Homology of G-Families of Quandles and Knotted Foams

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This work is related to on-going work with Atsushi Ishii and Masahico Saito.

Outline

1. G-family of quandles
2. Crossings, Vertices, and 2-chains
3. Moves to trivalent graphs, 3-chains, and boundaries
4. Computing the boundary maps algebraically
5. The cocycle conditions as moves to foams
6. The cocycle conditions algebraically
7. Coloring restrictions
8. An example knotted foam
9. Suggested further research

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G-family of quandles

after Ishii-Iwakiri-Jang-Oshiro. G is a group. Q is a set. $\forall g \in G$ $\exists$ binary operation $\triangleleft_g$ on Q s.t.

- $a \triangleleft_g a = a$
- $(a \triangleleft_g b) \triangleleft_h b = a \triangleleft_{gh} b$,
- $\triangleleft_{g^{-1}} = (\triangleleft_g)^{-1}$, $a \triangleleft_1 b = a$,
- $(a \triangleleft_g b) \triangleleft_h c = (a \triangleleft_h c) \triangleleft_{h^{-1}gh} (b \triangleleft_h c)$.

$(\forall a, b, c \in Q$, and $\forall g, h \in G)$.

Examples

1. Let H be a group. $G = \mathrm{Aut}(H)$ (acting on the right). Define $a \triangleleft_s b = (ab^{-1})s\ b$.
2. Specifically, $Q = (\mathbb{Z}/(p))^n$ — row vectors $G = \mathrm{SL}(n, \mathbb{Z}/(p))$, and $a \triangleleft_M b = aM + b - bM$.

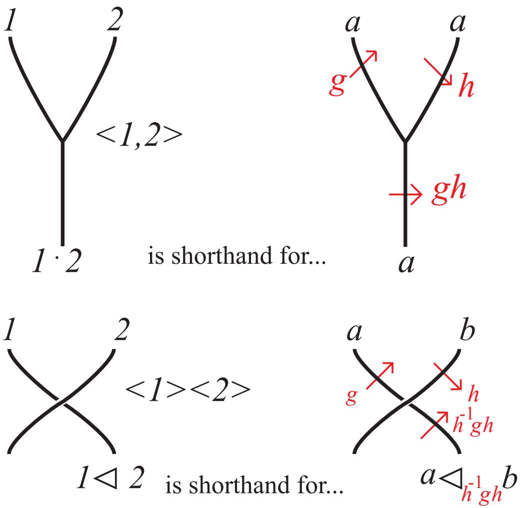
Lemma

Let (Q, G) denote a G family of quandles on Q. Then $G \times Q$ is a quandle under the binary operation $(g, a) \triangleleft (h, b) = (h^{-1}gh, a \triangleleft_h b)$.

This is called the *associated quandle* to the G-family.

In case $X = \{0, 1, 2\}$ $(= \mathbb{Z}/(3))$. Its automorphism group is $\mathbb{Z}/(2)$ which we write additively. Thus $a \triangleleft_1 b = 2b - a$ while $a \triangleleft_0 b = a$. defines a $\mathbb{Z}/(2)$-family of quandles.

The principle features of trivalent graphs are crossings and vertices. These are depicted here.



Partial Dictionary

$(a, b, c \in Q$, and $g, h, k \in G)$.

$$\langle 1, 2 \rangle \leftrightarrow ((a, g), (a, h))$$

$$\langle 1 \rangle \langle 2 \rangle \leftrightarrow ((a, g); (b, h))$$

$$\langle 1, 2, 3 \rangle \leftrightarrow ((a, g), (a, h), (a, k))$$

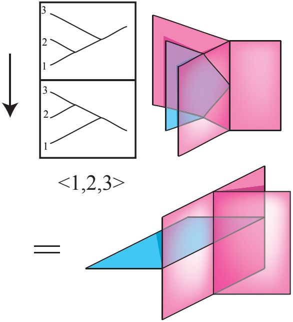
$$\langle 1, 2 \rangle \langle 3 \rangle \leftrightarrow ((a, g), (a, h); (b, k))$$

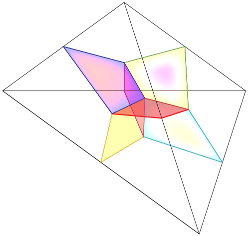
$$\langle 1 \rangle \langle 2, 3 \rangle \leftrightarrow ((a, g); (b, h), (b, k))$$

$$\langle 1 \rangle \langle 2 \rangle \langle 3 \rangle \leftrightarrow ((a, g); (b, h); (c, k))$$

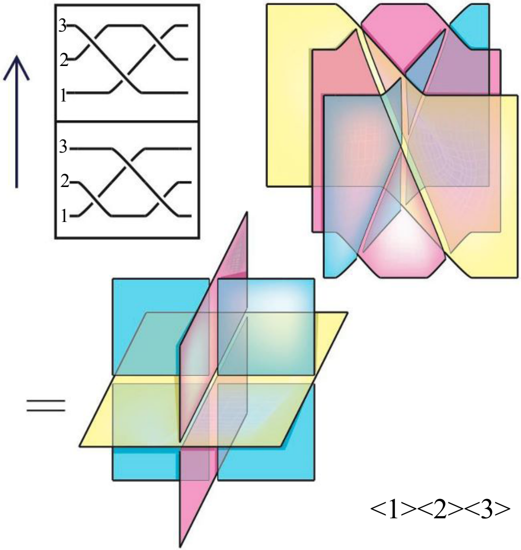
Analogy and definition

Classical knots are to knotted surfaces as knotted trivalent graphs are to knotted foams. Here I illustrate the space Y^2.





A 2-foam is a compact topological space for which every point has a nbhd. homeom. to a nbhd. of a point of Y^2.
A 2-foam has vertices, edges, and faces. Four edges are incident to a vertex. Three faces are incident to an edge. Six faces are incident at the vertex.



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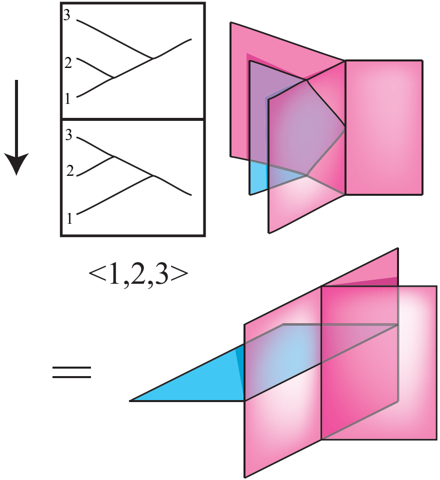
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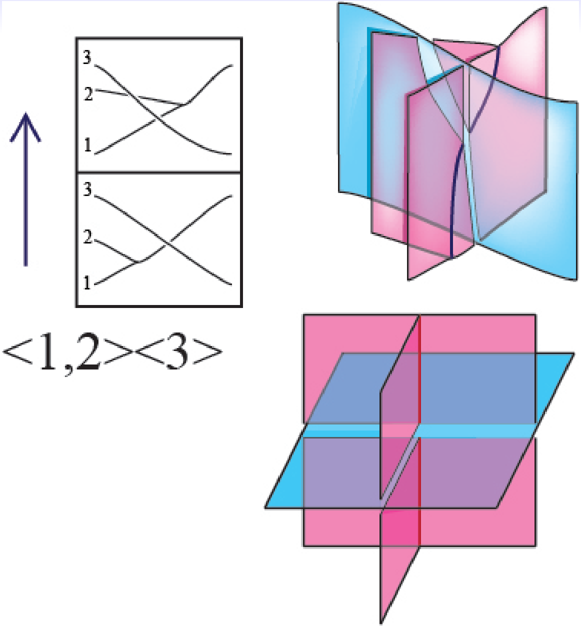
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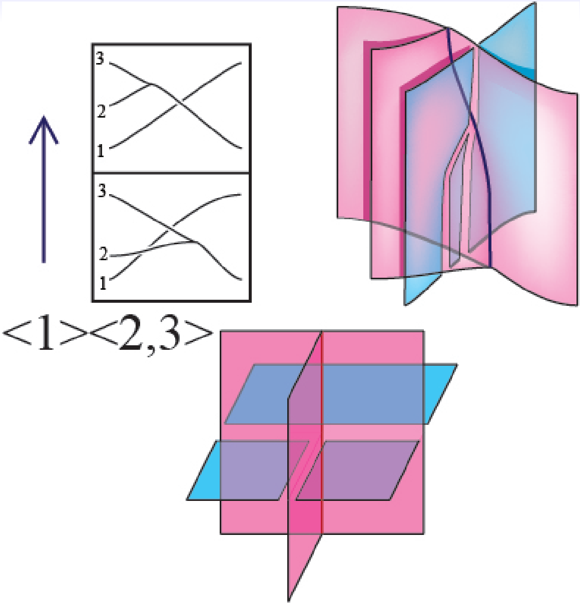
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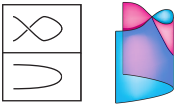
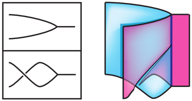
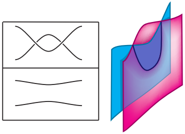
Leading remark

The next frame indicates some of the other ways of making a foam. These will not affect our homological calculations. Well, the branch point and the twisted vertex have to be accounted for. So we mod out by a degenerate subcomplex.







