Reidemeister/Roseman-type Moves to Foams

J. Scott Carter

University of South Alabama

Sept. 2012
Recent Advances in TQFT
Lisboã, Portugal

Acknowledgements

This talk was supported by the Ministry of Education Science and Technology (MEST) and the Korean Federation of Science and Technology Societies (KOFST).
This work is related to on-going work with Atsushi Ishii and Masahico Saito.

Outline

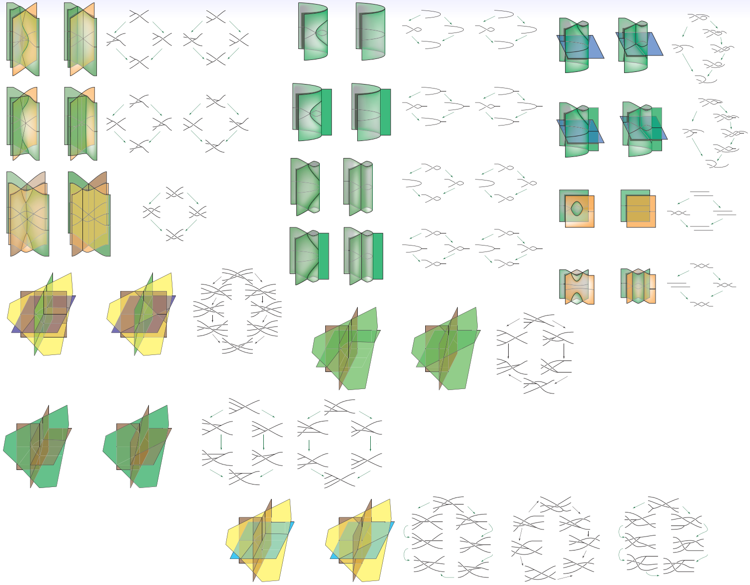
- Main Statement
- Definition of a foam
- Example of a knotted foam in 4-space
- Critical points, vertices, and crossings for spacial graphs
- Moves to spacial graphs
- Interpreting moves as critical portions of foams
- Critical points in one dimension higher
- Intersections

Main Result

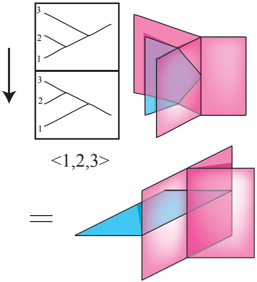
Theorem

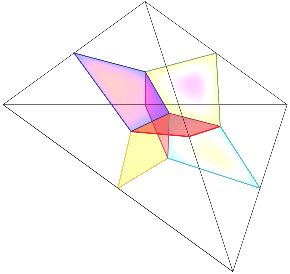
Let K_0 and K_1 be 2-foams w/out boundary embedded in 4-space with diagrams $D(K_i)$ for $i = 0, 1$. K_0 and K_1 are isotopic if and only if there is a sequence $D(K_{i/n})$ of diagrams for $i = 0, \ldots, n$ such that $D(K_{i/n})$ differs from $D(K(i-1)/n)$ by one of the RR-type moves that are listed below.*

*Reidemeister-Roseman



Analogy and definition





Classical knots are to knotted surfaces as knotted trivalent graphs are to knotted foams. Here I illustrate the space Y^2.

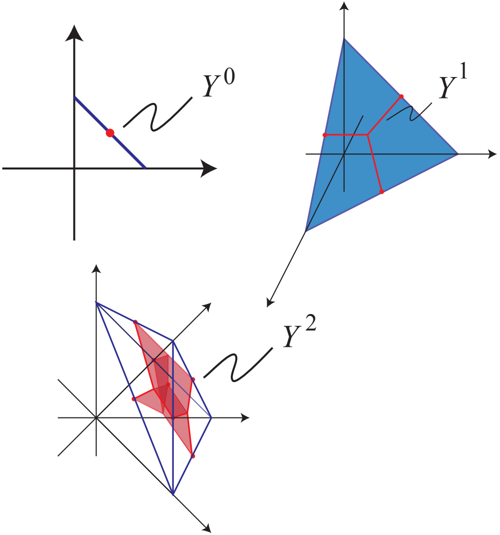
The space Y^2 will also be called *the associator*. In the general scenarios of categorification, we can define higher dimensional associators as geometric representations of the Stasheff polytopes. Here is the general construction.

The Space Y^n

Let $\Delta^{n+1} = \{\vec{x} \in \mathbb{R}^{n+2} : \sum x_i = 1 \ \& \ 0 \leq x_i\}$ denote the standard simplex. The space $Y^n \subset \Delta^{n+1}$ is defined as follows: $Y^0 = (\frac{1}{2}, \frac{1}{2})$. Take $\Delta^n_j = \{\vec{x} \in \Delta^{n+1} : x_j = 0\}$. Embed a copy, $Y^{n-1}_j \subset \Delta^n_j$. Cone $\cup_{j=1}^{n+2} Y^{n-1}_j$ to the barycenter $b = \frac{1}{n+2}(1, 1, \ldots, 1)$ of Δ^{n+1}.

$$Y^n = C\left(\cup_{j=1}^{n+2} Y^{n-1}_j\right).$$

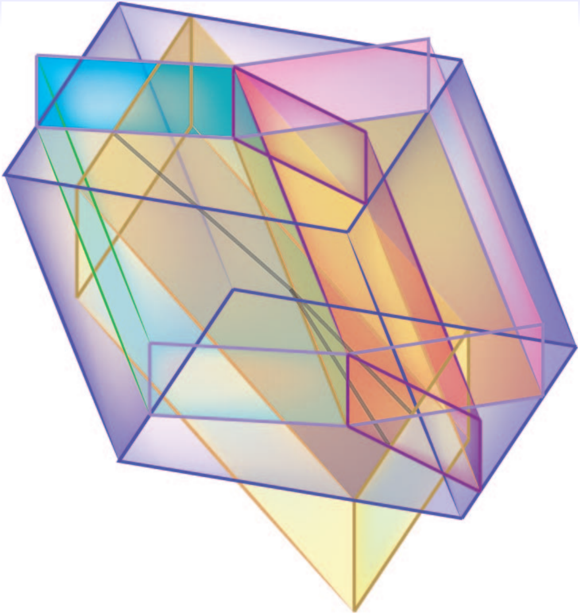
Y^0, Y^1, and Y^2

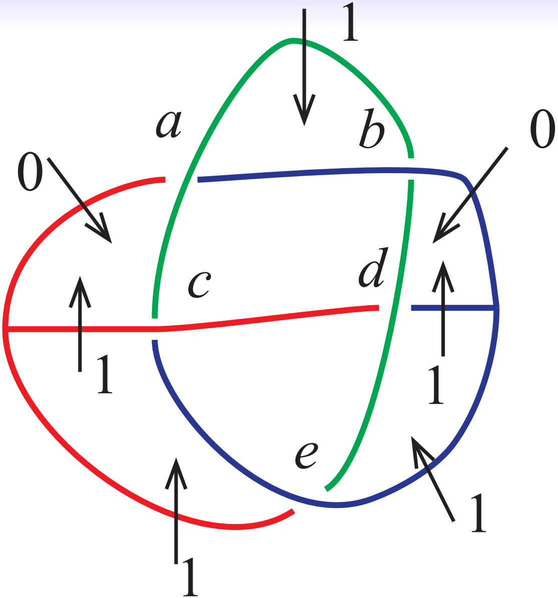


Definition of n-foams

Every point $y \in Y^n$ has a nbhd. that is homeom. to $Y^{n-k} \times D^k$. The union of these points is called *the k-stratum* — the union of these is a set of $\binom{n+2}{k}$ disks of dimension k, for $k = 1, \ldots, n$.
An *n-foam* is a top. sp. X for which each pt. $x \in X$ has a nbhd homeom. to a nbhd. of a point in Y^n.

A 2-foam is a compact topological space in for which every point has a nbhd. homeom. to a nbhd. of a point of Y^2.
A 2-foam has vertices, edges, and faces. Four edges are incident to a vertex. Three faces are incident to an edge. Six faces are incident at the vertex.

