Observables in two-dimensional BF theory

Sara Oriana Gomes Tavares
University of Nottingham

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Introduction and Motivation

We start with Witten's model for Yang-Mills in two dimensions (1988). The action can be written in a way that is only measure (not metric) dependent:

$$S[A] = -\frac{1}{4q^2}\int_\Sigma \mathrm{Tr}(F^2) = -\frac{1}{2q^2}\int_\Sigma d\mu \mathrm{Tr}(f^2)$$

- Σ is a 2d compact manifold not necessarily orientable. We also assume a principal G-bundle E over it;
- A is a $\mathfrak{g}$-valued connection, where $\mathfrak{g}$ is the Lie algebra of G;
- F is the curvature of A ($\mathrm{Ad}(E)$-valued two-form). One can always write $F = \rho f$, where ρ is a volume form (or density) and f and $\mathrm{Ad}(E)$-valued function;
- Tr: quadratic form on $\mathfrak{g}$.

Note: The explicit original metric dependence can be seen by writing $\int_\Sigma \mathrm{Tr}(F^2) = \int_\Sigma d\mu g^{ik} g^{jl} \mathrm{Tr}(F_{ij}F_{kl})$ (implicit summation).

Consider now a related action

$$I[A,B] = -i\int_\Sigma \mathrm{Tr}(BF) - \frac{q^2}{2}\int_\Sigma d\mu \mathrm{Tr}(B^2)$$

The two actions give rise to the same quantum theory in the sense the 'volume of the moduli space of flat connections' or 'vacuum expectation value' is the same:

$$Z(\Sigma) = \int DADBe^{-i\int_\Sigma \mathrm{Tr}(BF) - \frac{q^2}{2}\int_\Sigma d\mu \mathrm{Tr}(B^2)} = \int DAe^{-\frac{1}{4q^2}\int_\Sigma \mathrm{Tr}(F^2)}$$

Note: Heuristically one obtains the identity by 'completing the square' and performing the Gaussian integral over the B field.

The limit $q \to 0$ is BF theory (topological as we know). The purpose of this talk is to highlight possible directions to find

$$\langle \mathcal{O} \rangle := \int DADB\mathcal{O}[A,B]e^{-i\int_\Sigma \mathrm{Tr}(BF)}$$

What we know already:

- All such expectation values will be topological invariants;
- We know how to calculate the vacuum expectation value for all compact surfaces.

Note: The nomenclature used for 'moduli space of flat connections' is $L^2(\mathcal{A}_0/G)$. Step by step: $L^2(\mathcal{A})$ is the Hilbert space of functions of the connections, $L^2(\mathcal{A}/G)$ is the gauge invariant subspace; finally, to obtain $L^2(\mathcal{A}_0/G)$ we gauge fix $F = 0$.

BF theory in two dimensions

We have two pictures to reconciliate:

- Lattice gauge theory (independent of the lattice chosen);
- the standard TQFT approach.

As previously mentioned $L^2(\mathcal{A}_0/G)$ is the Hilbert space of gauge invariant functions of A, subject to the choice of gauge $F=0$. The inner product can be constructed from the usual Lie group non-degenerate bilinear form ($\int_G \bar{\psi}\phi$). Lattice gauge theory provides us with a basis for this space of functions: we take the radical step of replacing the smooth manifold Σ by a topologically equivalent simplicial complex $|K|$, a triangulation. Unfortunately, all the topological information of the principal bundle is lost in this manner and the result is interpreted as 'averaging over all possible principal G-bundles'. As Fig. 1 tries to depict, we will associate with each dual edge of the triangulation a parallel transport variable $\rho_a^b\left(e^{\int_{e_1} A}\right) = \rho_a^b\left(g\right), g \in G$. These variables