Boundaries

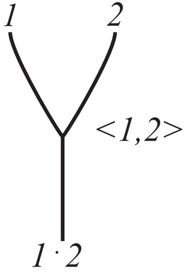
$$\begin{aligned}\partial\langle j+1, j+2, \ldots, j+k\rangle \\ = \ \lhd(j+1)\langle j+2, \ldots, j+k\rangle \\ + \sum_{\ell=1}^{k-1}(-1)^{\ell}\langle j+1, \ldots, (j+\ell)\cdot(j+\ell+1), \ldots, j+k\rangle \\ +(-1)^{k}\langle j+1, \ldots, j+k-1\rangle.\end{aligned}$$

$$\partial(PQ) = (\partial P)Q + (-1)^{\dim P} P(\partial Q).$$

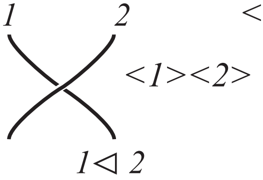
In part,

$$\partial\langle j+1\rangle = \lhd(j+1)\llcorner\!\lrcorner - \llcorner\!\lrcorner.$$

Boundaries



$$\partial\langle 1,2\rangle = \langle 2\rangle - \langle 1\cdot 2\rangle + \langle 1\rangle$$

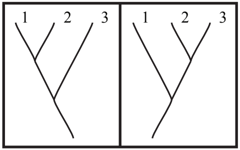


$$\partial\langle 1\rangle\langle 2\rangle = \langle 2\rangle - \langle 2\rangle - \langle 1 \triangleleft 2\rangle + \langle 1\rangle$$

Boundaries

<1,2>

<1 · 2, 3>

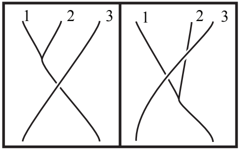


<2,3>

<1, 2 ·3>

$$\partial \langle 1,2,3\rangle = \langle 2,3\rangle - \langle 1\cdot 2, 3\rangle + \langle 1, 2\cdot 3\rangle$$
$$- \langle 1,2\rangle.$$

Boundaries



<1,2><3>

∂ <1,2><3> =

(∂<1,2>)<3> + <1,2>∂ <3>

=<2><3> -<1 · 2><3>

+ <1><3>

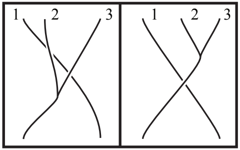
- <1,2> + <1◁3, 2◁3>

Boundaries

$<1><2>$

$<1 \lhd 2><3>$

$<2,3>$



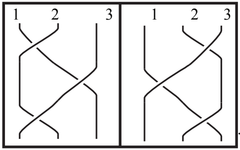
$<2,3>$

$<1><2\cdot 3>$

$<1><2,3>$

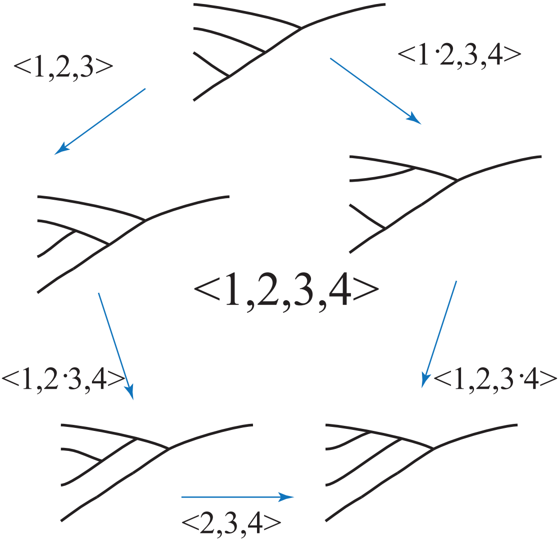
$$
\begin{aligned}
\partial <1><2,3> = \\
<2,3> - <2,3> \\
- <1><2> - \\
<1 \lhd 2><3> + \\
<1><2\cdot 3>
\end{aligned}
$$

Boundaries



∂ (<1><2><3>) =
<2><3>-<2><3>
-<1⊲2><3> +<1><3>
+<1⊲ 3><2⊲ 3>
- <1><2>

The boundaries of the expressions involving 4 variables correspond to the fundamental moves to foams.
In the next few slides, I will express these in movie move terms.



<1,2,3,4>
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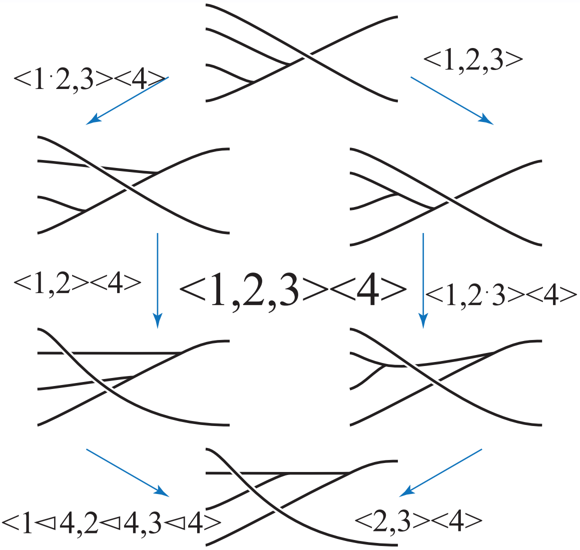
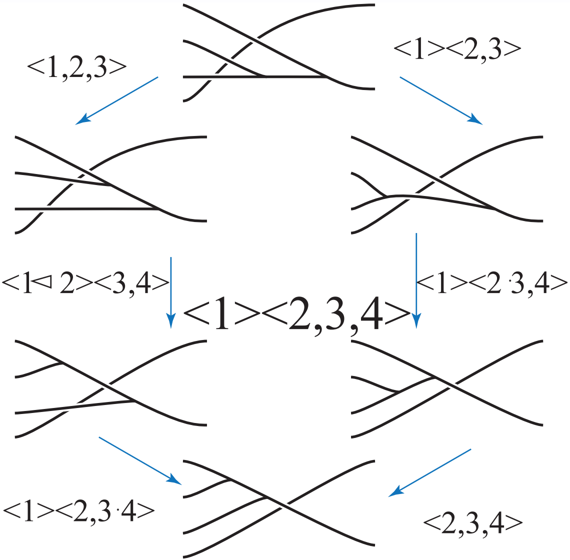
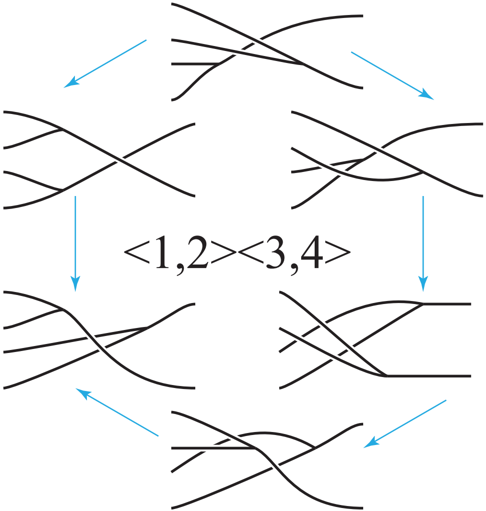
<1,2,3·4>
<2,3,4>



<1·2,3><4>
<1,2,3>
<1,2><4>
<1,2,3><4>
<1,2·3><4>
<1◁4,2◁4,3◁4>
<2,3><4>


<1,2,3>
<1><2,3>
<1⊲2><3,4>
<1><2·3,4>
<1><2,3,4>
<1><2,3·4>
<2,3,4>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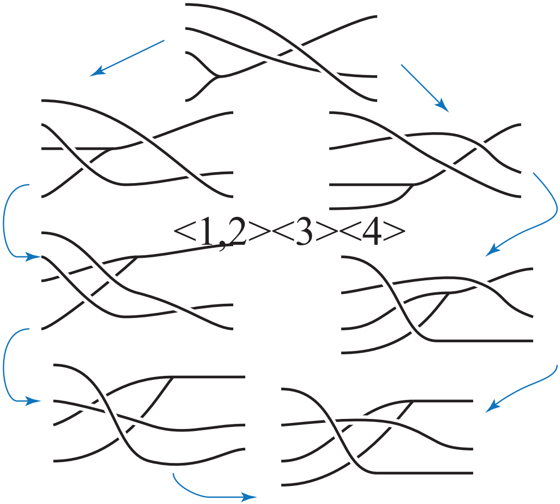
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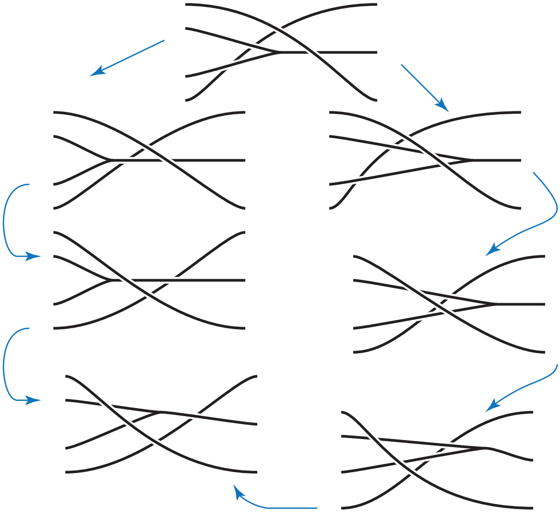

