Categorical Parallel-Transport

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Abstract

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There has been considerable interest and activity in the interface between geometry and category theory. In this talk we discuss the notions of categorical bundles and categorical connections on such bundles. We explore several examples, including a class of examples involving bundles of 'decorated' paths over spaces of paths.
This is joint work with Saikat Chatterjee and Amitabha Lahiri.

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Longer term goal: study stochastic parallel-transport in this framework.

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Then we will look at the notion of *categorical principal bundle*, which is a geometric structure that involves two gauge groups.

We will look at a central example: *decorated bundle*.

Following this I will define a *categorical connection* and give examples of such connections for decorated bundles.

Categorical Groups

A *categorical group* is a category $\mathbf{G}$ for which both the object set $\mathrm{Obj}(\mathbf{G})$ and the morphism set $\mathrm{Mor}(\mathbf{G})$ are groups, such that the source and target maps

$$s, t : \mathrm{Mor}(\mathbf{G}) \to \mathrm{Obj}(\mathbf{G})$$

are homomorphisms, and the identity-assigning map

$$\mathrm{Obj}(\mathbf{G}) \to \mathrm{Mor}(\mathbf{G}) : g \mapsto 1_g$$

is a homomorphism.

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Thus we have the map

$$\begin{aligned} \mathrm{Mor}(\mathbf{G}) &\simeq \mathrm{Obj}(\mathbf{G}) \times \ker s \\ f &\mapsto \left(s(f), f 1_{s(f)^{-1}} \right) \end{aligned} \tag{1}$$

Categorical Groups and Semi-direct Products

Conversely, we can reconstruct f from its source and $f1_{s(f)^{-1}}$:

$$f = f1_{s(f)^{-1}} \cdot 1_{s(f)}$$

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$$\begin{aligned} \mathrm{Mor}(\mathbf{G}) &\simeq \mathrm{Obj}(\mathbf{G}) \times \ker s \\ f &\mapsto \left(s(f), f1_{s(f)^{-1}}\right) \end{aligned} \tag{2}$$

is a bijection.

Categorical Groups and Semi-direct Products

Transfering the product operation in the group $\mathrm{Mor}(\mathbf{G})$ to $\mathrm{Obj}(\mathbf{G}) \times \ker s$ makes the latter into a semi-direct product

$$\mathrm{Mor}(\mathbf{G}) \simeq \mathrm{Obj}(\mathbf{G}) \rtimes_\alpha \ker s,$$

for a suitable homomorhism

$$\alpha : \ker s \; to \mathrm{Aut}\big(\mathrm{Obj}(\mathbf{G})\big).$$

Categorical Groups and Crossed Modules

For a categorical group $\mathbf{G}$ let

$$G = \mathrm{Obj}(\mathbf{G}) \quad \text{and} \quad H = \ker s \subset \mathrm{Mor}(\mathbf{G}). \tag{3}$$

Then we have the homomorphism

$$\alpha : G \to \mathrm{Aut}(H)$$

given by

$$\alpha(g) : h \mapsto 1_g h 1_{g^{-1}}$$

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$$\alpha(g) : h \mapsto 1_g h 1_{g^{-1}}$$

and the homomorphism

$$\tau : H \to G : h \mapsto t(h),$$

where $t(h)$ is the target of $h \in \mathrm{Mor}(\mathbf{G})$.

Categorical Lie Groups and Crossed Modules

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If G and H are Lie groups, and the homomorphisms α and τ are smooth then **G** is a *categorical Lie group*.
Equivalently, $\mathrm{Obj}(\mathbf{G})$ and $\mathrm{Mor}(\mathbf{G})$ are Lie groups, the maps s and t, as well as $g \mapsto 1_g$, are smooth.

Actions of Categorical Groups

A *right action* of a categorical group **G** on a category **P** is a functor

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that is a right action both at the level of objects and at the level of morphisms;
thus

$$\begin{aligned} \mathrm{Obj}(\mathbf{P}) \times \mathrm{Obj}(\mathbf{G}) \to \mathrm{Obj}(\mathbf{P}) \\ \mathrm{Mor}(\mathbf{P}) \times \mathrm{Mor}(\mathbf{G}) \to \mathrm{Mor}(\mathbf{P}) \end{aligned} \tag{4}$$

are right actions.

