Max Planck Institute for Mathematics
California Institute of Technology

super-A-polynomial

Sergei Gukov

based on: hep-th/0306165 (generalized volume conjecture)

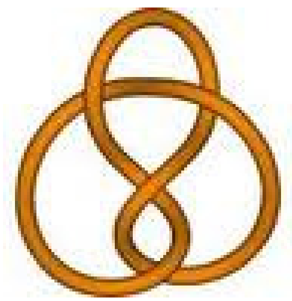
with T.Dimofte, arXiv:1003.4808 (review/survey)

with H.Fuji and P.Sulkowski, arXiv:1203.2182 (new!)

Kashaev's observation

[R. Kashaev, 1996]

knot K



invariant $\langle K \rangle_n \in \mathbb{C}$ labeled by a positive integer n

- defined via R-matrix
- very hard to compute

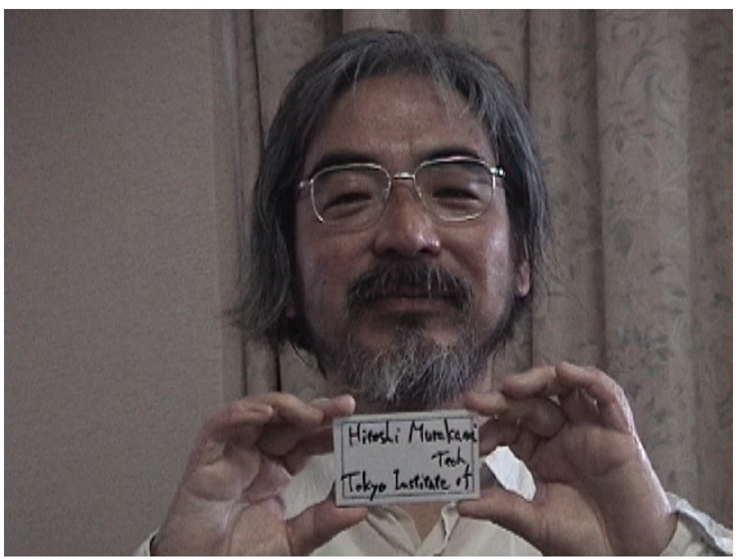
$$\lim_{n \to \infty} \frac{1}{n} \log \langle K \rangle_n = \mathrm{Vol}\,(S^3 \setminus K)$$

("volume conjecture")

A first step to understanding the Volume Conjecture

$\langle K \rangle_n = J_n(q)$ colored Jones polynomial

with $q = \exp(2\pi i/n)$



Hitoshi Murakami

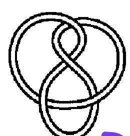
Jun Murakami (1999)

Colored Jones polynomial

$J_2(q) = J(q) =$ Jones polynomial

- In Chern-Simons gauge theory [E.Witten]

Wilson loop operator

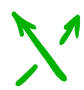
$\Big\langle$  $\Big\rangle_R =$ polynomial in q

R: 2-dimn'l representation of $SU(2)$

Colored Jones polynomial

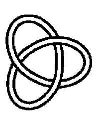
$J_2(q) = J(q) =$ Jones polynomial

- Skein relations:

$q^2 J($  $) - q^{-2} J($  $) = (q^{-1} - q) \cdot J($  $)$

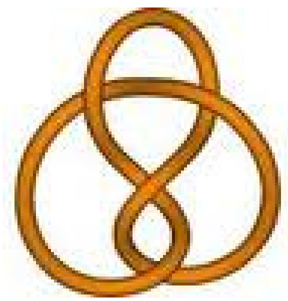
$J(\text{unknot}) = q^{-1} + q$

<u>Example:</u>

$J($  $) = q + q^3 + q^5 - q^9$

Colored Jones polynomial

knot **K** → **n**-colored Jones polynomial:



