TQFT and semi-classical limit

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3-dimensional quantum invariant

X an oriented closed 3-manifold, $G = \mathrm{SU}(2)$.

Witten's definition:

$$Z_k^W(X) = \int_{\Omega^1(X,\mathfrak{g})} e^{2i\pi k\, \mathrm{CS}_X(a)} da$$

with CS_X the Chern-Simons functional.

Witten (89), *Quantum Field theory and the Jones polynomial*.

Combinatorial definition:

$$Z_k^{RT}(X) = \left(\frac{\sqrt{2}\sin(\pi/k)}{k^{1/2}}\right)^{n+1} \tau_k^{-\operatorname{sgn}(A_L)} \sum_{\ell \in \{1,\dots,k-1\}^n} \left([\ell]! J_\ell^L\right)_{t=-e^{i\pi/2k}}$$

if $X = S^3(L)$ with $\pi_0(L) = \{L_1, \dots, L_n\}$ and

- J_ℓ^L the colored Jones polynomial
- $[\ell]!$ a factorial of quantum integers
- A_L the linking matrix and $\tau_k = \exp(3i\pi/4 - 3i\pi/2k)$.

Reshetikhin-Turaev (91), *Invariants of 3-manifolds via link polynomials and quantum groups*.
Blanchet-Habegger-Masbaum-Vogel (95), *Quantum field theories derived from the Kauffman bracket*

Witten asymptotic conjecture

Let X be an oriented closed 3-manifold. Then

$$Z_k^W(X) \sim \sum_{P \in \mathcal{M}(X)} e^{i\frac{\pi}{2}\eta(P)} |\tau(P)|^{\frac{1}{2}} e^{2i\pi k \,\mathrm{CS}_X(P)}$$

where

- $\mathcal{M}(X) = \{\text{flat } G\text{-principal bundle over } X\}/\sim$ and
- $\eta(P)$ and $\tau(P)$ are the eta invariant and the analytic torsion of the flat vector bundle $P \times_{\mathrm{ad}} \mathfrak{g}$.
- $\mathrm{CS}_X(P)$ is the Chern-Simons invariant of P.

Question: Has $Z_k^{RT}(X)$ this asymptotic behavior ?

Proof of Witten conjecture

- Lens spaces and mapping torus of hyperbolic diffeomorphism of the torus (Jeffrey, 92)
- Seifert manifolds (Rozanski 95-96, Hansen 05)
- Brieskorn homology sphere (Hikami 05)
- Mapping torus of finite order diffeomorphism of surface with genus $\geqslant 2$ (Andersen).
- Mapping torus of diffeomorphism of surface with genus $\geqslant 2$ satisfying a transversality assumption (C 10).

Theorem (Marché-C)

Let X be the manifold obtained by surgery with parameter p/q on the figure eight knot. We assume that p is not divisible by 4 and that for any $\rho \in \mathcal{X}(M)$, $H^1(M, \mathrm{Ad}_\rho) = 0$. Then

$$Z_k^{RT}(X) = \sum_{\rho \in \mathcal{M}(M)} e^{i\frac{m(\rho)\pi}{4}} k^{n(\rho)} a(\rho) e^{ik\,\mathrm{CS}(\rho)} + O(k^{-1})$$

where for any $\rho \in \mathcal{M}(X)$, $m(\rho)$ is an integer, $n(\rho) = 0$, $-1/2$ or $-3/2$ according to ρ is irreducible, no central abelian or central and

$$a(\rho) = \begin{cases} 2^{-1}\big(\mathbb{T}(\rho)\big)^{1/2} & \text{if } \rho \text{ is irreducible} \\ 2^{-1/2}\big(\mathbb{T}(\rho)\big)^{1/2} & \text{if } \rho \text{ is non-central abelian} \\ 2^{1/2}\pi/p^{3/2} & \text{if } \rho \text{ is central.} \end{cases}$$

with $\mathbb{T}(\rho)$ the Reidemeister torsion.

Topological Quantum Field Theory

TQFT with group $SU(2)$ and level $k \in \mathbb{N}^*$

- Closed surface $\Sigma^2 \to$ Hermitian space $V_k(\Sigma)$
- Manifold $M^3 \to Z_k(M) \in \begin{cases} \mathbb{C} \text{ if } \partial M = \emptyset \\ V_k(\partial M) \text{ otherwise} \end{cases}$

If M and N have the same boundary Σ,

$$Z_k(M \cup_\Sigma N^-) = \left\langle Z_k(M), Z_k(N) \right\rangle_{V_k(\Sigma)}$$

Knot state

The state of a knot K in S^3 is

$$Z_k(E_K) \in V_k(\Sigma)$$

where E_K is ($S^3 \backslash$ tubular neighborhood of K) and $\Sigma = \partial E_K$ is the peripheral torus.