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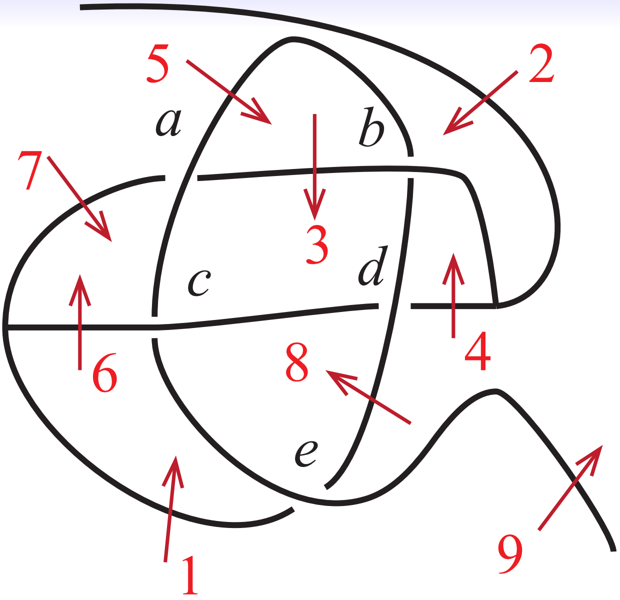
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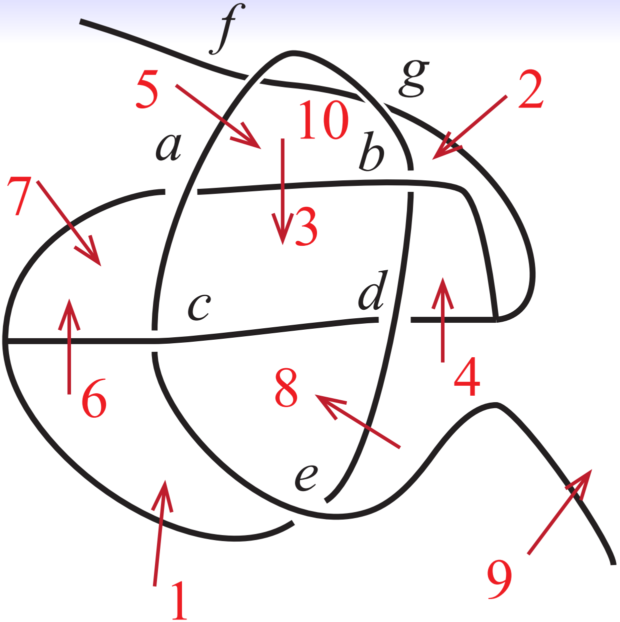

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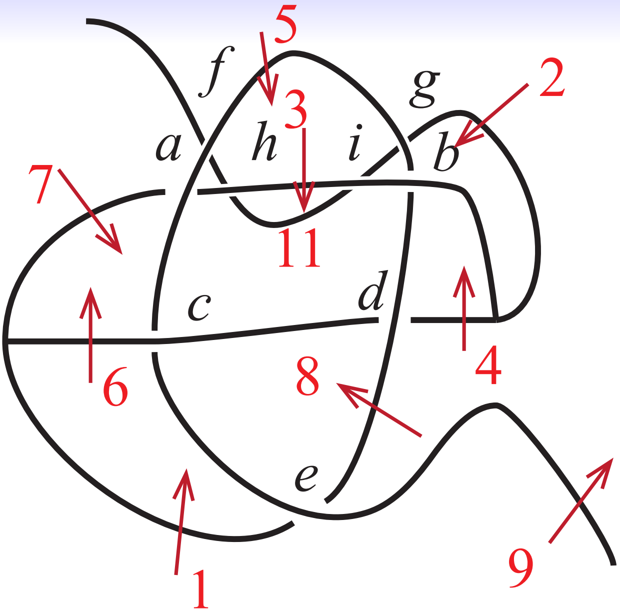

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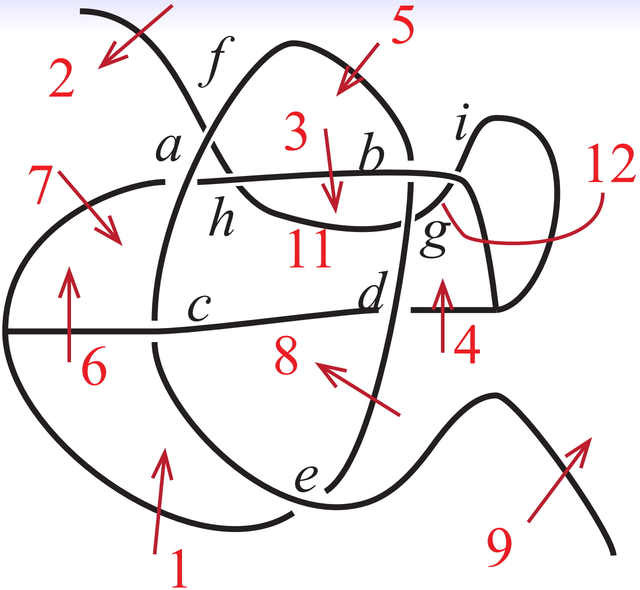

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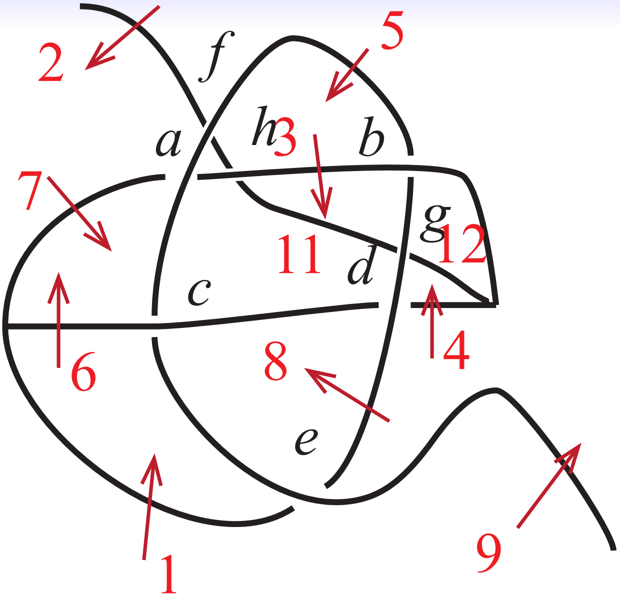

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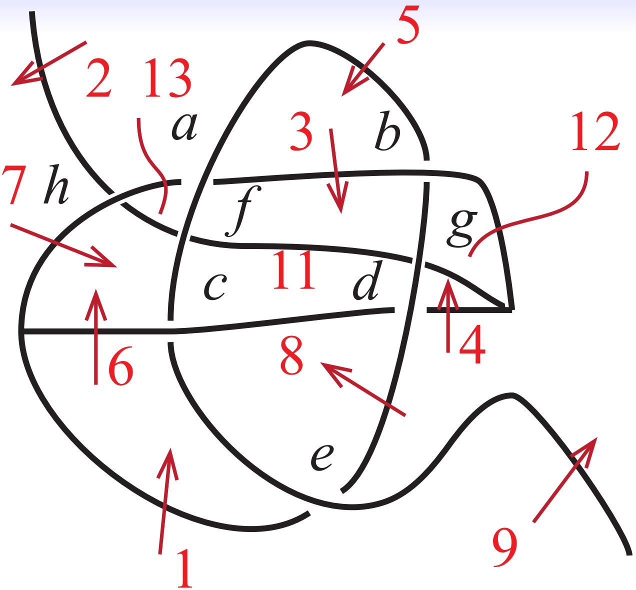

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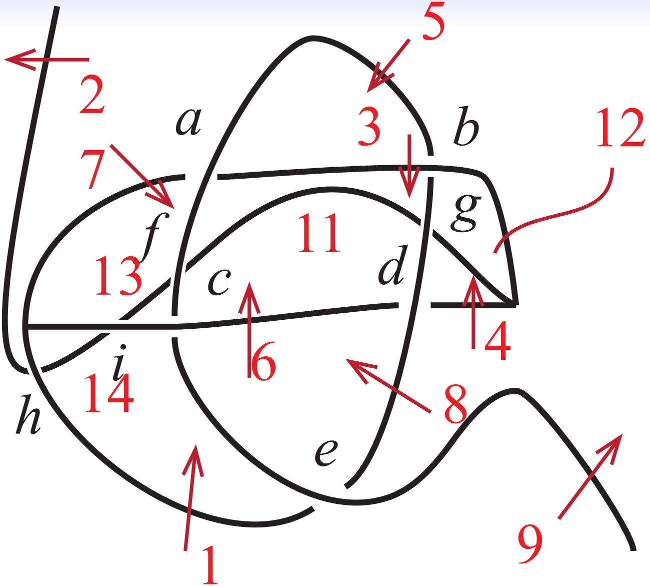

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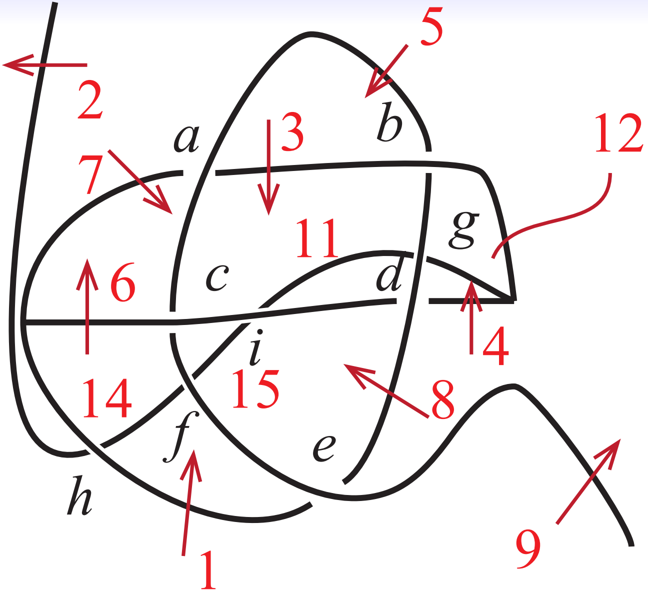