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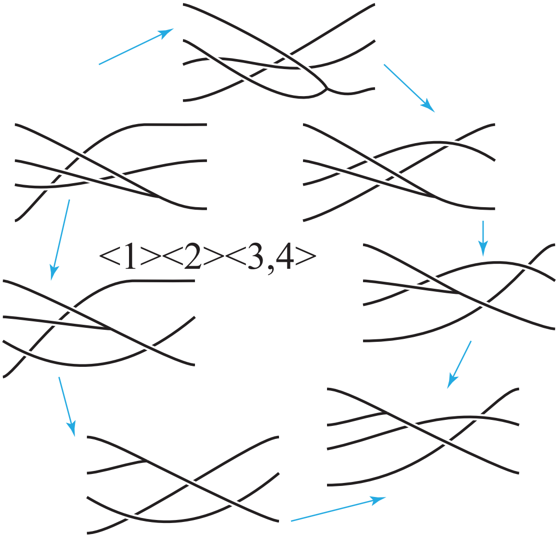
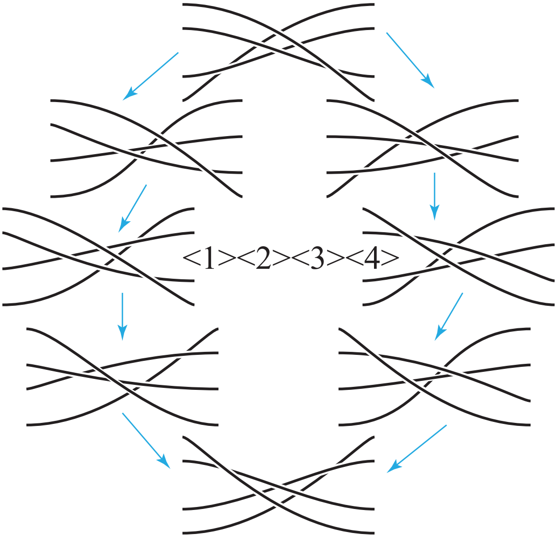

<1><2><3,4>


<1><2><3><4>

Boundaries, Commentary

In general, we can take boundaries of lin. combos. of exps. of the form:

$$\langle 1,\ldots,j_1\rangle\langle j_1+1,\ldots,j_1+j_2\rangle$$

$$\cdots\langle\sum_{\ell=1}^{k-1} j_\ell+1,\ldots,\sum_{\ell=1}^{k} j_\ell\rangle.$$

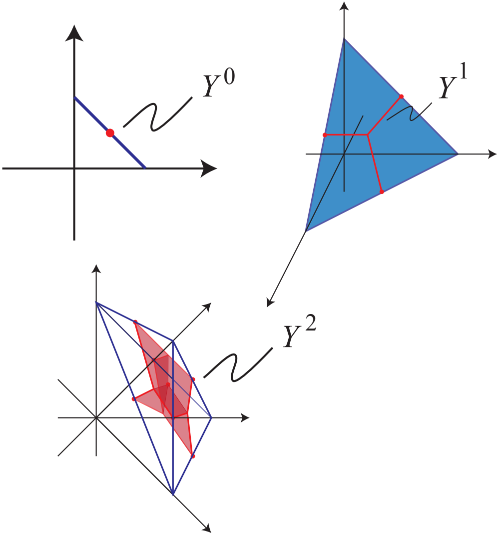
These expressions correspond to products of simplices. For example, Y corresponds to a triangle, X corresponds to a square, and the moves to trivalent graphs correspond to a tetrahedron, prism, prism, and cube.

The Space Y^n

Let $\Delta^{n+1} = \{\vec{x} \in \mathbb{R}^{n+2} : \sum x_i = 1 \ \& \ 0 \leq x_i\}$ denote the standard simplex. The space $Y^n \subset \Delta^{n+1}$ is defined as follows: $Y^0 = (\frac{1}{2}, \frac{1}{2})$. Take $\Delta^n_j = \{\vec{x} \in \Delta^{n+1} : x_j = 0\}$. Embed a copy, $Y^{n-1}_j \subset \Delta^n_j$. Cone $\cup_{j=1}^{n+2} Y^{n-1}_j$ to the barycenter $b = \frac{1}{n+2}(1, 1, \ldots, 1)$ of Δ^{n+1}.

$$Y^n = C\left(\cup_{j=1}^{n+2} Y^{n-1}_j\right).$$

Y^0, Y^1, and Y^2



Definition of n-foams

Every point $y \in Y^n$ has a nbhd. that is homeom. to $Y^{n-k} \times D^k$. The union of these points is called *the k-stratum* — the union of these is a set of $\binom{n+2}{k}$ disks of dimension k, for $k = 1, \ldots, n$. An *n-foam* is a top. sp. X for which each pt. $x \in X$ has a nbhd homeom. to a nbhd. of a point in Y^n.

Local pictures of knottings of an n-foam

Let $(j_1, j_2, \ldots, j_k)$ denote an ordered partition of $n+1$.
For example, when $n+1=3$, the partitions are (3),(2, 1), (1, 2), (1, 1, 1). When $n=4$, the partitions are (4), (3, 1), (1, 3), (2, 2), (2, 1, 1), (1, 2, 1), (1, 1, 2), and (1, 1, 1, 1).
For each such partition, we construct a local picture of a crossing as follows:

Local pictures of knottings

Let $(j_1, j_2, \ldots, j_k)$ denote an ordered partition of $n+1$. We write

$$\langle 1, \ldots, j_1 \rangle \langle j_1 + 1, \ldots, j_1 + j_2 \rangle$$

$$\cdots \langle \sum_{\ell=1}^{k-1} j_\ell + 1, \ldots, \sum_{\ell=1}^{k} j_\ell \rangle,$$

and consider $\prod_{\ell=1}^{k} \Delta^{j_\ell}$ as a fixed embedding in $\mathbb{R}^{n+1}$. Embed

$$Y_\ell^{j_\ell - 1} \subset \Delta^{j_\ell}.$$

...

Local pictures of knottings

Now take

$$\left[\cup_{\ell=1}^{k}\left(\Delta^{j_1}\times\cdots\times Y^{j_\ell-1}\times\cdots\times\Delta^{j_k}\right)\subset\mathbb{R}^{n+1}\times\{\ell\}\right],$$

and project this into $\mathbb{R}^{n+1}$. The factor ℓ is in the $(n+2)$nd coordinate and represents the relative height of each Y.

$$\left[\left(Y^{j_1-1} \times \Delta^{j_2} \cdots \times \Delta^{j_\ell} \times \cdots \times \Delta^{j_k}\right) \subset \mathbb{R}^{n+1} \times \{1\}\right],$$

$$\left[\left(\Delta^{j_1} \times Y^{j_2-1} \cdots \times \Delta^{j_\ell} \times \cdots \times \Delta^{j_k}\right) \subset \mathbb{R}^{n+1} \times \{2\}\right],$$

$$\cdots$$

$$\left[\left(\Delta^{j_1} \times \Delta^{j_2} \cdots \times Y^{j_\ell-1} \times \cdots \times \Delta^{j_k}\right) \subset \mathbb{R}^{n+1} \times \{\ell\}\right],$$

$$\cdots$$

$$\left[\left(\Delta^{j_1} \times \Delta^{j_2} \cdots \times \Delta^{j_\ell} \times \cdots \times Y^{j_k-1}\right) \subset \mathbb{R}^{n+1} \times \{k\}\right].$$

These project to a 0-dimensional multiple point in $\mathbb{R}^{n+1}$.

Definition of n-foams

Every point $y \in Y^n$ has a nbhd. that is homeom. to $Y^{n-k} \times D^k$. The union of these points is called *the k-stratum* — the union of these is a set of $\binom{n+2}{k}$ disks of dimension k, for $k = 1, \ldots, n$. An *n-foam* is a top. sp. X for which each pt. $x \in X$ has a nbhd homeom. to a nbhd. of a point in Y^n.