Categorical Principal Bundle

A *categorical principal bundle* is a functor

$$\mathbf{P} \to \mathbf{B} \tag{5}$$

along with a *right action* of a categorical group **G** on **P**:

$$\pi : \mathbf{P} \times \mathbf{G} \to \mathbf{P} \qquad \text{(a functor)},$$

which is free both at the level of objects and at the level of morphisms, and the action maps each fiber transitively into itself (again both on objects and on morphisms).

Categorical Principal Bundle: note

Note that we are not including local triviality in this definition.

Pathspaces

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Next,

$$\mathbf{P}_2(X) = \mathbf{P}_1\big(\mathbf{P}_1(X)\big),$$

a category whose objects are paths in X and morphisms are 'paths of paths' in X.
There are different choices available for formalizing these notions ...

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Let X be a manifold.

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For any box

$$I = \prod_{j=1}^{n} [a_j, b_j] \subset \mathbb{R}^n,$$

identify any map

$$f : I \to X$$

with any translate of f:

$$I + v \to X : t \mapsto f(t - v).$$

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$$I + v \to X : t \mapsto f(t - v).$$

Let

$$\mathrm{Mor}_n(X)$$

be the set of all C^∞ maps $I \to X$, identified up to translates, that are 'appropriately constant' near ∂I.

Pathspace Categories

We take

$$\mathbf{P}_n(X)$$

to be the category whose object set is $\mathrm{Mor}_n(X)$ and whose morphism set is $\mathrm{Mor}_{n+1}(X)$, with source and target being given by

$$\begin{aligned} s(f)(p_1, \ldots, p_n) &= f(p_1, \ldots, p_n, b_{n+1}) \\ t(f)(p_1, \ldots, p_n) &= f(p_1, \ldots, p_n, a_{n+1}) \end{aligned} \tag{6}$$

We define composition in the natural way. (This can also be done 'sideways'.)

Categorical Principal Bundles from Traditional Bundles

Let

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Let

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However, there is no convenient structure group for this.

Categorical Principal Bundles from Traditional Bundles

Instead now take **P** to have objects the points of P and morphisms of the form

$$(\gamma, p, q),$$

where

$$\gamma \in \mathrm{Mor}\big(\mathbf{P}_1(M)\big)$$

with $p \in P$ being the point on P above the source of γ, and q above the target of γ.

Categorical Principal Bundles from Traditional Bundles

Instead now take **P** to have objects the points of P and morphisms of the form

$$(\gamma, p, q),$$

where

$$\gamma \in \mathrm{Mor}\big(\mathbf{P}_1(M)\big)$$

with $p \in P$ being the point on P above the source of γ, and q above the target of γ. This gives us a categorical principal

bundle with structure group $\mathbf{G}_o$, whose object set is G and whose morphisms are $g_1 \to g_2$ for *all* $g_1, g_1 \in G$.

Horizontal Path Bundles

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be a traditional principal G-bundle with a connection form $\overline{A}$.

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$$\mathbf{B} = \mathbf{P}_1(M).$$

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be the category whose objects are the points of P and whose morphisms are $\overline{A}$-horizontal paths in P. (Thus it is a subcategory of $\mathbf{P}_1(P)$.)

Then the discrete categorical group $\mathbf{G}_d$, whose object set is G, acts on the right on $\mathbf{P}_1^{\overline{A}}(P)$ and then

$$\mathbf{P}_1^{\overline{A}}(P) \to \mathbf{P}_1(M)$$

is a principal categorical bundle with structure group $\mathbf{G}_d$.

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Let

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be a principal G-bundle with a connection form $\overline{A}$.