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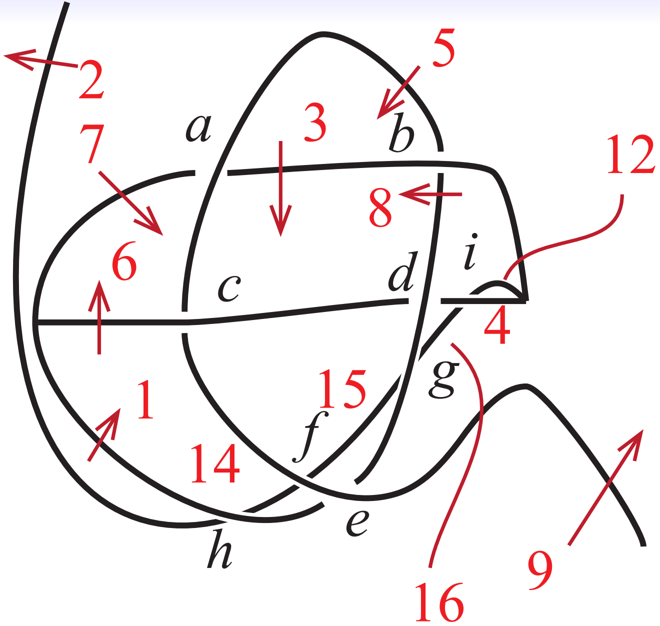

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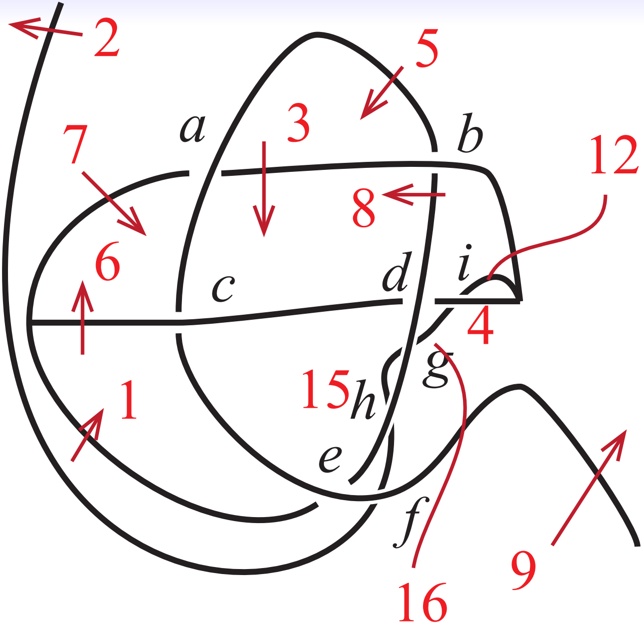
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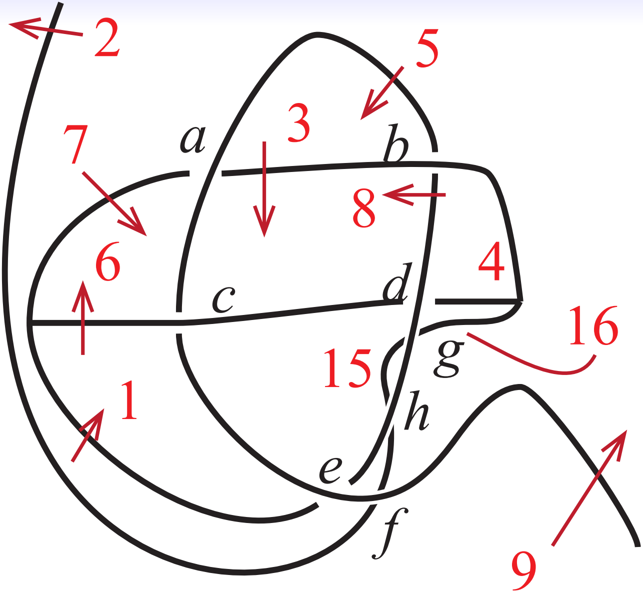

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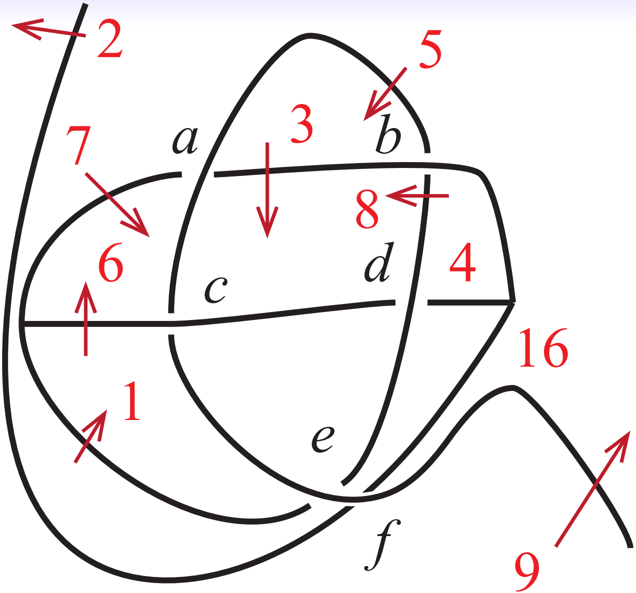
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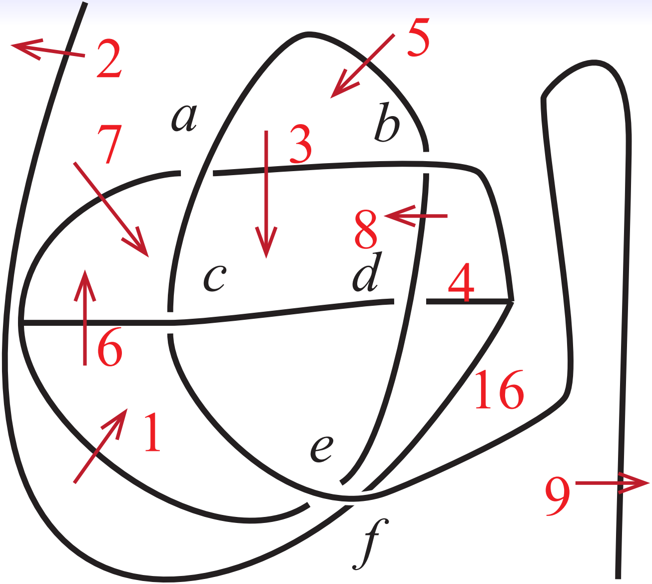

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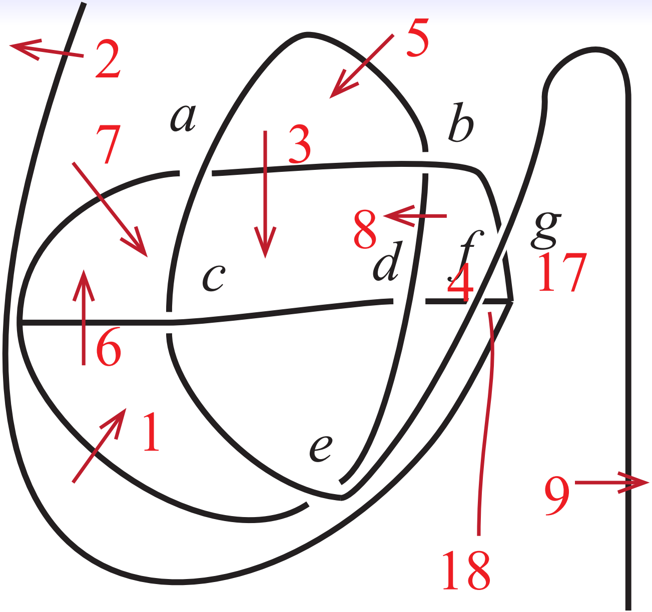