$J_n(K;q) \in \mathbb{Z}[q, q^{-1}]$

R = **n**-dimn'l representation of **SU(2)**

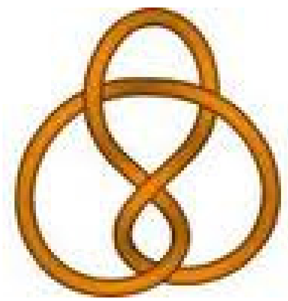
- "Cabling formula":

$$J_{\oplus_i R_i}(K;q) = \sum_i J_{R_i}(K;q)$$

$$J_R(K^n;q) = J_{R^{\otimes n}}(K;q)\,,$$

Colored Jones polynomial

knot **K** → **n**-colored Jones polynomial:



$J_n(K;q) \in \mathbb{Z}[q, q^{-1}]$

R = **n**-dimn'l representation of **SU(2)**

$$J_1(K;q) = 1\,,$$
$$J_2(K;q) = J(K;q)\,,$$
$$\mathbf{2}^{\otimes 2} = \mathbf{1} \oplus \mathbf{3} \Rightarrow J_3(K;q) = J(K^2;q) - 1\,,$$
$$\mathbf{2}^{\otimes 3} = \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{4} \Rightarrow J_4(K;q) = J(K^3;q) - 2J(K;q)$$
$$\ldots,$$

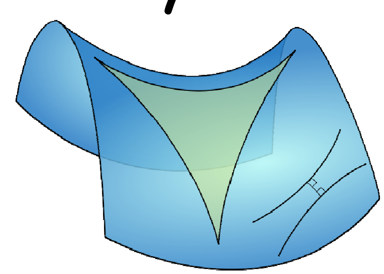
Volume Conjecture

Murakami & Murakami:

$$\langle K \rangle_n = J_n(K; q = e^{2\pi i/n})$$

$$\lim_{n \to \infty} \frac{2\pi \log |J_n(K; q = e^{2\pi i/n})|}{n} = \mathrm{Vol}(M)$$

quantum group invariants (combinatorics, representation theory) ↔ classical hyperbolic geometry



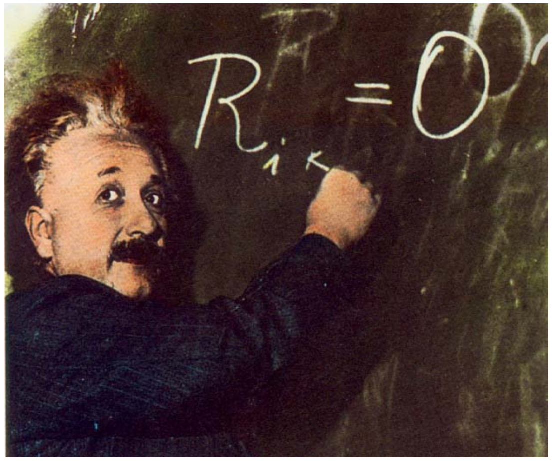
Physical interpretation of the Volume Conjecture

[S.G., 2003]

- analytic continuation of **SU(2)** is **SL(2,ℂ)**

$$\lim_{n\to\infty} \frac{2\pi \log |J_n(K; q = e^{2\pi i/n})|}{n} = \mathrm{Vol}(M)$$

- classical **SL(2,ℂ)** Chern-Simons theory = classical 3d gravity (hyperbolic geometry)



Physical interpretation of the Volume Conjecture

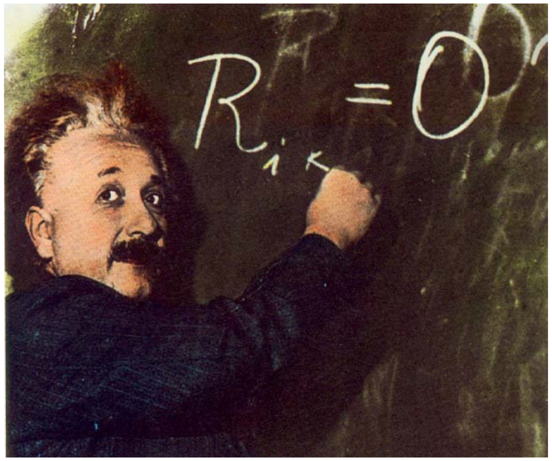
[S.G., 2003]

constant negative curvature metric on **M** ⟷ flat **SL(2,ℂ)** connection on **M = S³ \ K**

$$R_{ij} = -2g_{ij}$$

$$d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0$$

- classical **SL(2,ℂ)** Chern-Simons theory = classical 3d gravity (hyperbolic geometry)