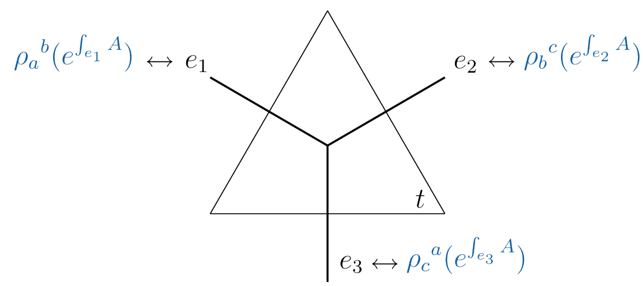


Figure 1: A 1-simplex t of the triangulation together with the dual information

are matrix elements of the irreducible representations of the Gauge group G (ρ labels such a representation). If each vertex is also given the structure of an intertwiner (details omitted) the networks hence formed will be dense in $L^2(\mathcal{A}_0/G)$. This would be enough to calculate the vacuum expectation value in the world of lattice gauge theory, via

$$Z(\Sigma) = \sum_{i\in I} \sum_{a,b,c=1}^{\dim i} \prod_{\text{triangles } t} \int_G {(\rho_i)_a}^b(g){(\rho_i)_b}^c(h){(\rho_i)_c}^a(l), \quad ghl = 1_G$$

and by imposing triangulation independence. As a physicist I like this formula because it resembles the original functional integral:

$$Z(\Sigma) = \int DADBe^{-i\int_\Sigma \mathrm{Tr}(BF)} = \int DA\delta(F).$$

However, my mathematician half has some reservations towards using it, since we have all the technology of TQFTs at our disposal. Recall TQFTs in two dimensions require the association of circles with vector spaces, cobordisms with linear maps, a tensor product etc. In fact, because of the product requirement, what we really need is a special associative algebra A: one that comes with a non-degenerate linear map $\varepsilon : A \to k$.

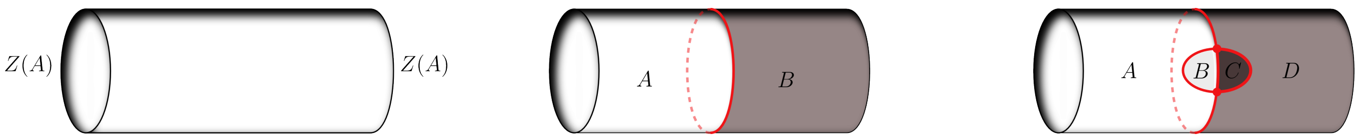
TQFT's in its full realization allow for too much generality (nilpotent algebras). As can be seen in the work by Fukuma, Hosono and Kawai we need the algebra to be symmetric and separable (semi-simple in most cases). This will underly our TQFT from lattice gauge theory. For the example at hand, the algebra we are interested is the group algebra. To see this we would note that the matrix elements of representations in $\mathrm{Irr}(G)$ (a set dense in $L^2(\mathcal{A}_0/G)$) is also dense in $\mathbb{C}[G]$. The most natural candidate might then seem to be $\mathbb{C}[G]$ (with evaluation at the identity as its linear functional), but it turns out we really need $\mathbb{R}[G]$. This comes from the treatment of non-orientable surfaces (as explored by Karimipour and Mostafazadeh). It requires additional structure: a $\star$-operation (an anti-homomorphism that is an involution). In the case at hand, this $\star$ would be the linear extension of the inverse map of G, that forces us to take the algebra to be $\mathbb{R}[G]$. The volume of moduli space of flat connections would be given by

$$\sum_{i\in I} f_i (\dim i)^{\chi_\Sigma}, \tag{1}$$

with f_i given by -1, 0 or 1 if the representation labelled by i is of quaternionic, complex or real type.