Local pictures of knottings of an n-foam

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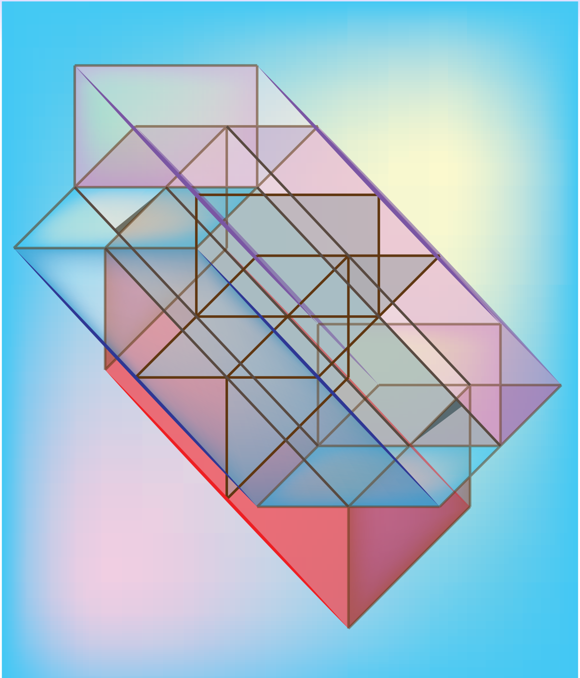
For example, when $n+1=3$, the partitions are (3),(2, 1), (1, 2), (1, 1, 1). When $n=4$, the partitions are (4), (3, 1), (1, 3), (2, 2), (2, 1, 1), (1, 2, 1), (1, 1, 2), and (1, 1, 1, 1).
For each such partition, we construct a local picture of a crossing as follows:



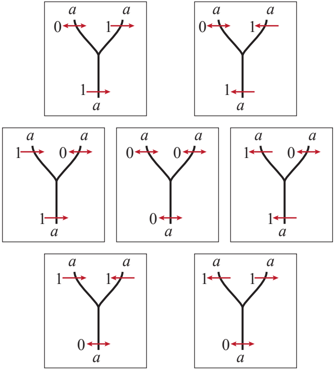
Theorem

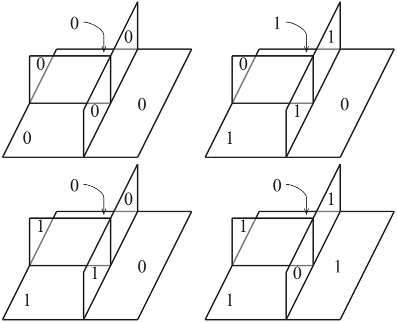
There is a non-trivial cocycle invariant of knotted foams.

Example

Let $G = \mathbb{Z}_2 = \{0, 1\}$, let $X = \{0, 1, 2\}$, and let the associated quandle be $Q = X \times G$. The quandle actions on R_3 are $a \triangleleft_0 b = a$, $a \triangleleft_1 b = 2b - a$, and the action on Q is $(a, g) \triangleleft (b, h) = (a \triangleleft_h b, g)$.

Coloring rules





Example

Mochizuki's 3-cocycle $\theta_p : X^3 \to \mathbb{Z}_3$ can be written as

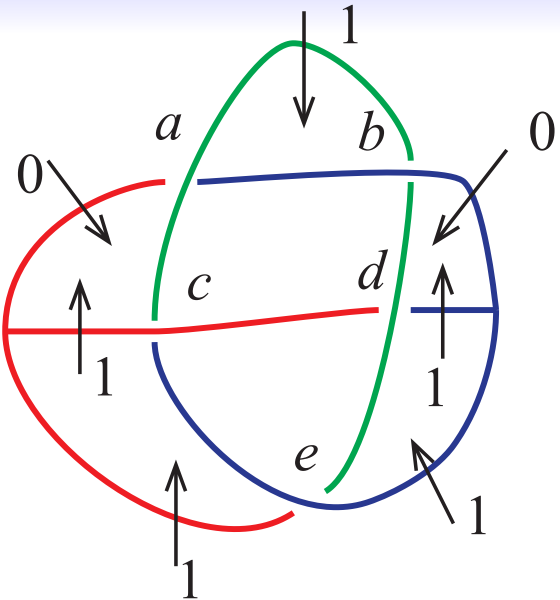
$$\theta_3(a, b, c) := (a - b)(c^3 + c^2 b + b^2 c).$$

Define

$$\theta((x_1, g_1), (x_2, g_2), (x_3, g_3))$$
$$= \begin{cases} \theta_3(x_1, x_2, x_3) & \text{if } \ g_1 = g_2 = g_3 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma

The function θ satisfies the cocycle conditions.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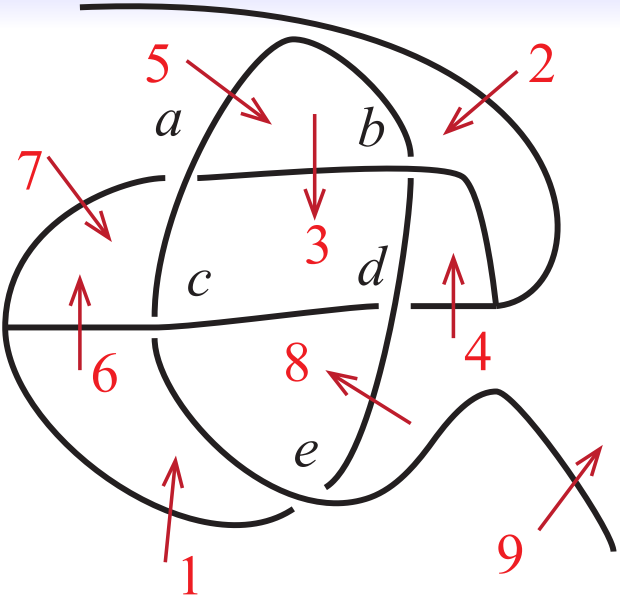

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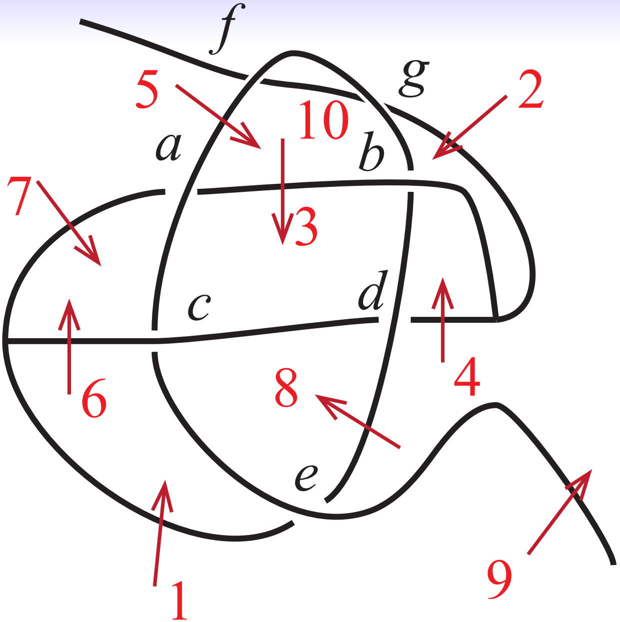
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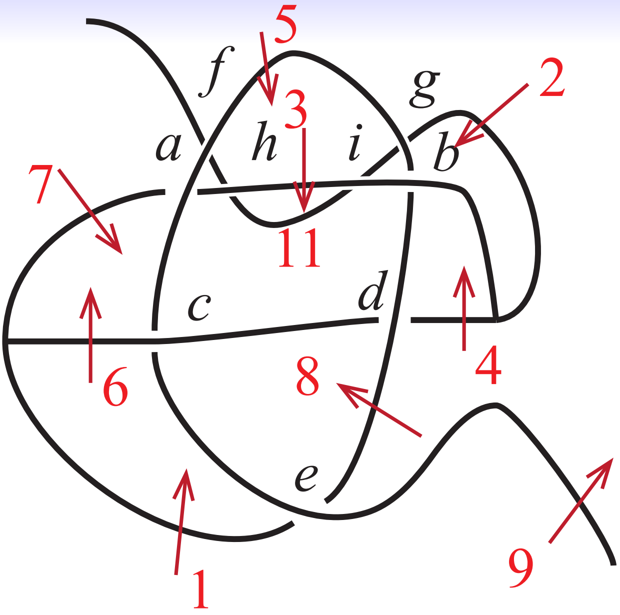

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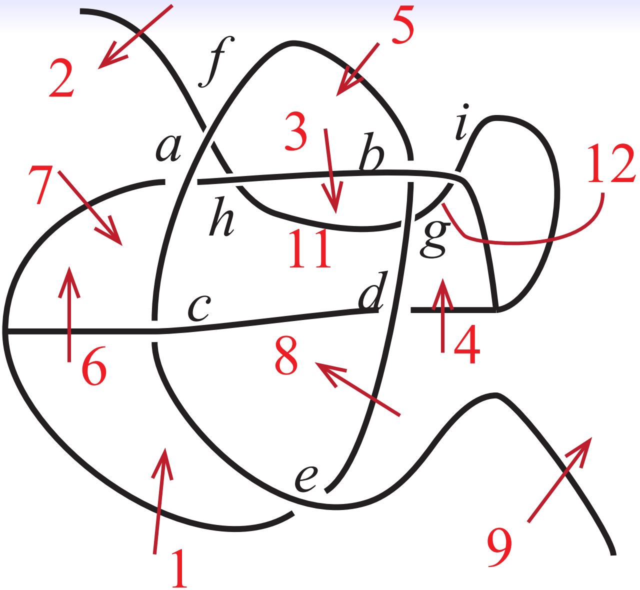