Now consider a categorical group **G** for which

$$\mathrm{Obj}(\mathbf{G}) = G, \qquad \text{and} \qquad \ker s = H,$$

where H is a second Lie group. Thus, as before,

$$\mathrm{Mor}(\mathbf{G}) = G \rtimes_{\alpha} H.$$

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and the morphisms are the pairs

$$(\tilde{\gamma}, h) \in \mathrm{Mor}\big(\mathbf{P}_1^{\overline{A}}(P)\big) \times H,$$

where $\tilde{\gamma}$ comes from any $\overline{A}$-horizontal path in P and h is any element of H.

Decorated Bundle: Source and Target

We define source and targets for $\mathbf{P}_1^{\overline{A}}(P)^{\mathrm{dec}}$ by

$$\begin{aligned} s(\tilde{\gamma}, h) &= s(\tilde{\gamma}) \\ t(\tilde{\gamma}, h) &= t(\tilde{\gamma})\tau(h)^{-1} \end{aligned} \tag{7}$$

where $\tau(h) = t(h) \in G$ is the target of $h \in H \subset \mathrm{Mor}(\mathbf{G})$.

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where $\tau(h) = t(h) \in G$ is the target of $h \in H \subset \mathrm{Mor}(\mathbf{G})$. The identity morphism at p is given by the pairing of the constant path at p with the identity element of H.

Decorated Bundle: Composition

We define composition in $\mathbf{P}_1^{\overline{A}}(P)^{\mathrm{dec}}$ by

$$(\tilde{\gamma}_2, h_2) \circ (\tilde{\gamma}_1, h_1) = (\tilde{\gamma}_2 1_{\tau(h_1)} \circ \tilde{\gamma}_1, h_2 h_1) \tag{8}$$

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Theorem

$\mathbf{P}_1^{\overline{A}}(P)^{\mathrm{dec}}$ *is a category.*

Decorated Bundle: Right Action

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$$(\tilde{\gamma}, h) \cdot (g_1, h_1) = \left(\tilde{\gamma} g_1, \alpha(g_1^{-1})(h_1^{-1} h)\right), \tag{9}$$

where, on the right, $\tilde{\gamma} g_1$ is the usual right-translate of $\tilde{\gamma}$ by g_1.

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Theorem

The formula (9) along with the given right action of G on P form a right action of **G** *on* $\mathbf{P}_1^{\overline{A}}(P)^{\mathrm{dec}}$.

The Decorated Principal Bundle

With all these observations in place we can state our first main result:

Theorem

If $\overline{A}$ is a connection on a principal G-bundle $\pi : P \to M$ and if $\mathbf{G}$ is a categorical Lie group with associated Lie grossed module (G, H, α, τ) then the construction above gives a principal categorical bundle

$$\mathbf{P}_1^{\overline{A}}(P)^{\mathrm{dec}} \to \mathbf{P}_1(M)$$

with structure categorical group $\mathbf{G}$.

The Decorated Principal Bundle: A Comment

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However, the source and target maps, as well as the composition law, in the category $\mathbf{P}_1^{\overline{A}}(P)$ involve the group H, which makes it less likely that it is an associated bundle in a way that would truly separate the roles of H and G.

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However, the source and target maps, as well as the composition law, in the category $\mathbf{P}_1^{\overline{A}}(P)$ involve the group H, which makes it less likely that it is an associated bundle in a way that would truly separate the roles of H and G.

Moreover, the categorical connections we explore below are *not* obtained using only a categorical connection on some principal bundle involving only G.

Categorical Connection

By a *categorical connection* ω on $\pi : \mathbf{P} \to \mathbf{B}$ we mean a rule that associates to each morphism

$$\gamma : x \to y \qquad \text{in } \mathrm{Mor}(\mathbf{P})$$

and each $u \in \pi^{-1}(x)$ a unique morphism, the *horizontal lift*,

$$\tilde{\gamma}_u^{\omega} \in \mathrm{Mor}(\mathbf{P})$$

with source u, such that:

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with source u, such that:

(i)
$$\tilde{\gamma}_{ug}^\omega = \tilde{\gamma}_u^\omega g$$

(ii) the horizontal lift of a composite $\delta \circ \gamma$ is the composite of horizontal lifts (with appropriate sources/initial-points).

