Outline

Contents

1 Motivation

Heuristic path integrals give single geometric picture for otherwise disparate results, including:

- Donaldson theory from 4D topological Yang-Mills
- Topology of moduli spaces of flat connections in 3D (Casson invariant) from supersymmetric 3D Yang-Mills
- Topology of moduli spaces of flat connections in 2D (intersection pairings) from 2D Yang-Mills
- Gauss-Bonnet-Chern, Atiyah-Singer and Hirzebruch index theorems from supersymmetric quantum mechanics (SUSYQM)

In each, a path integral localizes to compute a topological invariant of a finite-dimensional space.

2 Path integral "proof" of the Gauss-Bonnet-Chern theorem

"Facts" about path integrals (imaginary time) Fact 1 – Path integrals represent operator kernels

Given

- an operator H acting on a space of functions,
- with K_H the integral kernel of the corresponding heat operator e^{-tH},

$$K_H(x, y; t) = \int_{\substack{\sigma(0)=y \\ \sigma(t)=x}} e^{-S(\sigma,t)} \mathcal{D}\sigma.$$

- $S = \int_0^t L(\sigma; s)\, ds$ is the classical action,

- L is the classical Lagrangian associated to H via the Legendre transformation, and
- the integral is taken over all paths σ satisfying the given endpoint conditions.

"Facts" about path integrals Fact 2 – Steepest descent approximation

As t goes to zero (from above),

$$\int_{\substack{\sigma(0)=y\\ \sigma(t)=x}} e^{-S(\sigma,t)}\mathcal{D}\sigma = e^{-S(\sigma_{\mathrm{cl}},t)} \left(\frac{1}{\pi}\right)^{1/2} \int_{\substack{\bar\sigma(0)=0\\ \bar\sigma(t)=0}} e^{-S_Q(\bar\sigma,t)}\mathcal{D}\bar\sigma \; [1+\mathcal{O}(t)]$$

- $\sigma_{\mathrm{cl}}(x,y,t)$ solves the Euler-Lagrange equations for L.
- S_Q is the quadratic term in the expansion of S about this solution.

"Facts" about path integrals Fact 3 – Quadratic actions are nice

For example

$$\int_{\substack{\sigma(0)=0\\ \sigma(t)=0}} e^{-\frac{1}{2}\int_0^t |\dot\sigma|^2\,ds}\mathcal{D}\sigma = \left[\frac{1}{\left(-\frac{d^2}{dt^2}\right)}\right]^{1/2}$$

With zeta-function regularization,

$$\det\left(-\frac{d^2}{dt^2}\right) = \prod_{n=1}^{\infty}\left(\frac{n\pi}{t}\right)^2 = 2t$$

Gauss-Bonnet-Chern (GBC) The action

$$S(\sigma,\Pi,\Psi) = \int_0^t \frac{1}{2}\left|\dot\sigma\right|^2 - i\left\langle \Pi, \nabla_{\dot\sigma}\Psi\right\rangle + \frac{1}{4}(\Pi, R(\Psi,\Psi)\cdot\Pi)\,ds$$

- $\sigma : [0,t] \to M$ for M a Riemann manifold
- Ψ is an *anti-commuting* section of the tangent bundle over the image of σ.
- Π is an anti-commuting section of the cotangent bundle over the image of σ.

Anti-commuting vector field? $\int_{\substack{\sigma(0)=y\\ \sigma(t)=x}} e^{-S(\sigma,\Pi,\Psi,t)}\mathcal{D}\sigma\mathcal{D}\Pi\mathcal{D}\Psi$ = kernel of what?