Physical interpretation of the Volume Conjecture

[S.G., 2003]

constant negative curvature metric on **M** ⟷ flat SL(2,ℂ) connection on **M = S³ \ K**

$$R_{ij} = -2g_{ij}$$

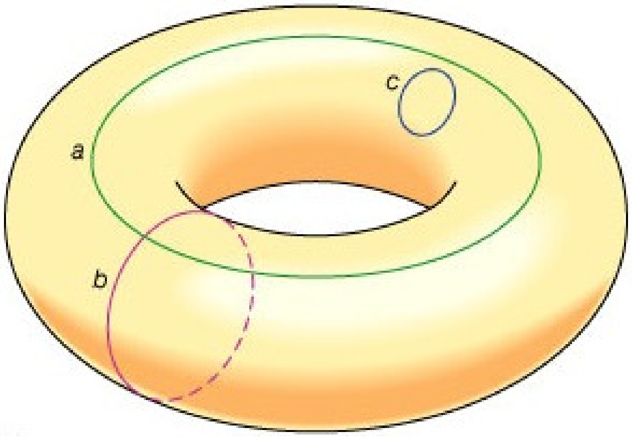
$$d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0$$

classical solution in 3D gravity with negative cosmological constant ($\Lambda = -1$)

classical solution in CS gauge theory

Example: unknot = BTZ black hole

K = unknot ➡ M = solid torus

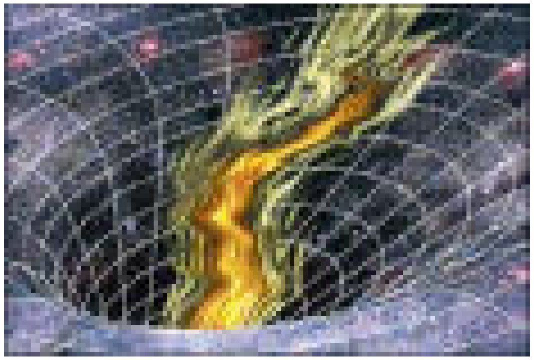


Euclidean BTZ black hole:

$$ds^2 = N^2 d\tau^2 + N^{-2} dr^2 + r^2 \left(d\phi^2 + N^\phi d\tau \right)^2$$

$$N = \sqrt{r^2 - M - \frac{J^2}{4r^2}} \quad , \quad N^\phi = -\frac{J}{2r^2}$$

$$r_\pm^2 = \frac{M}{2} \left[1 \pm \sqrt{1 + \left(\frac{J}{M} \right)^2} \right]$$



Physical interpretation of the Volume Conjecture

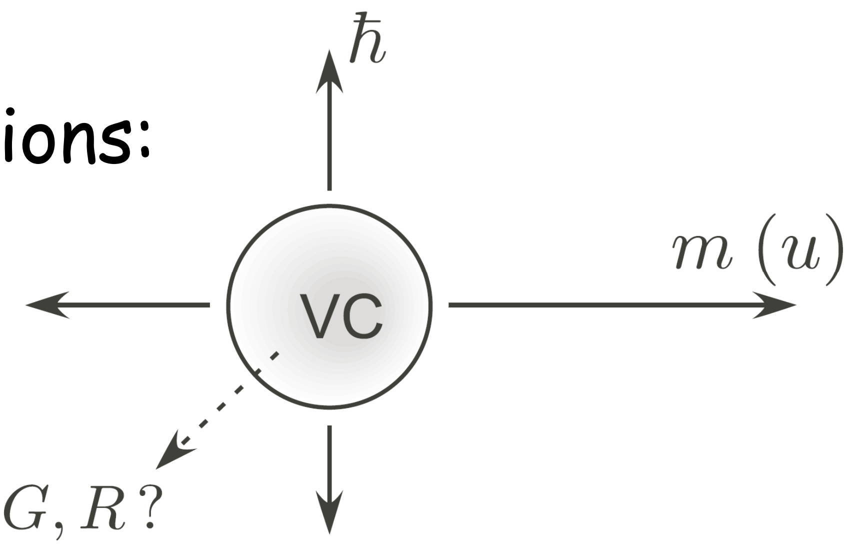
[S.G., 2003]