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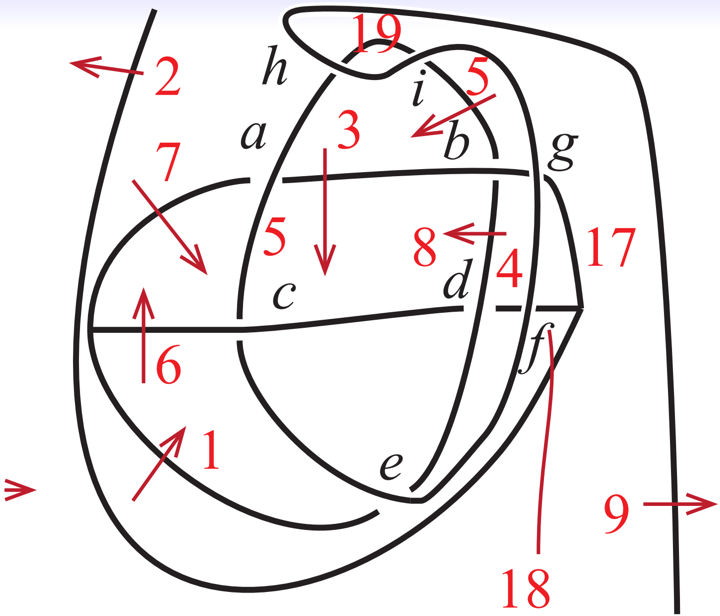

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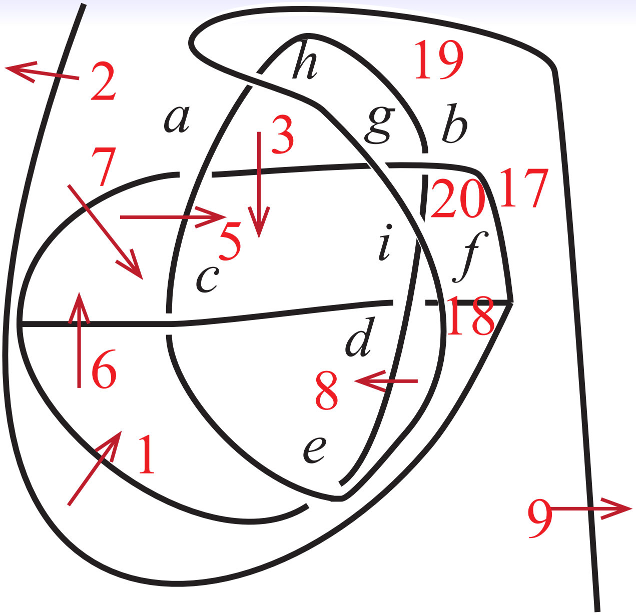
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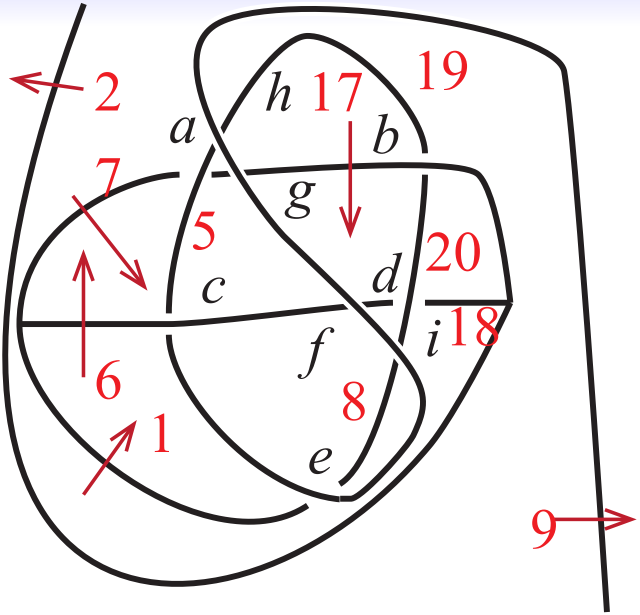
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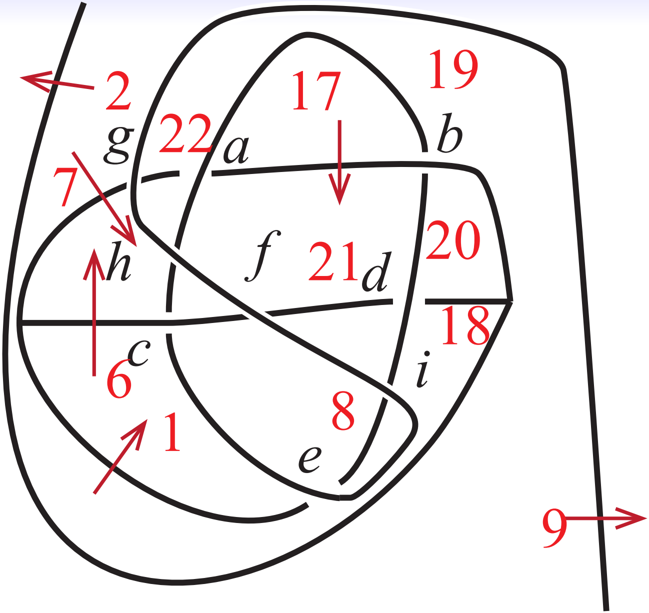

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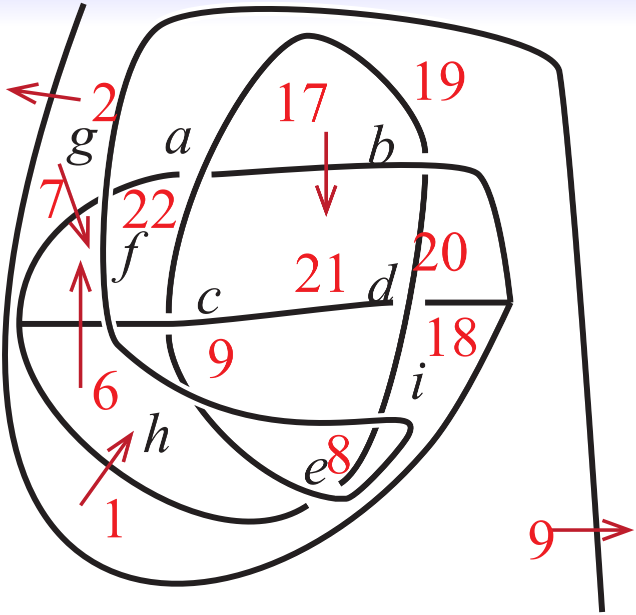
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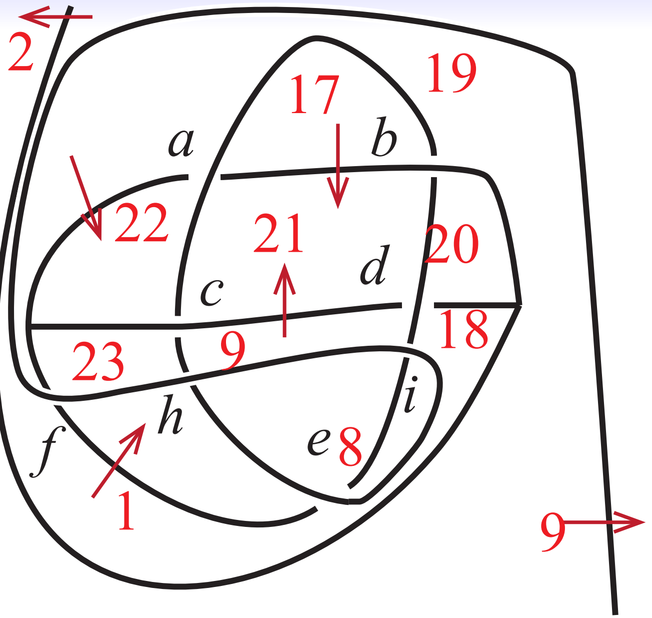

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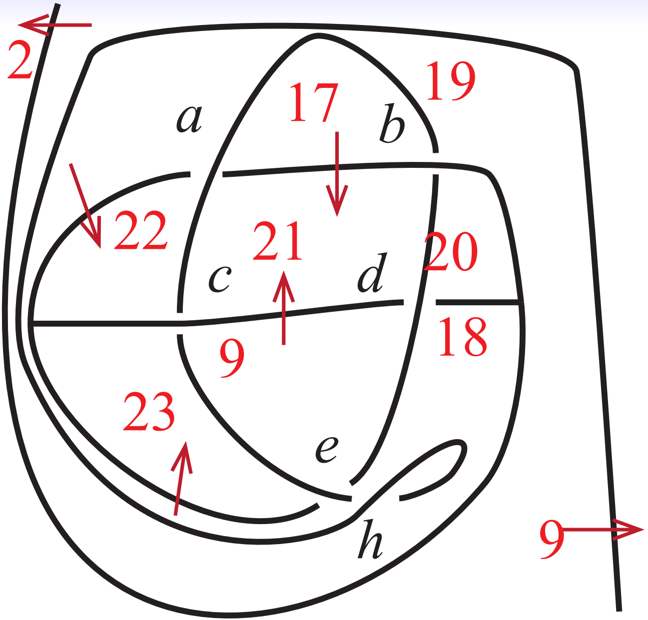