This kernel acts on $f(x,\psi)$ for $x \in M$, $\psi \in T_xM$

- Locally, $\psi = \psi^\mu \frac{\partial}{\partial x^\mu}$

- Expand $f(x,\psi^1,\ldots,\psi^n)$ in Taylor series: $f(x,\psi)=\sum_{k=0}^{n}\sum_{\mu_1<\mu_2<\cdots<\mu_k} f_{\mu_1\mu_2\cdots\mu_k}(x)\,\psi^{\mu_1}\psi^{\mu_2}\cdots\psi^{\mu_k}$

$\psi^\mu \mapsto dx^\mu$ gives an isomorphism between functions $f(x,\psi)$ and differential forms.

$$K_\Delta(x,y,\psi_x,\psi_y;t)=\int e^{-S(\sigma,\Pi,\Psi,t)}\mathcal{D}\Pi\mathcal{D}\Psi\mathcal{D}\sigma,$$

(with endpoint conditions on σ and Ψ) represents the kernel of $e^{-\frac{t}{2}\Delta}$ for Δ the Laplace-Beltrami operator on forms.

Berezin integration and supertrace $\int K_\Delta(x,x,\psi,\psi;t)\,d\psi dx=\chi(M)$

Define $\oint f(\psi)d\psi$ as the volume of the top-degree piece of ψ. [1ex] In particular,

$$\int f(x,\psi)d\psi dx=\int_M f,\text{and}\oint K(\psi,\psi)d\psi=\mathrm{str}(\mathfrak{k}),$$

for $\mathfrak{k}$ the endomorphism with kernel $K(\psi,\xi)$. [1ex] Then

$$\begin{aligned}\int K_\Delta(x,x,\psi,\psi;t)\,d\psi dx &= \sum_k \mathrm{tr}\left(e^{-\frac{t}{2}\Delta_{2k}}\right)-\mathrm{tr}\left(e^{-\frac{t}{2}\Delta_{2k+1}}\right)\\ &\equiv \mathrm{Str}\left(e^{-\frac{t}{2}\Delta}\right)\end{aligned}$$

where Δ_p is the restriction of Δ to $\bigwedge^p M$. [1ex] Because $(d+d^*)$ pairs the even and odd-degree eigenforms of Δ with non-zero eigenvalues, the right-hand side reduces to

$$\sum_k \dim\ker\Delta_{2k}-\dim\ker\Delta_{2k+1}=\chi(M),$$

independent of t.

Path integral proof of GBC Alvarez-Gaumé 1983

Assembling, and using the first two "facts"

$$\begin{aligned}\chi(M) &= \lim_{t\to 0}\int K_\Delta(x,x,\psi,\psi;t)\,d\psi dx\\ &= \lim_{t\to 0}\int\left[\int_{\text{based loops}} e^{-S(\sigma,\Pi,\Psi,t)}\,\mathcal{D}\Pi\mathcal{D}\Psi\mathcal{D}\sigma\right]d\rho d\psi dx\\ &= \lim_{t\to 0}\int e^{-S(\sigma_{\mathrm{cl}},\Pi_{\mathrm{cl}},\Psi_{\mathrm{cl}},t)}\,d\rho d\psi dx\\ &\quad\times\left(\frac{1}{\pi}\right)^{n/2}\int e^{-S_Q}\,\mathcal{D}\bar{\Pi}\mathcal{D}\bar{\Psi}\mathcal{D}\bar{\sigma}.\end{aligned}$$

$\sigma_{\mathrm{cl}}=x$, $\Pi_{\mathrm{cl}}=\rho$ $\Psi_{\mathrm{cl}}=\psi$; last path integral is $\left(\frac{1}{2t}\right)^{n/2}$, so [1ex] $\chi(M)=\left(\frac{1}{2\pi t}\right)^{n/2}\int e^{\frac{t}{4}(\rho,R(\psi,\psi)\cdot\rho)}\,d\rho d\psi dx$, or $\chi(M)=\left(\frac{1}{2\pi}\right)^{n/2}\int\mathrm{Pfaff}(R)$.

