (\sigma \otimes 1) = (1 \otimes \sigma) \circ (\sigma \otimes 1) \circ (1 \otimes \sigma)$$

which allows to define anti-symmetriser operators

$$A^{(r)} : (\Omega^1(\mathrm{SU}_q(2)))^{\otimes r} \circlearrowleft$$

so that the exterior algebra is

$$\Omega^r(\mathrm{SU}_q(2)) = (\Omega^1(\mathrm{SU}_q(2)))^{\otimes r} / \mathrm{Ker}\, A^{(r)}$$

- The classical braiding is the flip $\sigma = \tau$, with $\sigma^2 = 1$. The properties of the (classical) wedge product originates from this, and one has

$$A^{(r)}_{\tau}\, \omega = r!\, \omega, \qquad\qquad \forall\, \omega \in \Omega^r(S^3)$$

- The (quantum) wedge product depends on σ, and one has a specific spectral decomposition on $\Omega(\mathrm{SU}_q(2))$.

$$A^{(r)}\, \omega = \lambda_{\omega}\, \omega,$$

- Differential calculi on S^2_q are induced via the $\mathrm{U}(1)$-coaction.

and the fibration?

- A formulation of principal bundles where a Hopf algebra $\mathcal{H}$ (the gauge group) *coacts* on a total space algebra $\mathcal{P}$ has been developed by [BM,H,D].

- In order to have a consistent notion of vertical vectors, horizontal forms, and connections, differential calculi on $\mathcal{P}$ and $\mathcal{H}$ must be compatible.

- It is then possible to define sections of associated bundles, which are elements of finite projective modules over the base ${}^{\mathcal{H}}\mathcal{P}$ of the bundle.

For each 3d calculus considered over $\mathrm{SU}_q(2)$ we have

- a 1d calculus on $\mathrm{U}(1)$,
  a 2d exterior algebra over S^2_q.

They are compatible

For the 4d calculus over $\mathrm{SU}_q(2)$ we have

- a 1d calculus on $\mathrm{U}(1)$,
  a 3d exterior algebra over S^2_q.

They also are compatible

so we have connections, and covariant derivative

The missing ingredient is then a suitable Hodge duality on $\Omega(\mathrm{SU}_q(2))$ and $\Omega(\mathrm{S}^2_q)$.

- In the classical setting one starts from a tensor and contractions

$$g = g^{ab} X_a \otimes X_b \qquad\qquad g(\omega^a, \omega^b) = g^{ab}$$

so to define

$$\star : \Omega^k(G) \to \Omega^{N-k}(G) \qquad\qquad \star(\phi) = \frac{1}{k!} g(\phi, \mu)$$

and to prove that

$$\star^2 \propto (-1)^{k(N-k)} \qquad \Leftrightarrow \qquad g^{ab} = g^{ba}$$

$$[\star, *] = 0 \qquad \Leftrightarrow \qquad g^* = g$$

- The degeneracy of $\star^2$ characterizes the symmetry of g.

And in the quantum setting?!

- We start from a tensor

$$g \,=\, g_{ab} X_a \otimes X_b$$

on the quantum tangent space of the calculus on $\mathrm{SU}_q(2)$ and a volume form μ.

- We define a contraction operator

$$T(g,\, \lambda,\, \mu) \,:\, \Omega^r(\mathrm{SU}_q(2)) \,\longrightarrow\, \Omega^{N-r}(\mathrm{SU}_q(2))$$

- Is g symmetric?! Is the contraction operator a Hodge duality?!

- We relate this two questions.

- We define the bilinear g σ-symmetric, provided the square of the contraction operator T has the same degeneracy of the antisymmetrisers $A^{(r)}$.

- We define the bilinear g real, provided the contraction operator T commutes with the $*$-hermitian conjugation:

$$[T, *] = 0$$

- Do non degenerate σ-symmetric and real bilinear g's exist? Yes.

- We then say, T are the Hodge operators corresponding to real and symmetric bilinear form g, with $T^2(1) = \mathrm{sgn}(\det g)$

example: for the $4D$ calculus

- Following this formalism one has

$$(T^{\pm})^2(\omega) = (-1)^{k(4-k)} \left((\operatorname{sgn} g) \frac{\lambda^{\pm}_{\omega}}{\lambda^{\pm}_{\omega^*}} \right) \omega.$$

It appears as a natural generalisation of the classical relation to the quantum setting, where the braidings $\sigma^{\pm}$ associated to the calculus on $\mathrm{SU}_q(2)$ have a non trivial spectrum.

- Moreover:

$$T^{+}T^{-}(\omega) = T^{-}T^{+}(\omega) = (-1)^{k(4-k)} (\operatorname{sgn} g)\, \omega.$$

This relation shows the closest similarity to the classical.

- The Laplacian coming from a specific choice of such a g is the Casimir

$$\begin{aligned} \Box\, x \;&=\; -a\{L_+L_- + q^2 L_- L_+ + (1+q^2) L_z L_z - 2(q^2-1) L_0 L_z\} \triangleright x \\ &=\; a\, C_q \triangleright x, \end{aligned}$$

example: Yang-Mills equations on S^2_q

- Given Hodge dualities on the family of $\Omega(\mathrm{SU}_q(2))$, via a reduction procedure we induce Hodge duality on $\Omega(\mathrm{S}^2_q)$.

- For each Hopf fibration we describe the set of connections on the associated bundles via a connection $\omega = \xi(n)\,(\omega_z + A)$ 1-form; the corresponding curvature is
$$\Omega^2(\mathrm{S}^2_q) \ni F = \mathrm{d}\,\omega + \omega \wedge \omega$$

- Gauge transformations are given by:
$$A' = \frac{1}{\xi(n)}\,(\xi(n)\,e^{-i\,f} A\,e^{i\,f} - \mathrm{d}(e^{-i\,f})\,e^{i\,f})$$

- Non perturbative solutions of the Yang-Mills equations are still given by the Grassmann (monopole) connection $A = 0$.

- Is it possible to write non perturbative solutions of the YM equations on $\mathrm{SU}_q(2)$?!

- Is it possible to understand how to introduce a Chern-Simons action for a field theory on this class of 3D non commutative spaces?

Some references

1. AZ, *Hodge duality operators on left covariant exterior algebras* arXiv:1112.6383;

2. AZ, *A class of hodge duality operators over the quantum $SU(2)$*, J. Geom. Phys. 2012

3. G.Landi, AZ, *Calculi, Hodge operators and Laplacians on a quantum Hopf fibration*, Rev.Math.Phys. 2011;

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