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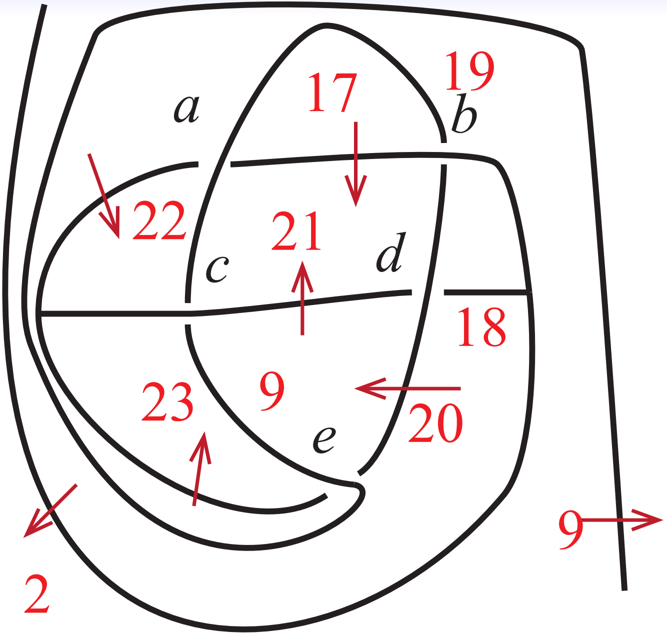

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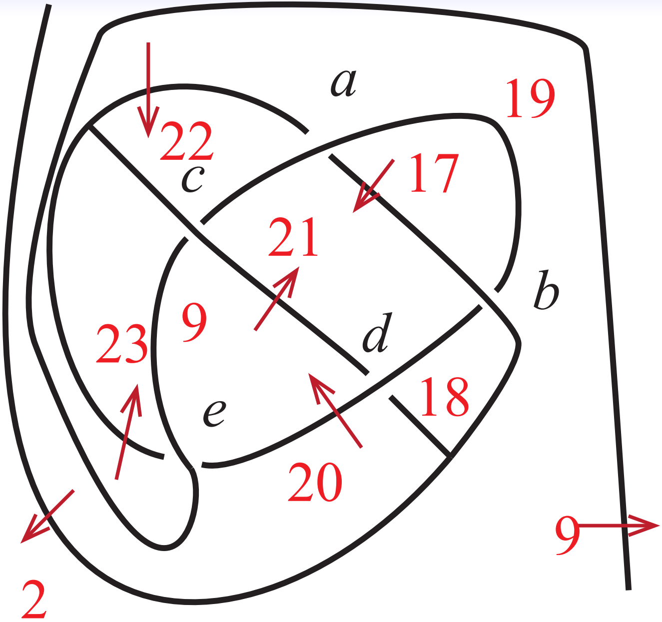

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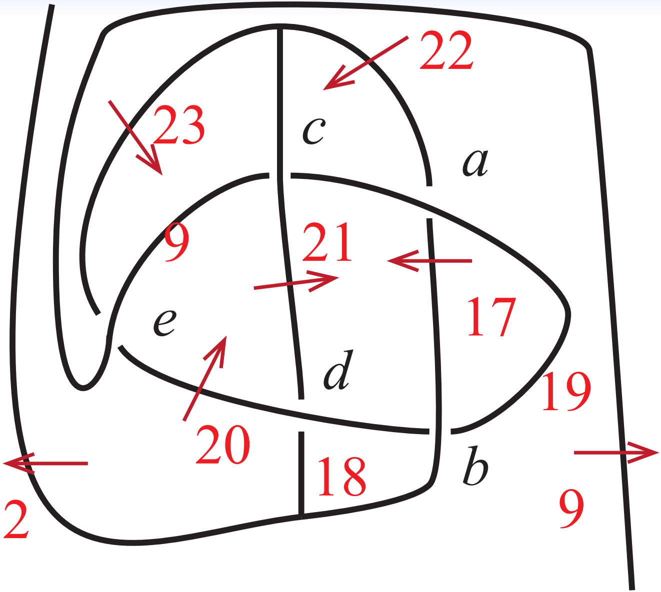
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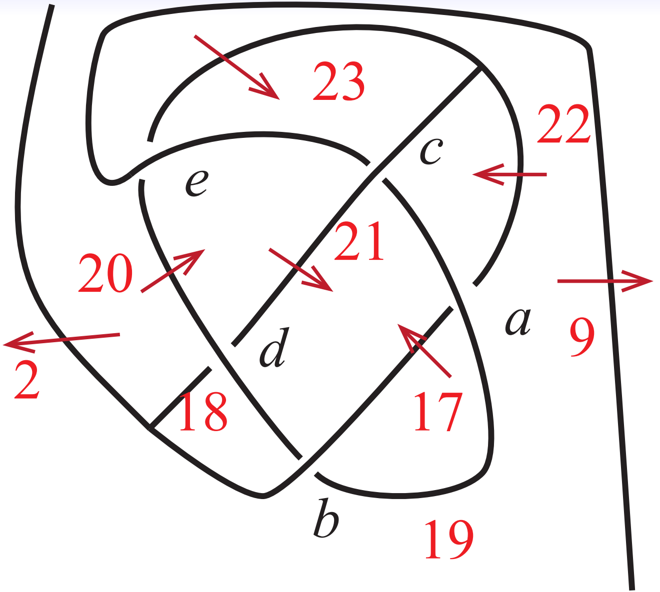

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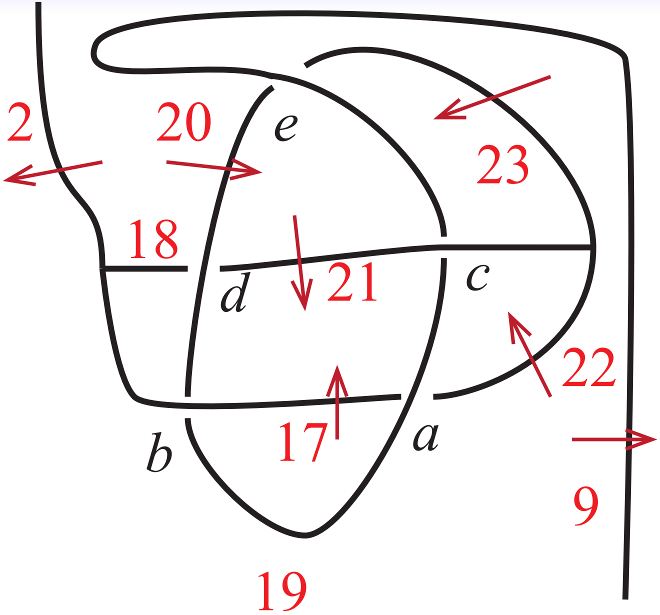

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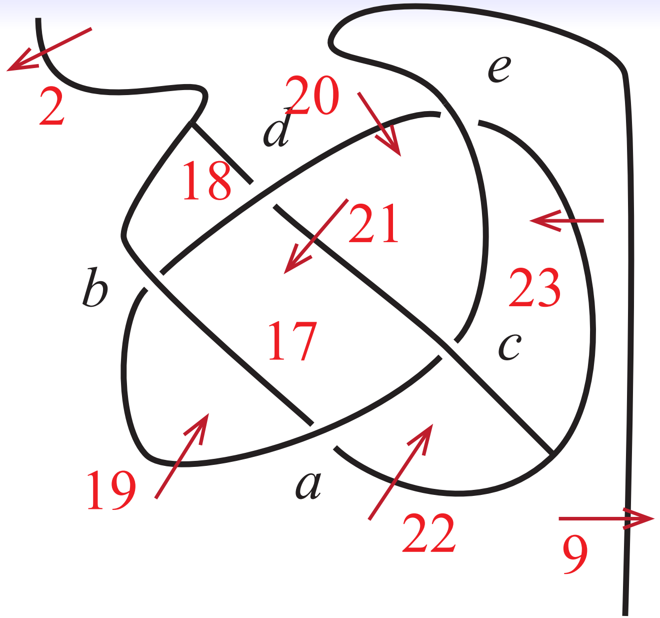
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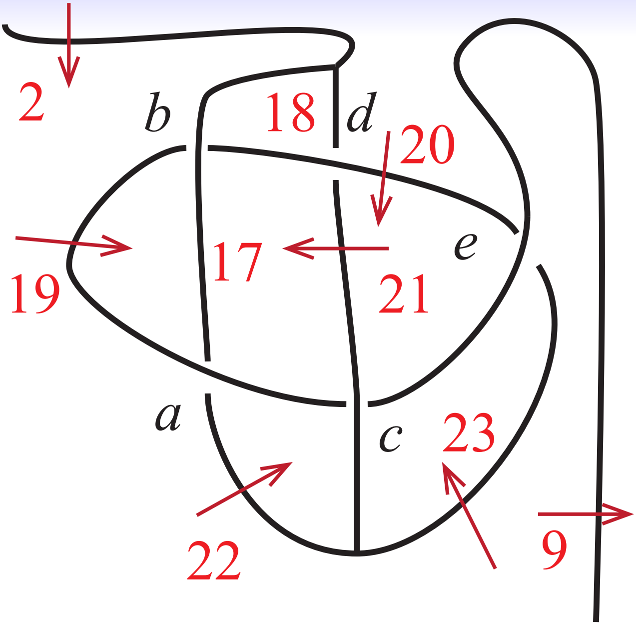
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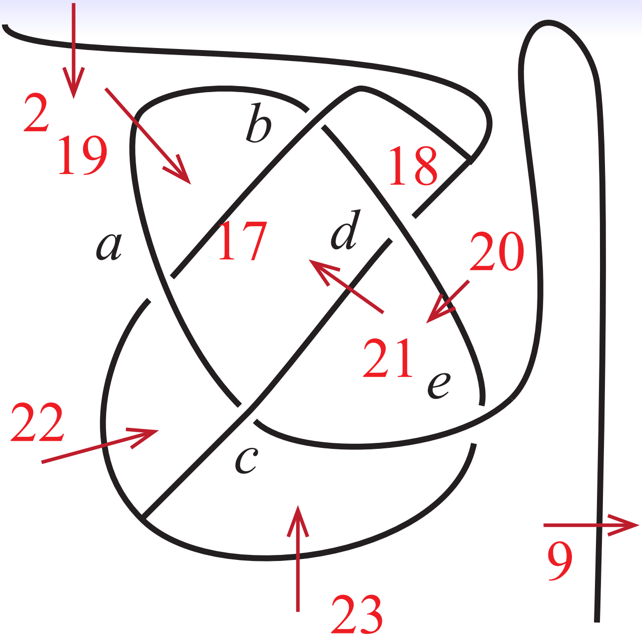