TQFTs with defects

Expression (1) is also the one we would obtain from lattice gauge theory calculations. Then what is the advantage of using the TQFT formalism? The answer is its power: we know how to treat (finite) surfaces with boundary. In particular, take a look at the cylinder map depicted in Fig. 2a. One can conclude that it is an operator (linear map) $A \to A$. In fact, we can write with no loss of generality $Z_A \to Z_A$ because it is a surjective projection onto Z_A. When we look at our case of interest, $A = \mathbb{R}[G]$ we can in fact conclude it is an operator that preserves gauge invariance! A loop variable g is invariant under the adjoint action $UgU^{-1}, U \in G$ and at $Z_{\mathbb{R}[G]}$ lie exactly the functions satisfying $f(g) = f(UgU^{-1})$. The interesting part is that there are many



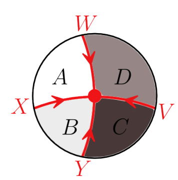
(a) Cylinder 'bare' map

(b) 'Loop' operator

(c) 'Theta' operator

Figure 2: Several possible 'gauge invariant' operators associated with a cylinder

more operators yet to be given a physical interpretation. A whole panoply has been introduced by Davydov, Kong and Runkel (2011) (Fig. 2b and 2c). To each directed defect line we associate an algebra bimodule. In lattice gauge theory, the axioms of a bimodule appear from the topological restrictions exactly as the separable Frobenius algebra appears for the bare theory. The vertices (see Fig. 3) are again associated with intertwiner maps now from a (cyclic invariant) tensor product of bimodules to the underlying field (these are isomorphisms in several important cases).



$\longleftrightarrow \; \circlearrowleft_A X \otimes_B Y \otimes_C V \otimes_D W \to k$

Figure 3: An example of a defect intertwiner

Remark to the categorical inclined minds

TQFT's with defects (at least from lattice gauge theory) form a monoidal bicategory (with duals), known as Bim (objects are separable Frobenius k-algebras A, B, morphisms are bimodules ${}_AX_B$, ...). The spaces of cyclic invariant tensor products can be obtained using the "round trace" defined by Willerton in analogy with the usual trace of monoidal categories (with duals). The invariant subspaces correspond to the first Hochschild homology class $HH_0(A, X)$. Something remarkable happens for separable Frobenius algebras: the invariants $HH_0(A, X)$ and $HH^0(A, X)$ match (this is directly related to the fact the sub-vector space $A/[A, A]$ is in fact isomorphic to the ideal $Z(A)$). This means they are also given by the "diagonal trace" discovered independently by Ganter, Kapranov '07 and Willerton and defined for any 2-category (and that is conjugation invariant). This seems to point towards an equivalence of the two traces, in certain bicategories, yet to be found.

Example (constrained BF theory)

I would like to end my talk with something more palpable, with a little help from Yang-Mills à la Witten. Consider the following vacuum expectation value:

$$\tilde{Z}(\Sigma) = \int DADBD\lambda e^{-\frac{g^2}{2}\int_\Sigma \lambda(\mathrm{Tr}(B^2) - iR^2) - i\int_\Sigma \mathrm{Tr}(BF)}.$$

We can reinterpret it has

$$\langle\mathcal{O}\rangle = \int DADB\mathcal{O}[B]e^{-i\int_\Sigma \mathrm{Tr}(BF)}, \qquad \mathcal{O}[B] = \int D\lambda e^{-\frac{g^2}{2}\int_\Sigma \lambda(\mathrm{Tr}(B^2)-iR^2)}$$

We have been able to conclude $\tilde{Z}(\Sigma) = f_\imath(\dim \imath)^{\chi_\Sigma}$ if $iR^2 = c_2(\imath)$ or zero otherwise; it was then not difficult to find the counterpart of $\mathcal{O}$ as a topological defect. It is a loop as in Fig. 2b associated to the representation labeled by $\imath$ regarded as an $\mathbb{R}[G]$ bimodule.

The expressions for operators associated to BF theory regarded as topological defects have been found. By construction they are all topological invariants and in the next months it is our goal to find their counterpart in terms of operators $\mathcal{O}[A,B]$ of quantum field theory.