Traditional Connections as Categorical Connections

Recall that for a traditional principal G-bundle $\pi : P \to M$ we have the categorical principal bundle

$$\mathbf{P} \to \mathbf{P}_1(M)$$

where the objects of $\mathbf{P}$ are the points of P and a morphism $f : p \to q$ of $\mathbf{P}$ is a triple

$$(\gamma, p, q),$$

where $\gamma \in \mathbf{P}_1(M)$, and $p, q \in P$ lie above $s(\gamma)$ and $t(\gamma)$, respectively.

Traditional Connections as Categorical Connections

A traditional connection on $\pi : P \to M$ provides a horizontal lift path on P initiating at p; denoting its final point by $\tau_\gamma(p)$ we take the "categorical" horizontal lift of γ to be

$$\big(\gamma, p, \quad \underbrace{\tau_\gamma(p)}_{\text{p.t. of p along } \gamma} \quad\big)$$

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$$\left(\gamma, p, \underbrace{\tau_\gamma(p)}_{\text{p.t. of p along } \gamma}\right)$$

Thus a traditional connection on $\pi : P \to M$ gives a categorical connection.

Categorical Connections on Decorated Bundles

Let $\overline{A}$ be a connection form on a traditional principal G-bundle $\pi : P \to M$, and **G** a categorical Lie group with associated Lie crossed module $(G, H, \ldots)$.

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If we are given

$$\text{an } H\text{-valued 1-form } C \text{ on } P$$

with a suitable equivariance property then we obtain

$$h^* : \mathrm{Mor}\big(\mathbf{P}_1^{\overline{A}}(P)\big) \to H : \tilde{\gamma} \mapsto \text{“PathOrdered} \exp(-\int_{\tilde{\gamma}} C)\text{”}$$

Categorical Connections on Decorated Bundles

Now let us define the horizontal lift map

$$\begin{aligned} \mathrm{Mor}(\mathbf{B}) &\to \mathrm{Mor}\Big(\mathbf{P}_1^{\overline{A}}(P)^{\mathrm{dec}}\Big) \\ \gamma &\mapsto \Big(\tilde{\gamma}_u, h^*(\tilde{\gamma}_u)\Big) \end{aligned} \tag{10}$$

where $\tilde{\gamma}_u$ is the $\overline{A}$-horizontal lift of γ with initial point $u \in P$.

Categorical Connections on Decorated Bundles

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where $\tilde{\gamma}_u$ is the $\overline{A}$-horizontal lift of γ with initial point $u \in P$.

Theorem

The horizontal lifting procedure (10) is a categorical connection on the categorical bundle

$$\mathbf{P}_1^{\overline{A}}(P)^{\mathrm{dec}} \to \mathbf{P}_1(M).$$

Decorated Bundles over Pathspaces

As usual we start with a traditional connection form $\overline{A}$ on a traditional principal G-bundle $\pi : P \to M$, and a Lie crossed module

$$(G, H, \alpha, \tau),$$

corresponding to a categorical Lie group $\mathbf{G}_1$.

Decorated Bundles over Pathspaces

Using a second connection form A on P and an $\mathrm{Lie}(H)$-valued 2-form B on P (with suitable properties) we have the 1-form on the space of paths on P given by

$$\omega_{(A,B)}(\tilde{v}) = A\big(\tilde{v}(t_0)\big) + \tau\left[\int_{t_0}^{t_1} B\left(\tilde{\gamma}'(t), \tilde{v}(t)\right)\, dt\right] \tag{11}$$

where $\tilde{v}$ is a vector field along the path $\tilde{\gamma} : [t_0, t_1] \to P$.

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where $\tilde{v}$ is a vector field along the path $\tilde{\gamma} : [t_0, t_1] \to P$.

This form specifies a connection on the bundle

$$\mathbf{P}_2^{\overline{A}}(P) \to \mathbf{P}_2(M),$$

where on the left we only take $\overline{A}$-horizontal paths on P as objects, and the structure categorical group is discrete with object set G.