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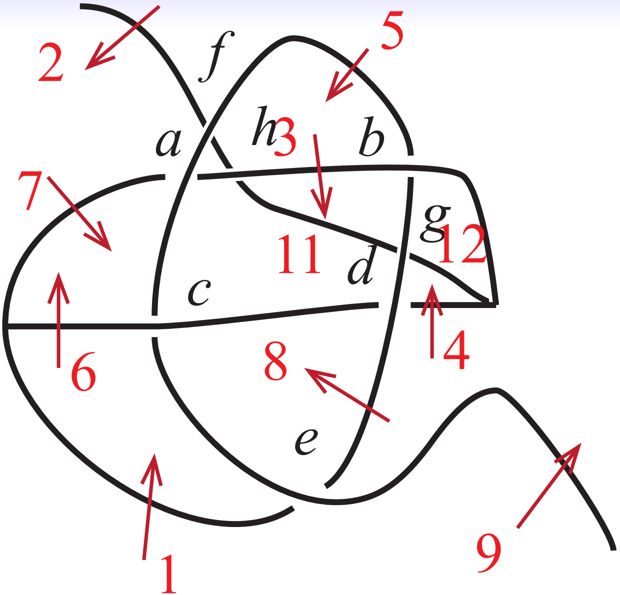

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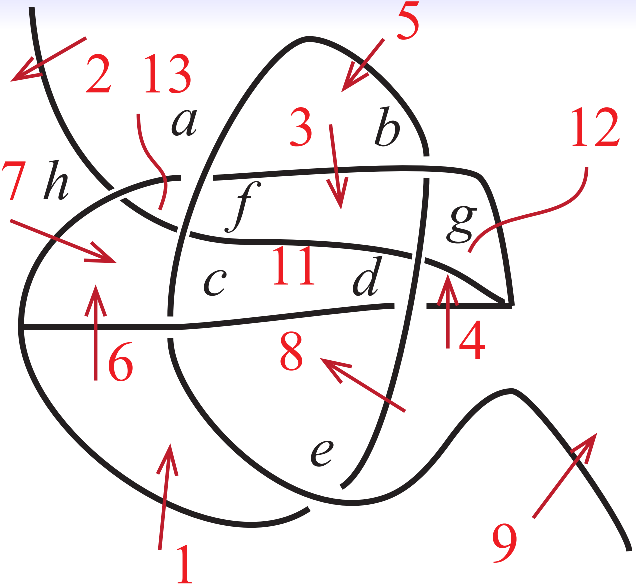

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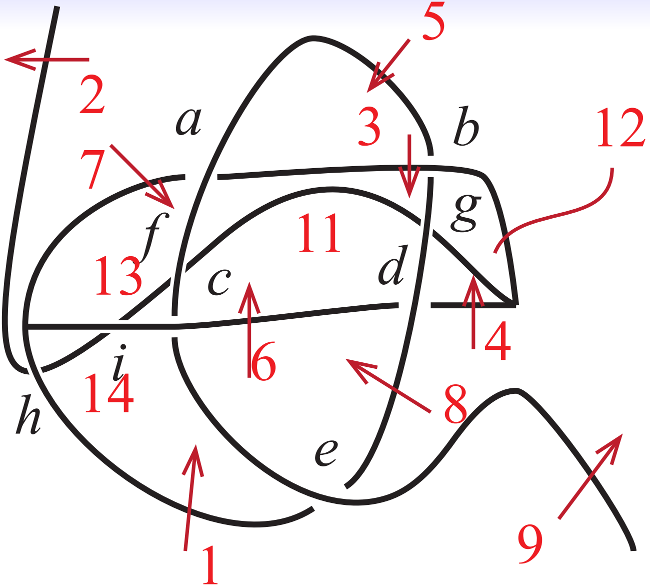

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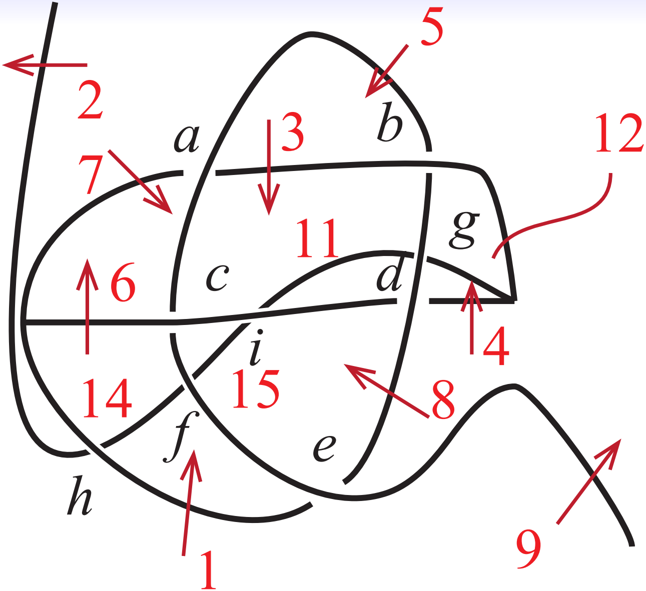

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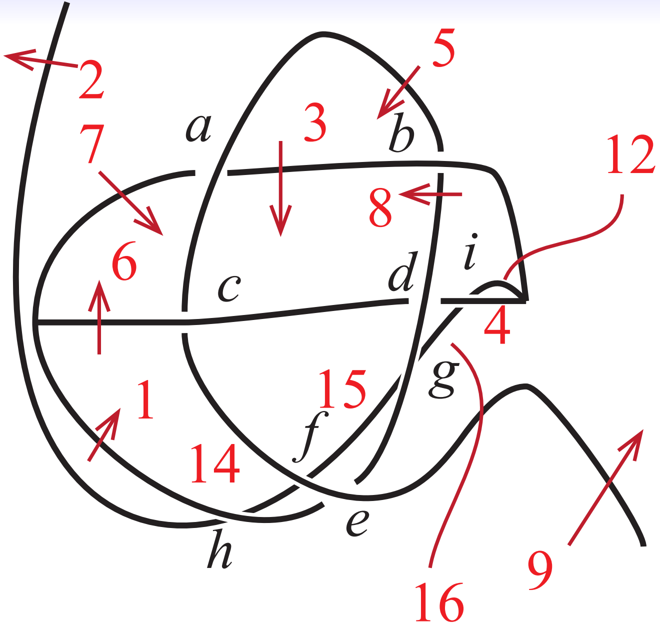

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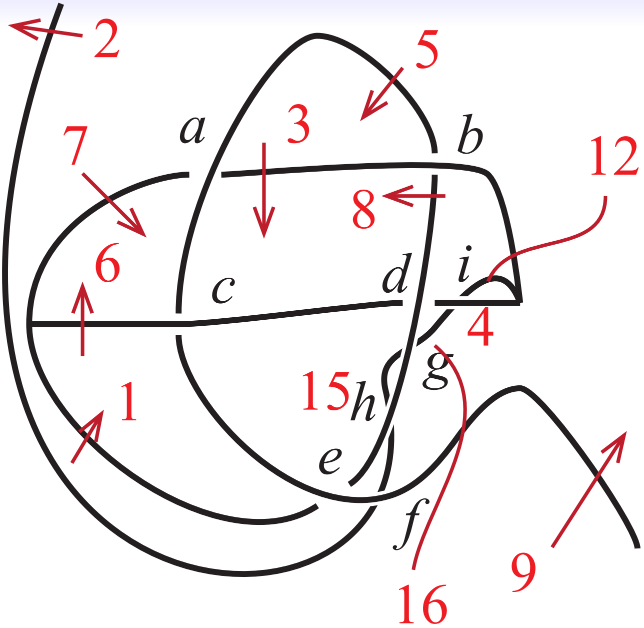
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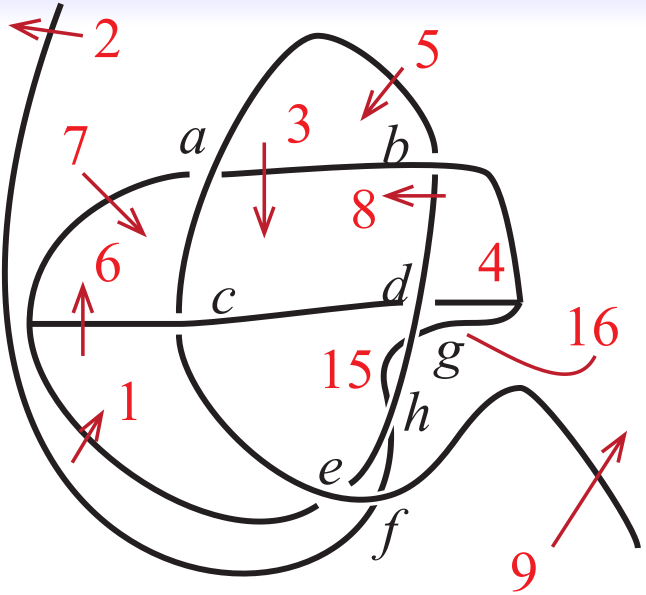