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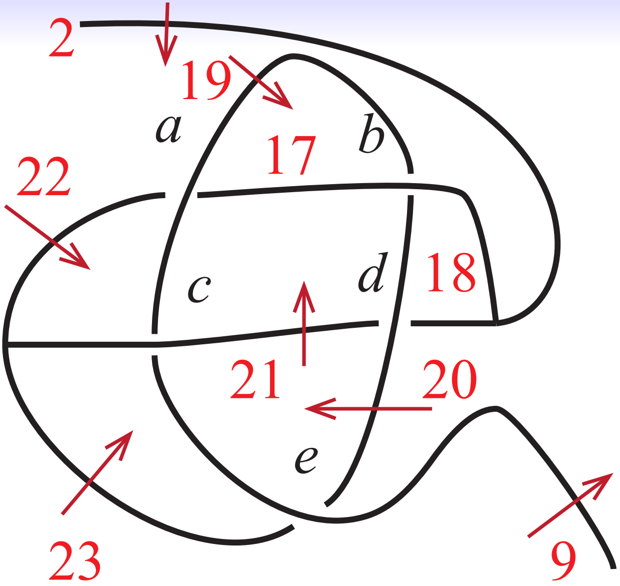
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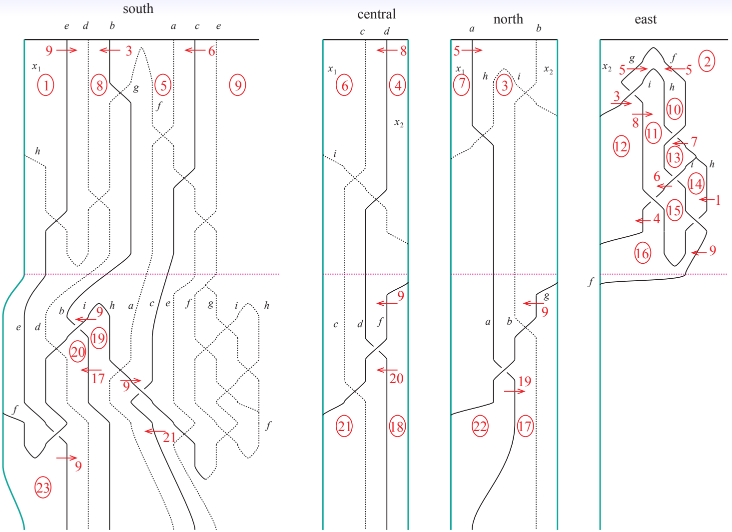
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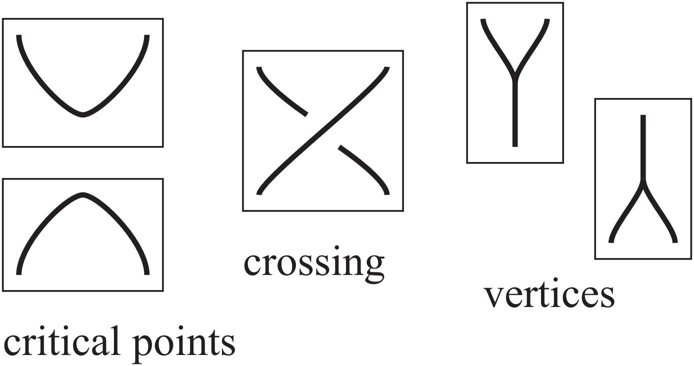
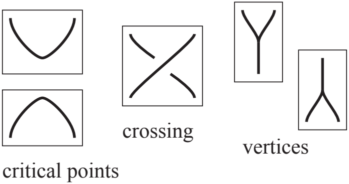
Developing the analogy

First, we consider the local pictures involved in knotted trivalent graphs.



Diagrams for trivalent graphs

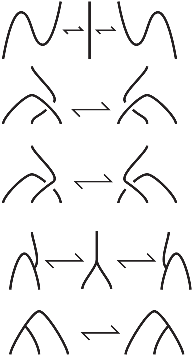


In dim 0-mfld $\subset$ 1-mfld, we have vertices. In one dim larger, they may be created or annihilated by critical points (index 0 or 1), or they may cross (1-mfld $\cap$ 1-mfld $\subset$ 2-mfld). Trivalent vertices are considered as 0-dim'l $\subset$ 2-dim'l.

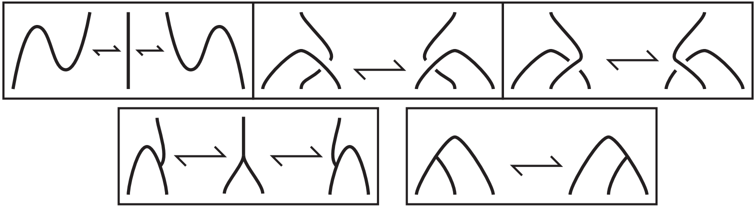
The basic philosophy of the Reidemeister-type moves is to carefully quantify the critical points and intersections in a picture that is one dimensional larger. By observing the codimension 1 singularities in one more dimension, we observe the Reidemeister moves.
Before we exploit this, I want to turn to the interactions between critical points and crossings.

Types of moves: Part I

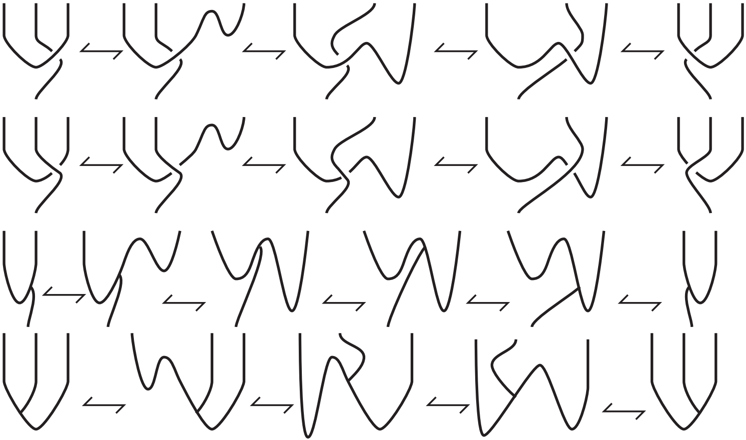
These moves do not change the underlying topology of the plane containing the diagram:



Cat. and Geom. Conseq.



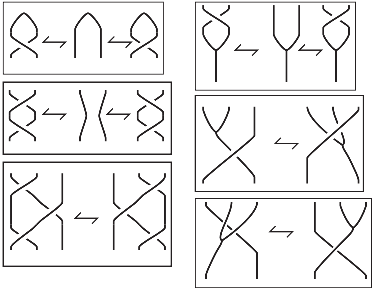
1. The zig-zag move gives a categorical dual or adjoint map. 2. A comultiplication can be defined in terms of a multiplication. 3. The upside-down versions of the last four moves follow easily.