Decorated Bundles over Pathspaces

Now suppose

$$\mathbf{G}_2$$

is a categorical group whose object group is the morphism group of $\mathbf{G}_1$:

$$\mathrm{Obj}(\mathbf{G}_2) = \mathrm{Mor}(\mathbf{G}_1),$$

and let K be the subgroup of $\mathrm{Mor}(\mathbf{G}_2)$ consisting of the morphisms with source being the identity.

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and let K be the subgroup of $\mathrm{Mor}(\mathbf{G}_2)$ consisting of the morphisms with source being the identity.

Using the forms $\overline{A}$, A, and B on P, we can construct a doubly decorated principal bundle

$$\mathbf{P}_2(P)^{\mathrm{dec,dec}} \to \mathbf{P}_2(M)$$

with structure categorical group $\mathbf{G}_2$.

Decorated Bundles over Pathspaces

The categorical bundle

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bears a nice relationship to

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$$\begin{aligned} \mathrm{Obj}\big(\mathbf{P}_2(M)\big) &= \mathrm{Mor}\big(\mathbf{P}_1(M)\big) \\ \mathrm{Obj}(\mathbf{G}_2) &= \mathrm{Mor}(\mathbf{G}_1) \\ \mathrm{Obj}\big(\mathbf{P}_2(P)^{\mathrm{dec,dec}}\big) &= \mathrm{Mor}\big(\mathbf{P}_1^{\overline{A}}(P)^{\mathrm{dec}}\big) \end{aligned} \tag{12}$$

Decorated Bundles over Pathspaces

The categorical bundle

$$\mathbf{P}_2(P)^{\mathrm{dec,dec}} \to \mathbf{P}_2(M)$$

bears a nice relationship to

$$\mathbf{P}_1^{\overline{A}}(P)^{\mathrm{dec}} \to \mathbf{P}_1(M).$$

$$\begin{aligned} \mathrm{Obj}\big(\mathbf{P}_2(M)\big) &= \mathrm{Mor}\big(\mathbf{P}_1(M)\big) \\ \mathrm{Obj}\big(\mathbf{P}_2(P)^{\mathrm{dec,dec}}\big) &= \mathrm{Mor}\big(\mathbf{P}_1^{\overline{A}}(P)^{\mathrm{dec}}\big) \end{aligned} \tag{12}$$

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The categorical bundle

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$$\begin{aligned}
\mathrm{Obj}\big(\mathbf{P}_2(M)\big) &= \mathrm{Mor}\big(\mathbf{P}_1(M)\big) \\
\mathrm{Obj}(\mathbf{G}_2) &= \mathrm{Mor}(\mathbf{G}_1) \\
\mathrm{Obj}\big(\mathbf{P}_2(P)^{\mathrm{dec,dec}}\big) &= \mathrm{Mor}\big(\mathbf{P}_1^{\overline{A}}(P)^{\mathrm{dec}}\big)
\end{aligned} \tag{12}$$

Decorated Bundles over Pathspaces

Using a $\mathrm{Lie}(K)$-valued 2-form C_2 on P, we can construct a categorical connection on

$$\mathbf{P}_2(P)^{\mathrm{dec,dec}} \to \mathbf{P}_2(M)$$

by applying the same general procedure that was used for $\mathbf{P}^{\mathrm{dec}}_{\overline{A}}$.

Some comparisons

The approach to categorical bundles and connections we have presented here differs from other approaches:

- Our base and bundle categories do not have any smooth or even topological structures.
- We do not use any trivializations.
- Our connections are defined through parallel-transport and not through forms. Of course, suitable forms of various degrees would give rise to connections in our sense.
- We do not impose any flatness condition in the definition of the categorical connection. There might be several different such conditions of interest.

Link to paper

This talk is based on the paper
`http://arxiv.org/pdf/1207.5488.pdf`
with a more frequently updated version at
`https://www.math.lsu.edu/~sengupta/papers/CLS2geom.pdf`

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