Moral:

$$\lim_{n\to\infty}\left(\text{SU(2) Chern-Simons}\right) \simeq \text{SL(2,C) Chern-Simons}$$

Classical limit $q = \exp(2\pi i/n) \to 1$

- leads to many generalizations:

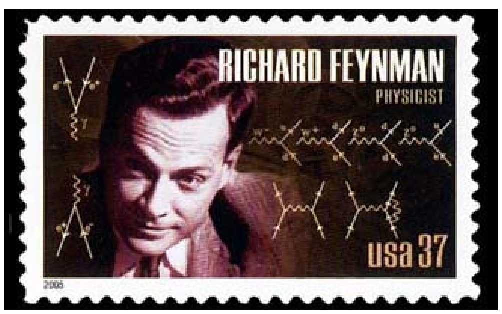


Physical interpretation of the Volume Conjecture

<u>Generalization 1:</u> 't Hooft limit:

$$q = e^{\hbar} \to 1\,, \qquad n \to \infty\,, \qquad q^n = e^u \equiv x \quad \text{(fixed)}$$

$$J_n(K; q = e^{\hbar}) \quad \overset{n\to\infty}{\underset{\hbar\to 0}{\sim}} \quad \exp\left(\frac{1}{\hbar}S_0(u) + \ldots\right)$$



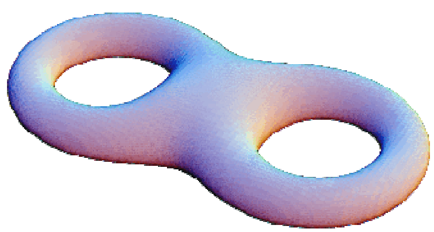
partition function of
SL(2,ℂ) Chern-Simons theory

Generalized Volume Conjecture

$S_0(u)$ = classical action of
SL(2,ℂ) Chern-Simons theory

$\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, M)$: space of solutions

$$dA + A \wedge A = 0$$ on $S^3 \setminus K$



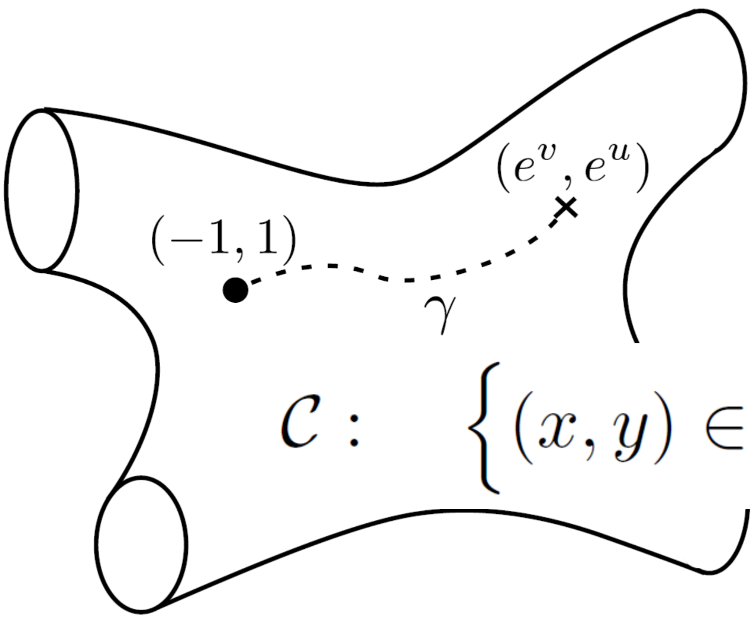
A-polynomial
of a knot K

$$\mathcal{C}: \quad \left\{ (x,y) \in \mathbb{C}^* \times \mathbb{C}^* \middle| A(x,y) = 0 \right\}$$