3 Path integral proof of the Gauss-Bonnet-Chern theorem

3.1 Overview

A rigorous path integral Overview

Use a partition P of $[0,t]$ into N subintervals to define $\text{Path}_N \subset M^{N+1}$, the space of continuous paths composed of N geodesic segments [1ex] Approximate the path integral by discretizing:

$$\int_{\text{Path}_N} e^{-S(\sigma,\Pi,\Psi)}\,\mathcal{D}\Pi\mathcal{D}\Psi\mathcal{D}\sigma = K(t_1) * K(t_2) * \cdots * K(t_N),$$

where

$$K(x,y,\psi_x,\psi_y,t) = \oint \exp\left[-\frac{1}{2t}\,|\dot\sigma|^2 + i\overline{\langle \rho, (\nabla_{\dot\sigma}\Psi)\rangle} - \frac{t}{8}\overline{(\rho, R(\Psi,\Psi\psi_x)\cdot\rho)}\right] d\rho$$

for σ the geodesic from y to x, with $\exp_y \dot\sigma = x$,and

$$K(t_1) * K(t_2) = \int\oint K(x,z,\psi_x,\psi_z;t_1)K(z,y,\psi_z,\psi_y;t_2)\,d\psi_z dz$$

This is time-slicing

$$\int e^{-\int_0^t L(\sigma)\,ds}\mathcal{D}\sigma \sim \int e^{-\sum \overline{L}(\sigma_i,\sigma_{i-1},t_i)t_i}\prod J\,d\sigma_i$$

with the approximation presumably improving under refinement.[1ex] Define $K = Je^{\overline{L}}$. [1ex] Note that ”Fact 1” and the trivial partition give

$$K_\Delta(t) \sim K(t)$$

for small t.

Specifying K

Choose $K(x,y,\psi_x,\psi_y;t) = \oint(2\pi t)^{-n/2}$

$$\times\exp\left[-\frac{|\vec{x}_y|^2}{2t} - \frac{t\mathfrak{r}}{6} + \frac{1}{12}\text{Ricci}(\vec{x}_y,\vec{x}_y) + i\left\langle\rho, \mathfrak{P}^x_y\psi_x - \psi_y\right\rangle + \frac{t}{8}(\rho, R[\psi_y,\psi_y]\,\rho) + \frac{t}{8}\left(\mathfrak{P}^x_y\rho, R[\psi_x,\psi_x]\,\mathfrak{P}^x_y\rho\right)\right]d\rho$$

when x and y are close enough to be within the injectivity radius of each other and $K = 0$ otherwise. Here $\exp_y(\vec{x}_y) = x$, and $\mathfrak{P}$ denotes parallel transport. In Riemann normal coordinates centered at y,

$$K = \oint (2\pi t)^{-n/2} \exp\Bigg[-\frac{|\vec{x}_y|^2}{2t} - \frac{t\mathfrak{r}}{6} + \frac{1}{12}\mathrm{Ricci}(\vec{x}_y, \vec{x}_y) + i\,\langle \rho, \psi_x - \psi_y \rangle$$
$$+ \frac{i}{6}\,\langle \rho, R[\vec{x}_y, \psi_x]\,\vec{x}_y \rangle + \frac{t}{8}(\rho, R[\psi_x, \psi_x]\,\rho) + \frac{t}{8}(\rho, R[\psi_y, \psi_y]\,\rho) \Bigg] d\rho,$$

after dropping complicating terms that do not affect the limit.

"Facts" 1 & 2 are now theorems, and imply a local form of GBC

Theorem 1. *The fine-partition limit K^∞ of the approximate path integrals is well-defined and agrees with the heat kernel:*

$$\int e^{-S(\sigma,\Pi,\Psi)}\,\mathcal{D}\Pi\mathcal{D}\Psi\mathcal{D}\sigma \equiv K^\infty = \ker e^{-t\Delta/2}.$$

Theorem 2.

$$K^\infty = K + \mathfrak{E},$$

for an error term $\mathfrak{E}$ with $\mathrm{str}(\mathfrak{E}) \in \mathcal{O}\big(t^{2\epsilon/n}\big)$ for some positive ϵ.