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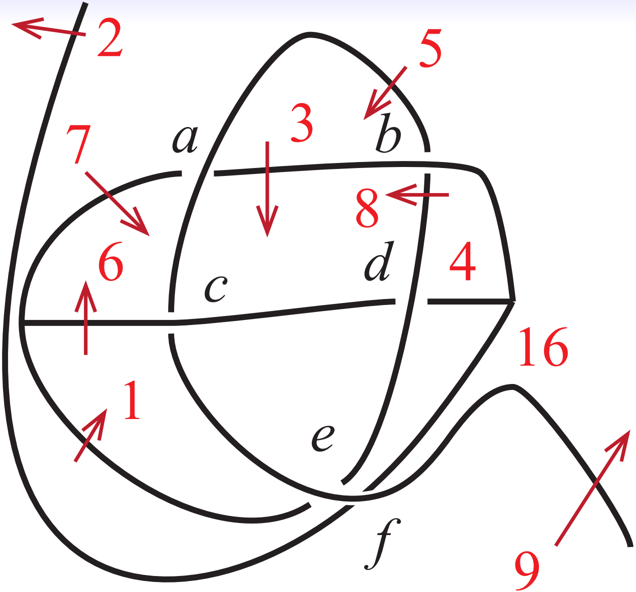
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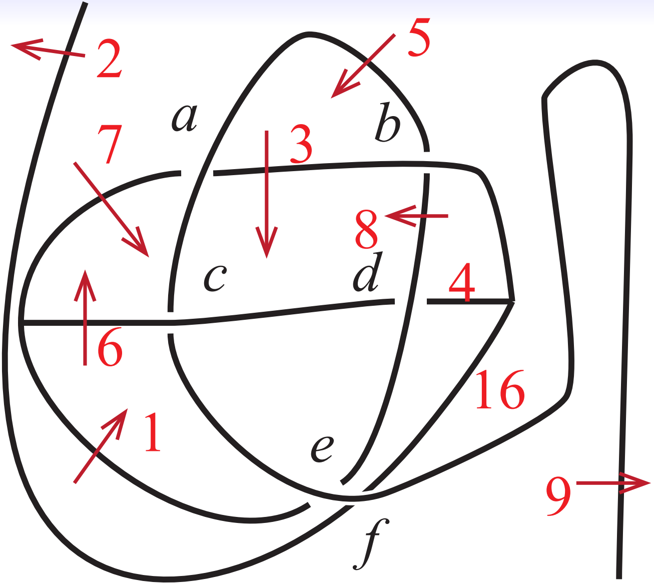

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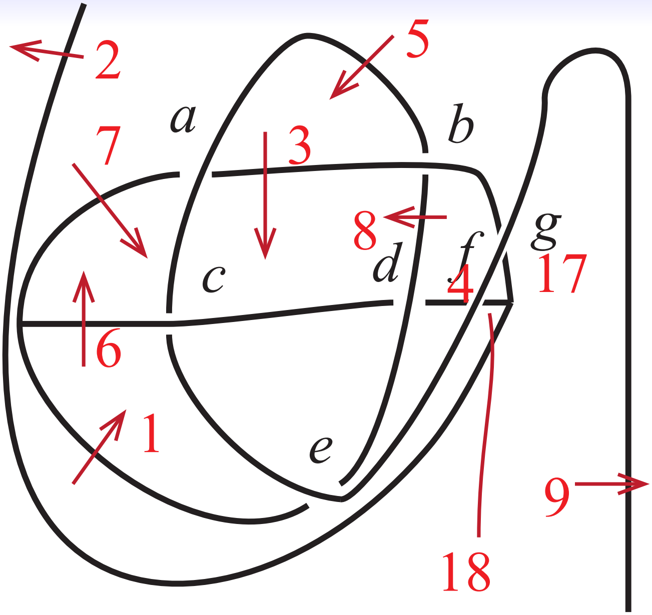

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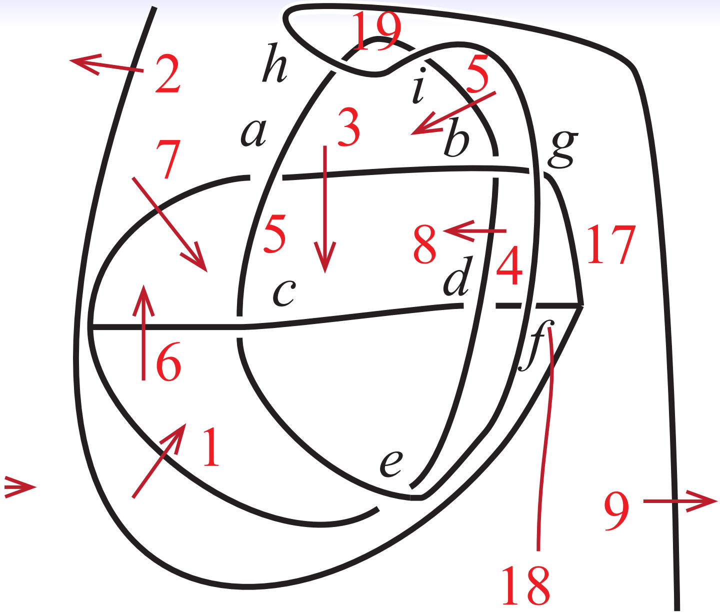

19
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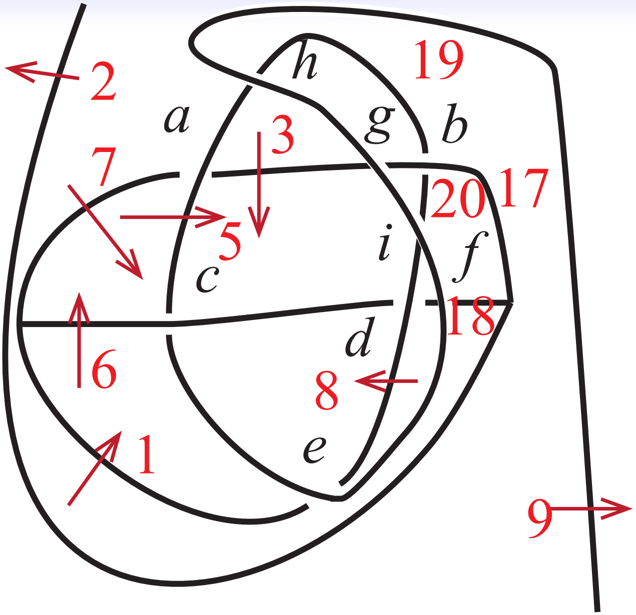
18



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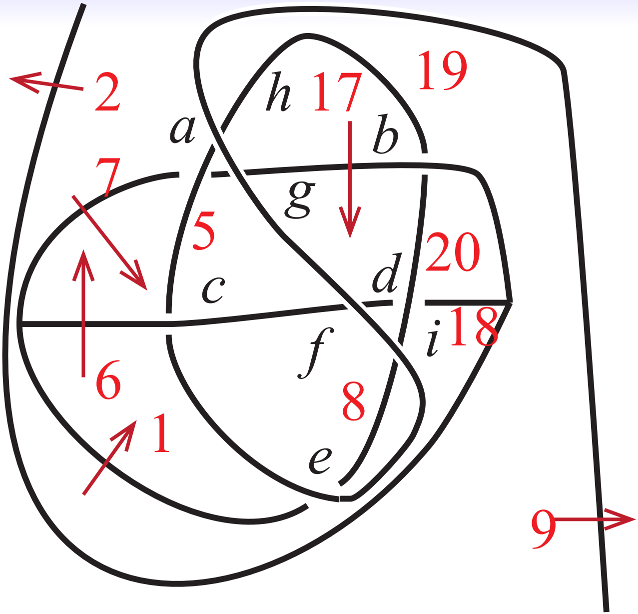