If $X = E_K \cup_\Sigma N$ with N a solid torus with boundary Σ,

$$Z_k(X) = \left\langle Z_k(E_K), Z_k(N) \right\rangle_{V_k(\Sigma)}.$$

Torus quantization and theta functions

(μ, λ) positive basis of $H_1(\Sigma, \mathbb{Z})$, (p, q) associated linear coordinates, $z = p + \tau q$ complex coordinate.

$$\mathcal{H}_k^{\text{alt}} = \left\{ f\, t^k \Big/ \begin{array}{l} f : \mathbb{C} \to \mathbb{C} \text{ holomorphic} \\ f(z+1) = f(z) \\ f(z+\tau) = f(z) e^{2i\pi k(2z+\tau)} \\ f(-z) = -f(z) \end{array} \right\}$$

where $t = e^{2i\pi(p+\tau q)q}$.
Then $(\Psi_\ell - \Psi_{-\ell}, 1 \leqslant \ell \leqslant k-1)$ is a basis of $\mathcal{H}_k^{\text{alt}}$ where

$$\Psi_\ell(z) = \Theta_{k,\tau}\left(z + \frac{\ell}{2k}\tau\right) e^{2i\pi(\ell z + \tau \frac{\ell^2}{4k})} t^k(z)$$

with

$$\Theta_{\tau,k}(z) = \sum_{n \in \mathbb{Z}} e^{4i\pi zkn + 2i\pi n^2 k\tau}$$

Knot state norms

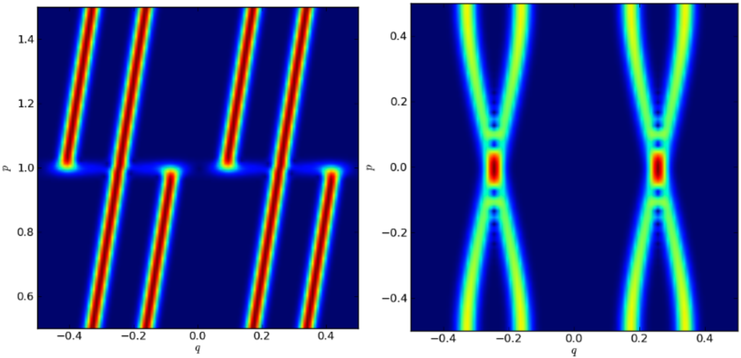

Figure: The pointwise norm of the knot state for the trefoil and the figure eight knot at level $k = 200$

Knot state asymptotics

Theorem (Marché-C;C)

Let K be the figure eight knot or a torus knot. Then for any $\rho \in \mathcal{M}(\Sigma)$, we have

- *if* $\rho \notin r(\mathcal{M}(E_K))$, $Z_k(E_K)(\rho) = \mathcal{O}(k^{-\infty})$.
- *if* $r^{-1}(\rho) = \{\rho_1, \ldots, \rho_N\}$ *and* ρ_j *is irreductible for any* j*, then*

$$Z_k(E_K)(\rho) \sim k^{3/4} \sum_{j=1,\ldots,N} e^{im_j\pi/4} \frac{\sqrt{\mathbb{T}(\rho_j)}}{4\pi^{3/4}} e^{2i\pi k\,\mathrm{CS}(\rho_j)}$$

  where the m_j *are integers.*

- *if* $r^{-1}(\rho) = \{\rho_0\}$ *and* ρ_0 *regular abelian, then*

$$Z_k(E_K)(\rho) \sim k^{1/4} e^{im\pi/4} \frac{\sqrt{\mathbb{T}(\rho_0)}}{2^{3/2}\pi^{3/4}} e^{2i\pi k\,\mathrm{CS}(\rho_0)}.$$

q-difference relation, $q = e^{-2i\pi/k}$

The knot state satisfies:

$$Q_k Z_k(E_K) = R_k Z_k^0,$$

where if K is the figure eight knot

$$\begin{aligned} Q_k =& (q^{-1}M^2 - qM^{-2})L + (qM^2 + q^{-1}M^{-2})L^{-1} \\ & + (M^2 - M^{-2})(-M^4 - M^{-4} + M^2 + M^{-2} + q^2 + q^{-2}), \\ R_k =& (M^5 + M^{-5} + M^3 + M^{-3} - (q^2 + q^{-2})(M + M^{-1})) \end{aligned}$$

and if K is the torus knot with parameter (a, b)

$$Q_k = \mathrm{id} - (q^{-1}M)^{-2ab}L^{-2}, \qquad R_k = \sum_{i=1}^{4} \epsilon_i q^{\epsilon_i}(q^{-1}M)^{-ab+p_i}$$

with $(\epsilon_i, p_i) = (1, -a - b)$, $(-1, -a + b)$, $(1, a + b)$, $(-1, a - b)$ for $i = 1, 2, 3, 4$.

Bibliography

1. *Knot state asymptotics I, AJ Conjecture and abelian representations*, arXiv:1107.1645, Marché-C.
2. *Knot state asymptotics II, Witten conjecture and irreducible representations* , arXiv:1107.1646, Marché-C.
3. *Torus knot state asymptotics*, arXiv:1107.4692, C.