Local GBC

In fact, by looking only at the top-form piece, these results lead to the local form

Theorem 3.

$$\Big[\lim_{t\to 0} K^\infty(x, x, \psi_x, \psi_x; t)\Big]_{\mathrm{top}} = (2\pi)^{-n/2}\,\mathrm{Pfaff}(R)$$

3.2 Proof sketch

K is an approximate heat kernel

Direct calculation gives

$$\begin{aligned} \frac{\partial K}{\partial t} + \frac{1}{2}\Delta_x K &= F_1 H(x,y;t) + F_2 t\mathcal{O}_t(t^\epsilon)\,, \text{ and} \\ \frac{\partial K}{\partial t} + \frac{1}{2}\Delta_y K &= F_3 H(x,y;t) + F_4 t, \mathcal{O}_t(t^\epsilon) \end{aligned}$$

where $H = (2\pi t)^{-n/2} e^{-\frac{|\vec{x}_y|^2}{2t}}$, and $|F_i|_t = \mathcal{O}(t^\epsilon)$. [1ex] Here the norm on the endomorphisms F_i is, in an orthonormal basis,

$$\left| \psi^{j_1} \cdots \psi^{j_k} \frac{\partial}{\partial \psi^{l_1}} \cdots \frac{\partial}{\partial \psi^{l_k}} \right|_t = \begin{cases} 1 & k \le 2 \\ t^{(k-2)(-1/2+\epsilon/n)} & k \ge 2 \end{cases}.$$

Operator convergence

This ensures that *as operators* the approximate path integrals $K^{*P} \equiv K(t_1) * K(t_2) * \cdots * K(t_N)$ converge to the heat operator: [1ex] For $f_0(x, \psi_x)$ piecewise continuous, and $t \geq 0$,

$$f(t) = \lim_{|P| \to 0} K^{*P} * f_0$$

is the unique solution to the heat equation $\partial f / \partial t = -\frac{1}{2} \Delta f$ with $f(0) = f_0$.

Propositions on products

- *Norms of products:* For $t = t_1 + t_2 > 0$,

$$\|K(t_1) * L\|_t \leq e^{\mathcal{O}\left(t_1^{\epsilon/n}\right) + \mathcal{O}\left(t^2\right)} \|L\|_{t_2} .$$

- *Almost semigroup:*

$$\|K(t_1) * K(t_2) - K(t)\|_t = \mathcal{O}\left(t^{1+\epsilon}\right) .$$

- *Cauchy* For sufficiently small $t > 0$, all partitions P of t, and all refinements Q of P,

$$\left\| K^{*Q} - K^{*P} \right\|_t = \mathcal{O}(t) \, |P|^{\epsilon} .$$

Supertrace of error

These suffice to prove

$$K_\Delta = K^\infty = K + \mathcal{O}_t\left(t^{1+\epsilon}\right) .$$

The error term is $\mathfrak{E} = F_1 H + t F_2$, for $|F_i|_t \in \mathcal{O}\left(t^{1+\epsilon}\right)$, so

$$\operatorname{str}(\mathfrak{E}) = \operatorname{str}(F_1) (2\pi t)^{-n/2} + \operatorname{str}(F_2) \, t.$$

Since $\left| [F_i]_{\text{top}} \right|_t \leq |F_i|_t$,

$$\operatorname{str}(F_i) \left| \psi^1 \cdots \psi^n \frac{\partial}{\partial \psi^1} \cdots \frac{\partial}{\partial \psi^n} \right|_t = \operatorname{str}(F_i) \, t^{(n-2)(-1/2+\epsilon/n)} \in \mathcal{O}\left(t^{1+\epsilon}\right) ,$$

and thus

$$\operatorname{str}(\mathfrak{E}) \in \mathcal{O}\left(t^{2\epsilon/n}\right) .$$

Full details

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