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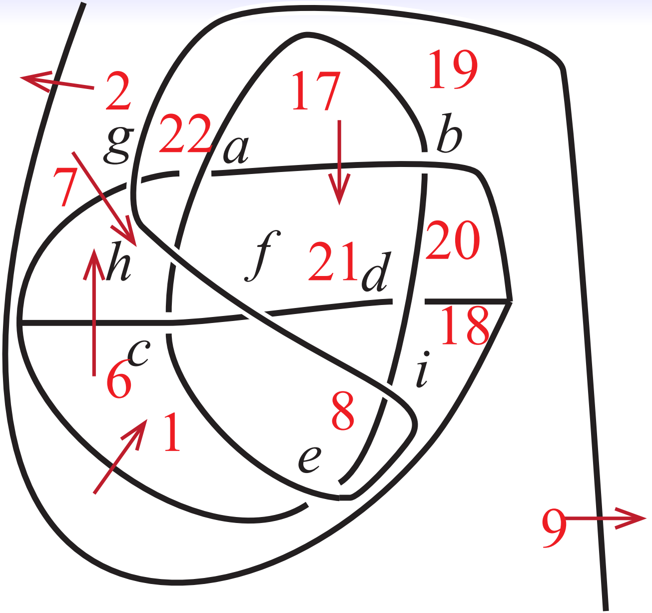
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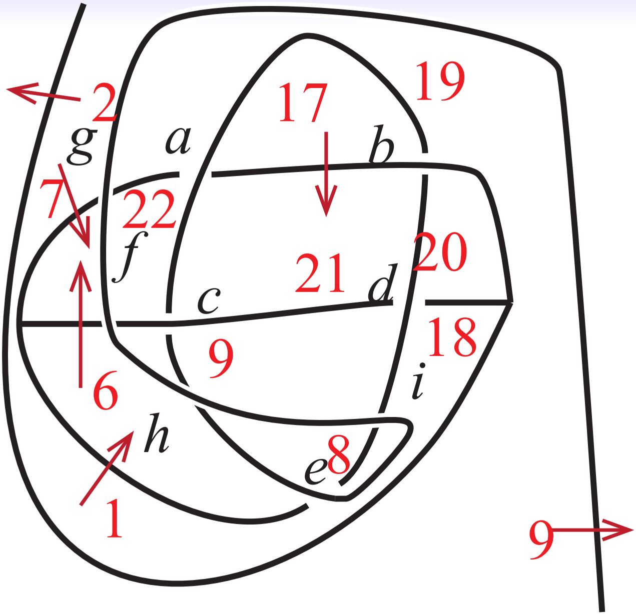

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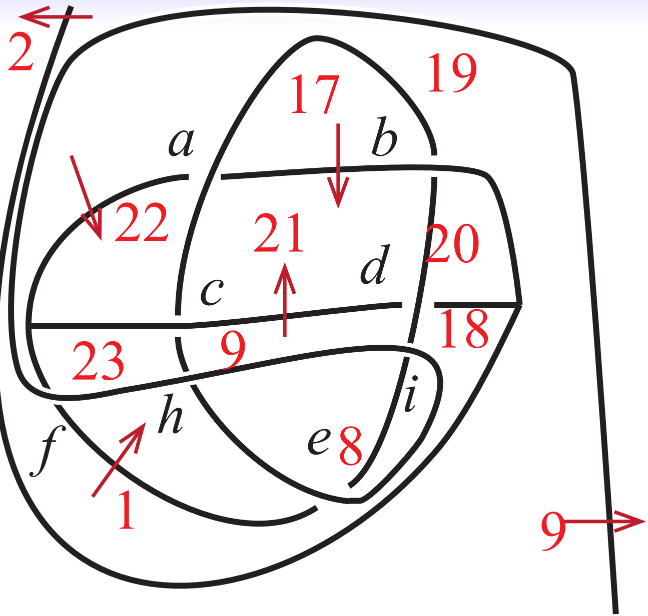
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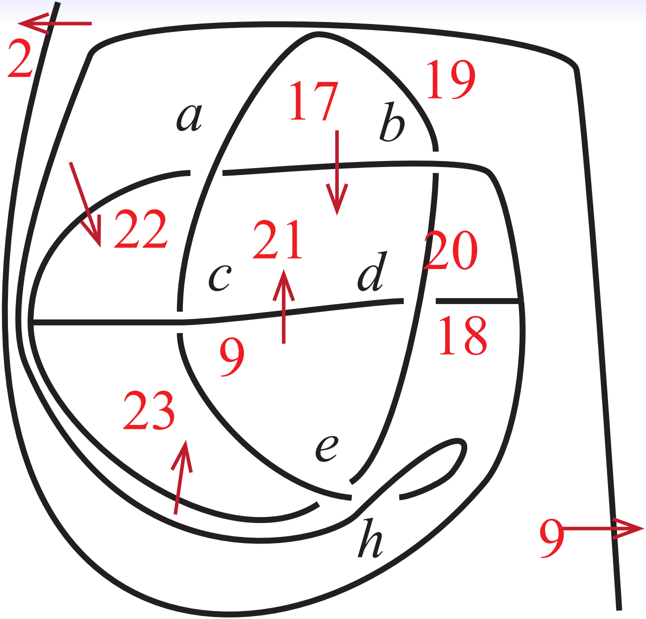

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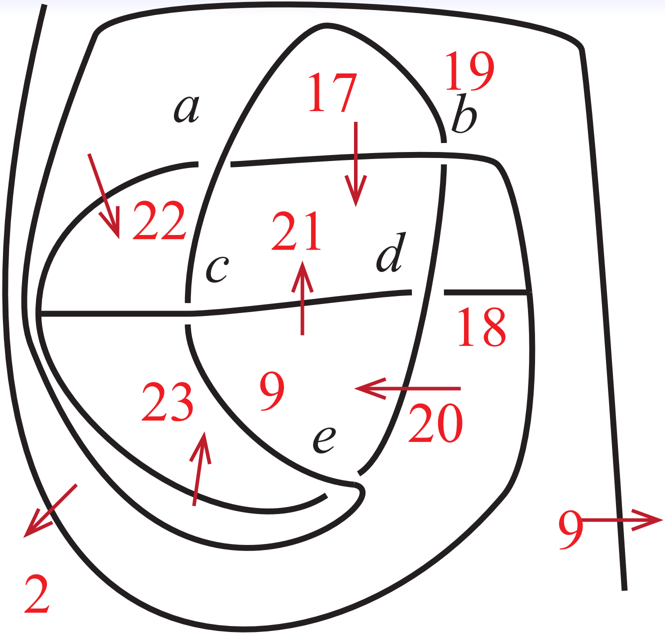

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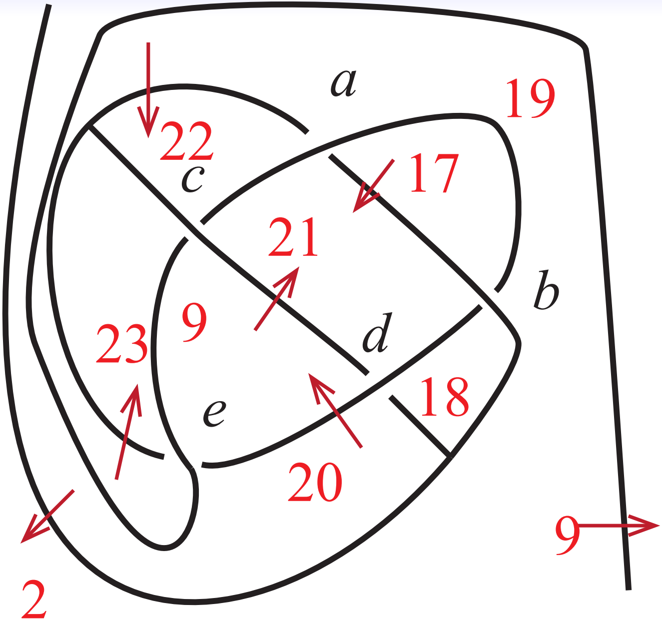
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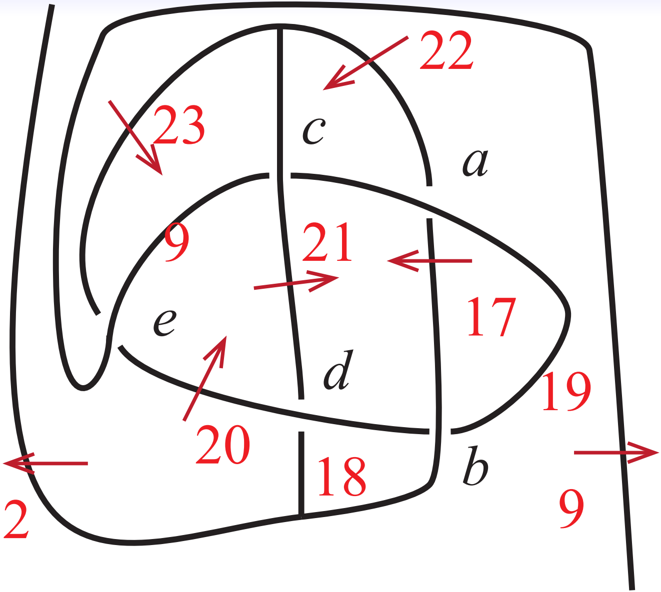
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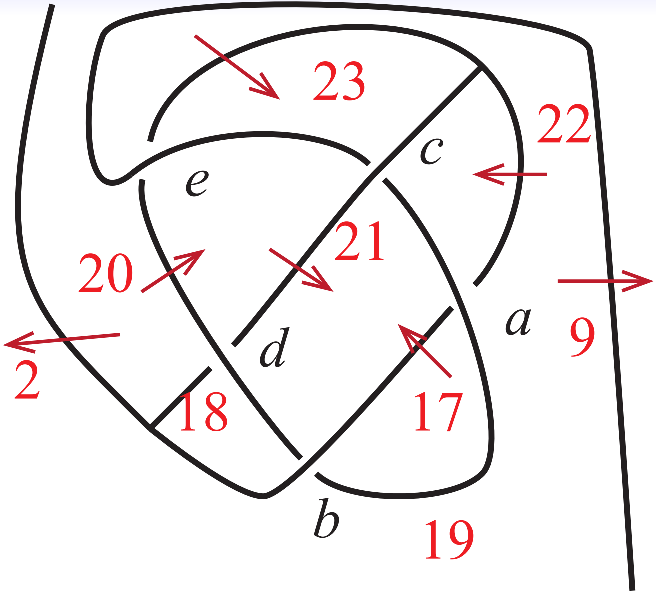