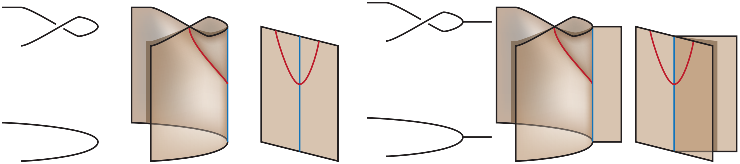
When we are looking at knotted trivalent graphs, we are looking at a type of braided Frobenius category. I hesitate to make the categorical construction explicit because it is beyond my categorical expertise. For the rest of the talk, I want to examine the Reidemeister moves for trivalent graphs, consider these as atomic pieces of knotted foams, and consider the Reidemeister moves for the foams. Ultimately, I believe that there is a complete categorical interpretation of this and it is within the suzerain of TQFTs. In my talk tomorrow, I will give homological interpretations to the moves and their higher dimensional versions.

Types of moves: Part II

These are the Reidemeister moves for knotted trivalent graphs.

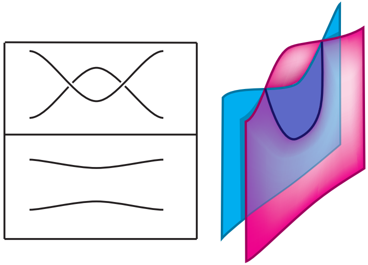


Critical Analysis: Type-I



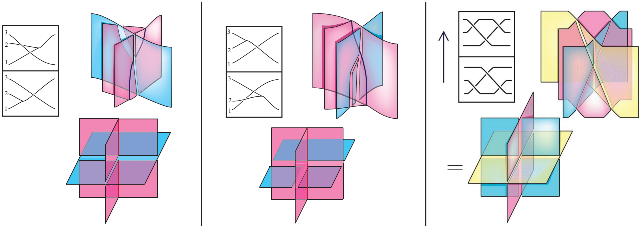
Both the type-I move and the twisted vertex are occurrences of a critical point on the double decker set — the pre-image of the double point set on the ambient surface/foam.

Critical Analysis: Type-II

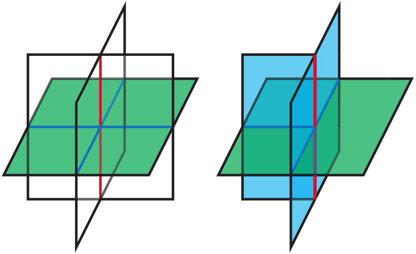


The type-II-move is a critical point of the double point set.

Intersection Analysis: Type-III +

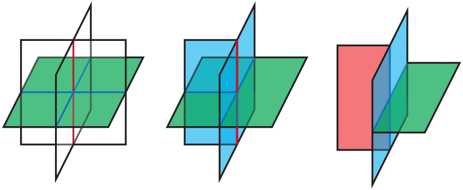


Intersection Analysis: Type-III +



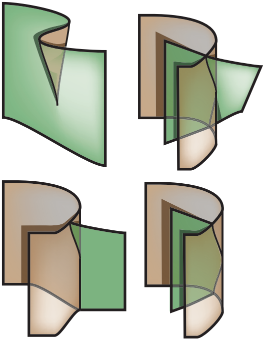
The type-III-move is the intersection b/2 a 1-dim'l set (crossing $\times[0,1]$) and a 2-dim'l sheet (arc $\times[0,1]$). Similarly, the YI-move and the IY-move have the same projection in which the vertex of the $Y \times[0,1]$ crosses a 2-dim'l sheet (arc $\times[0,1]$).

Foam Vertex



The vertex of a foam is a 0-dim'l point in 3-space. While we can think of the intersection b/2 the two arcs in the blue sheet, this is artificial — it depends on the particular drawing.

Small remark



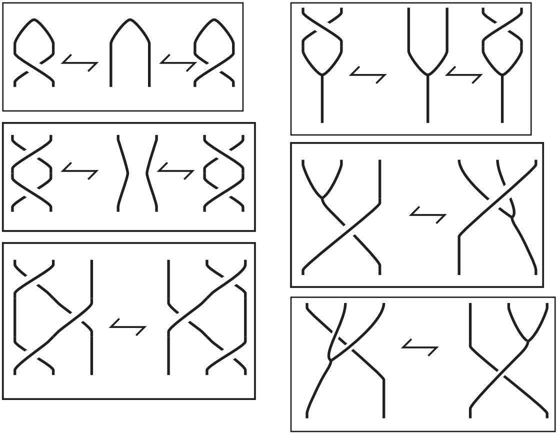
Each of the ψ-type moves has a interpretation as a surface. Frohman and Roseman are studying embedded foams and also only pay mild attention to these. I include them here, now, because they facilitate one's drawings of the knotted foams.

Recap

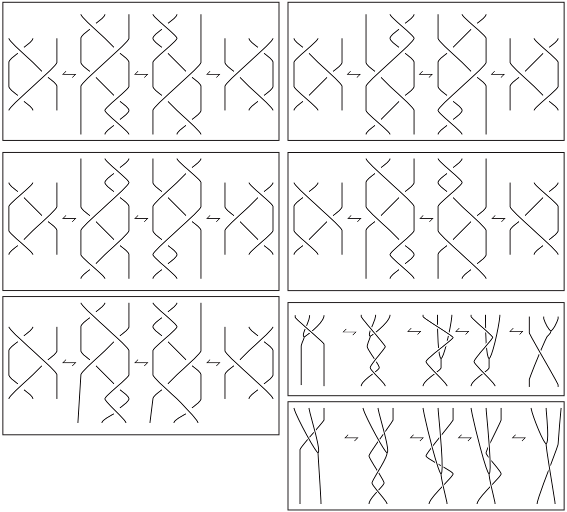
The Reid. moves for trivalent graphs correspond to critical points and intersections in one more dimension. Thus an isotopy between knotted graphs can be decomposed into critical pieces. By compactness there are finitely many of these. Each move is found from among these:

Recap

Each move to a knotted trivalent graph is found from among these:



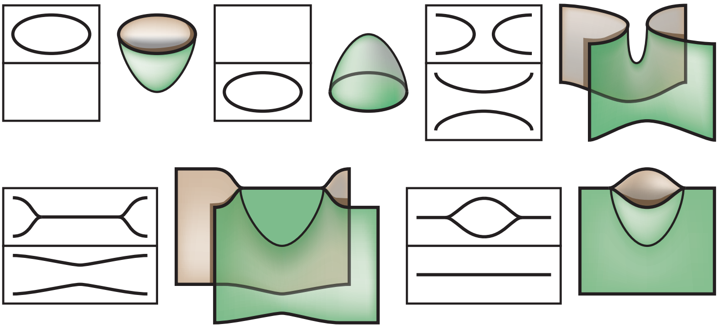
Turaev's trick



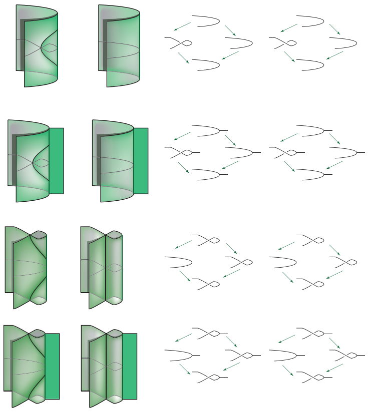
Roseman-type moves for knotted foams

In the next few slides, I am going to discuss the critical points and intersections in one higher dimension. So our foams are 2-dim'l. The branch points, twisted vertices, triple points, and intersections b/2 an edge and a transverse sheet are all 0-dim'l. so the critical pts. of 1-dim's sets characterize these moves. Before this I want to mention the critical points of foams.

Critical points of foams

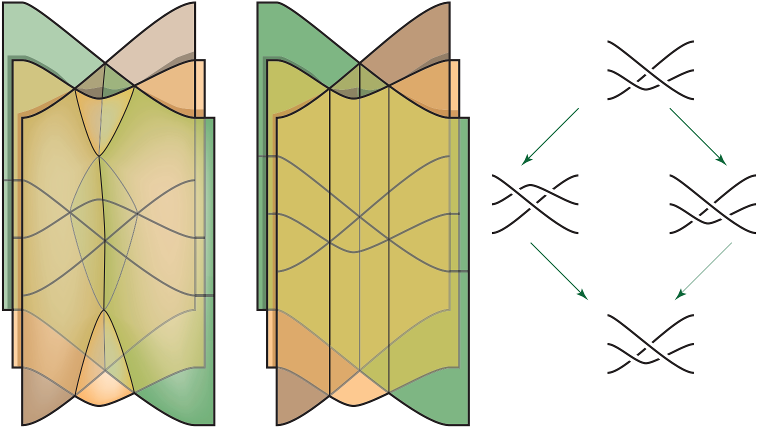


Critical points of the branch point set

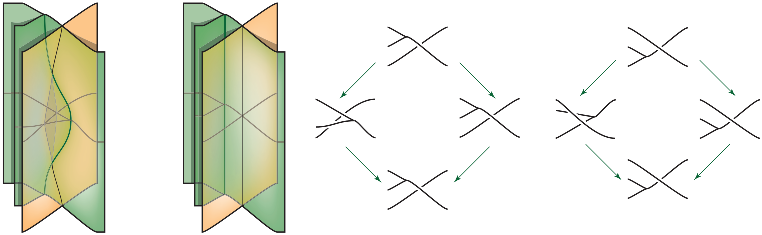


The critical points for arcs of branch points (and twist points) are illustrated here.

Critical points of the triple point set

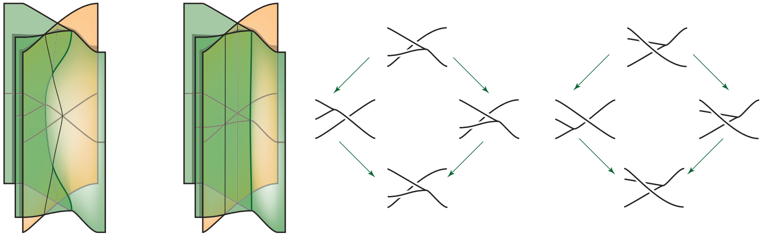


Critical points of the intersection set 1



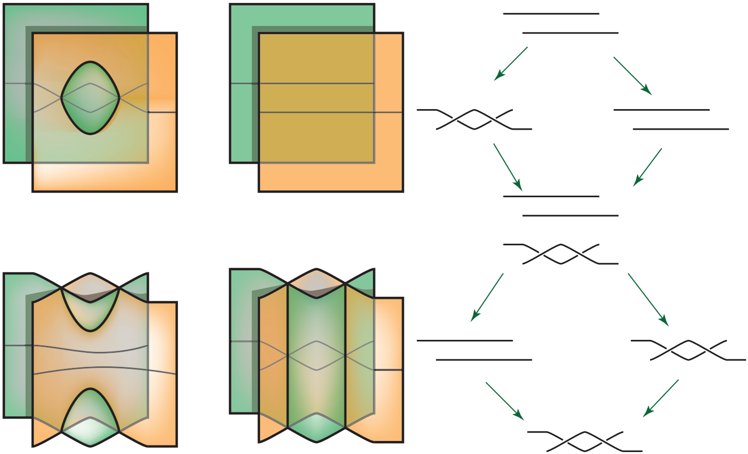
This move is a YI-bubble move. The crossing points between an edge and a transverse sheet are 0-dim'l. A critical point creates or annihilates such a pair.

Critical points of the intersection set 2



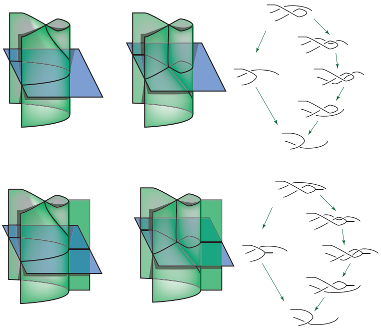
This move is a YI-saddle move. The crossing points between an edge and a transverse sheet are 0-dim'l. A critical point creates or annihilates such a pair.

Critical points of the double point set



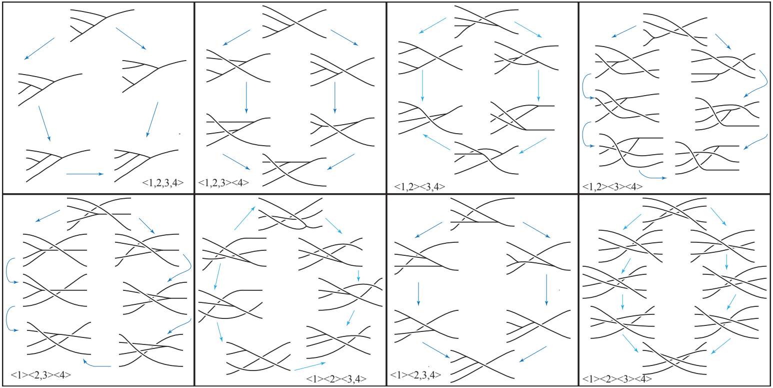
The double point set forms a surface in an isotopy. Its critical points are optima or saddles.

Int. pts. b/2 branch/twist set and trnsvs. sheet



The branch pt./twist vertex can pass through a transverse sheet. 1-dim $\cap$ 3-dim $\subset$ 4-dim.

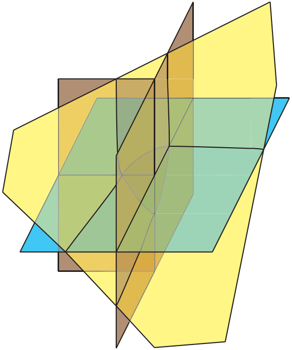
8 interesting moves

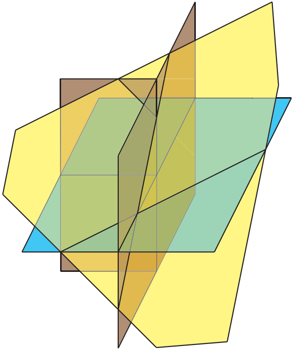


Caution

In the slide above, I listed the standard 3-2 move (compatibility among associators). This is *not* strictly a move for 2-foams. But if the foam is carrying an embedded 4-mfd in S^4, then the move is allowed.

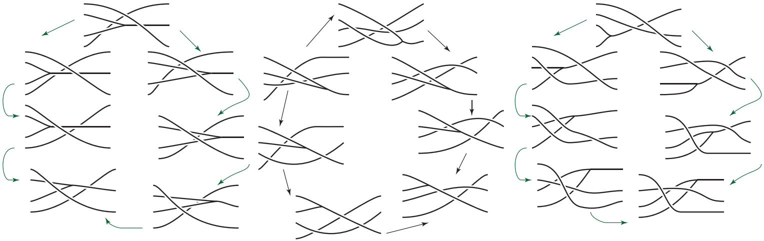
The YII-move



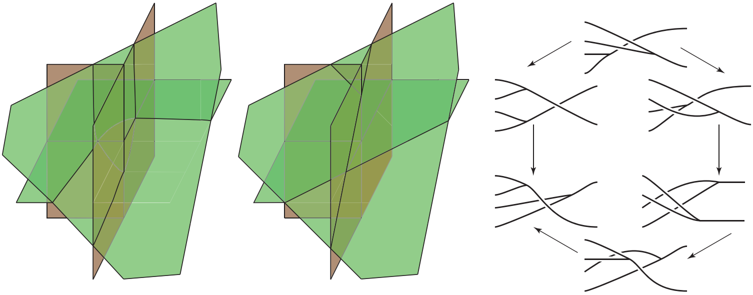


An edge of the form $Y^1 \times [0, 1]$ pass through an edge of double points. 2-dim $\cap$ 2-dim $\subset$ 4-dim.

YII, IYI, IIY

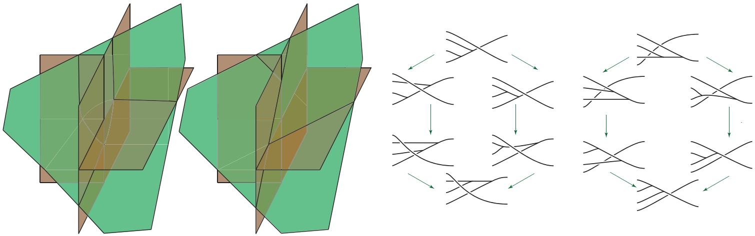


The YY-move



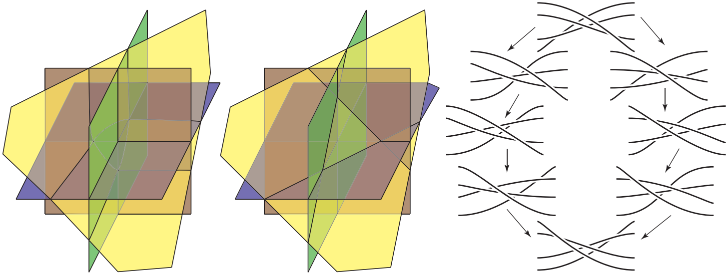
Two edges (each of the form $Y^1 \times [0,1]$) pass through each other.2-dim $\cap$ 2-dim $\subset$ 4-dim.

The YYI-move



A vertex (at the juncture of the foam Y^2) passes through a transverse sheet. 1-dim $\cap$ 3-dim $\subset$ 4-dim.

The Zamalochikov/Tetrahedral move



A triple point passes through a transverse sheet.

Thanks

That's my story and I am sticking to it.

감사합니다

ありがとう

謝謝

Obrigado!