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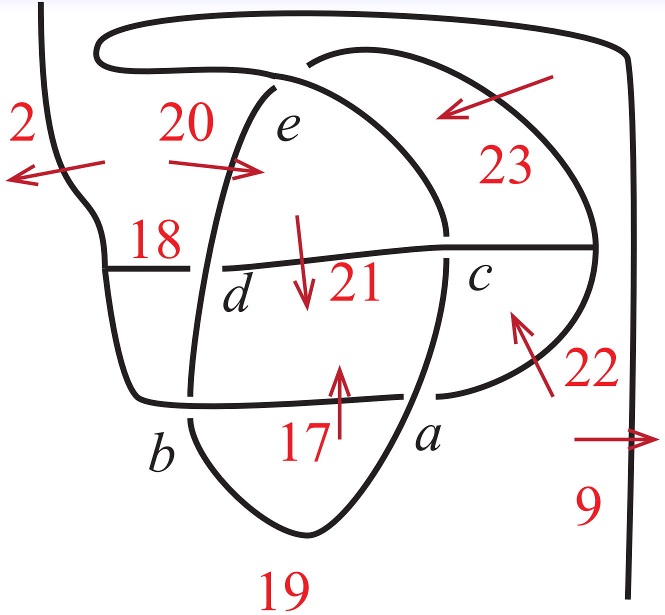

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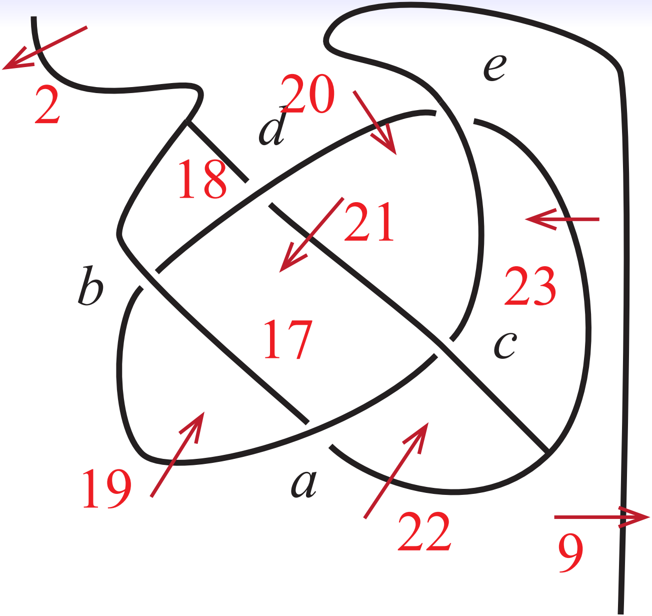
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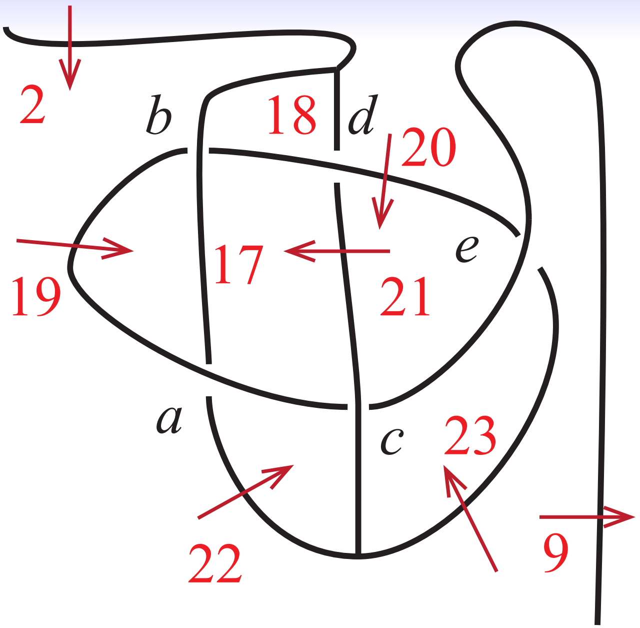

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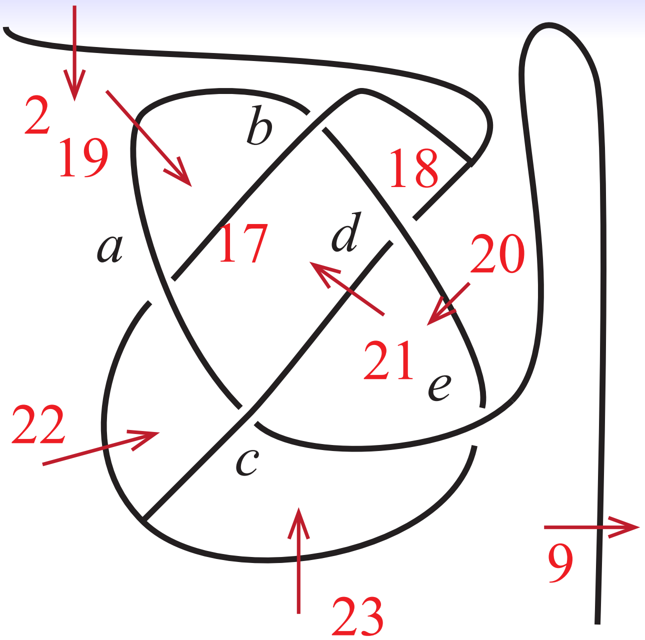

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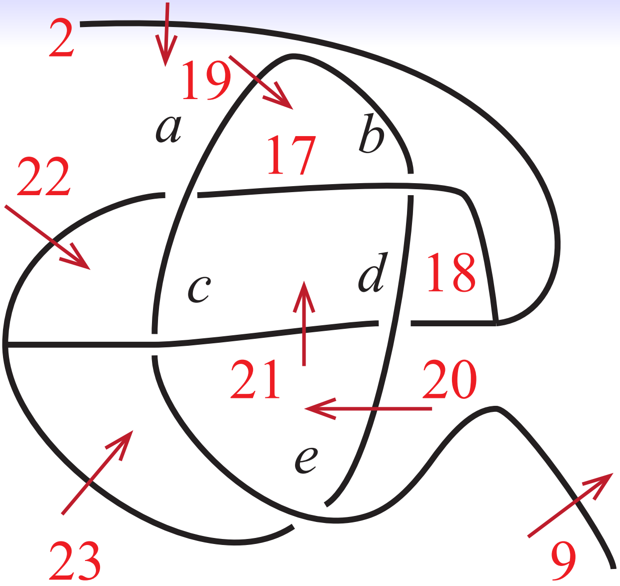
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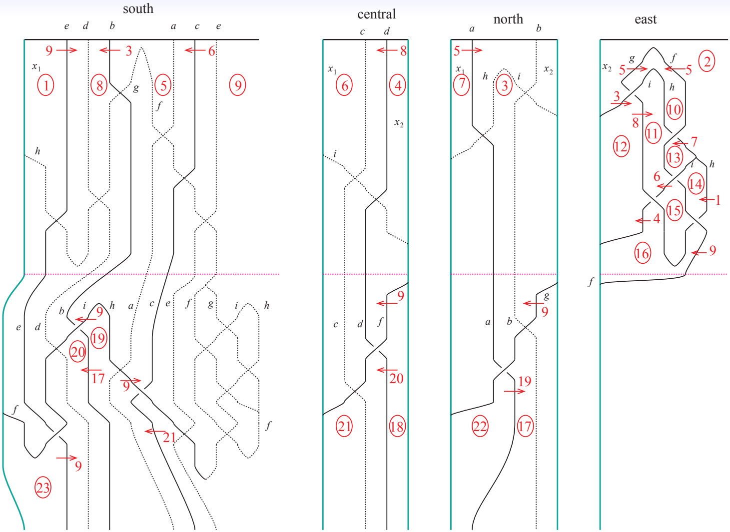
south
central
north
east

- The above movie was the first twist of a 2-twist-spin of the knotted trivalent graph 5_2.
- This is an example of a knotted foam.
- We have computed that the corresponding knotted foam has a non-trivial cocycle invariant by using a variation upon Mochizuki's 3-cocycle.

Suggested Further Reseach

- Spinning trivalent graphs — fibred?
- Use movies to construct interesting examples
- Use Nosaka and Inoue-Kabaya techniques to find more interesting cocycles
- Optimal number of crossings of each type
- Relations among different crossing types
- Higher dimensional knottings of foams

Thank you very much

감사합니다

ありがとう

謝謝

Obrigado