Algebraic curves and knots

$$S_0(u) \;=\; \int v du \;=\; \int \log y \frac{dx}{x}$$

$x = e^u \qquad y = e^v$



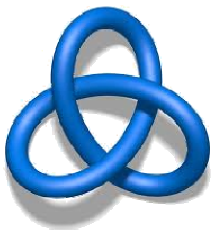
$$\mathcal{C}: \quad \left\{ (x,y) \in \mathbb{C}^* \times \mathbb{C}^* \,\middle|\, A(x,y) \;=\; 0 \right\}$$

A-polynomial of a knot K

Physical interpretation of the Volume Conjecture

[S.G., 2003]

<u>Generalization 2:</u>



knot **K**

n-colored Jones polynomial:

$J_n(K;q)$

recursion relation:

$$\alpha(q^n, q)\, J_{n-1} + \beta(q^n, q)\, J_n + \gamma(q^n, q)\, J_{n+1} \;=\; 0$$

rational functions

Quantum Volume Conjecture

using $x \equiv e^u = q^n = \text{fixed}$ [S.G., 2003]

we can write this recursion relation as:

$$\widehat{A}\, J_*(K;q) \;\simeq\; 0$$

where $\widehat{A}(\widehat{x},\widehat{y};q) = \;\alpha\widehat{y}^{-1} + \beta + \gamma\widehat{y}$

$$\widehat{x}J_n \;=\; q^n J_n$$

$$\widehat{y}J_n \;=\; J_{n+1}$$

so that

$$\widehat{y}\widehat{x} \;=\; q\widehat{x}\widehat{y}$$

Quantum Volume Conjecture

[S.G., 2003]

- In the classical limit **q→1** the operator $\widehat{A}(\widehat{x},\widehat{y};q)$ becomes **A(x,y)** and the way it comes about is that

$$x\ ,\ y \quad \rightsquigarrow \quad \widehat{x}\ ,\ \widehat{y}$$

$$A(x,y) \ = \ 0 \quad \rightsquigarrow \quad \widehat{A}(\widehat{x},\widehat{y})\, Z_{\mathrm{CS}}(M) \ = \ 0$$

- in the mathematical literature was independently proposed around the same time, and is know as the AJ-conjecture

[S.Garoufalidis, 2003]

Quantization and B-model

$$\log T(u) = \lim_{u_1 \to u_2 = u} \int \left(\frac{du_1\, du_2}{(u_1 - u_2)^2} - B(u_1, u_2) \right)$$

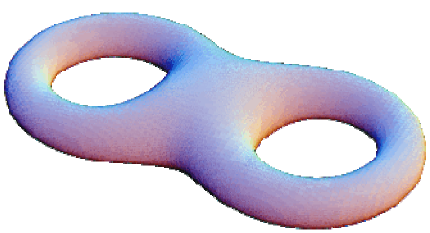
Bergman kernel

- simple formula that turns classical curves $A(x,y) = 0$ into quantum operators

[S.G., P.Sulkowski]

$$x\,,\,y \rightsquigarrow \hat{x}\,,\,\hat{y}$$

$$A(x,y) = 0 \rightsquigarrow \hat{A}(\hat{x},\hat{y})\, Z_{\mathrm{CS}}(M) = 0$$



$$\hat{A}(\hat{x},\hat{y};q) = \sum_{m,n} a_{m,n}\, q^{c_{m,n}}\, \hat{x}^m\, \hat{y}^n$$

Quantization and B-model

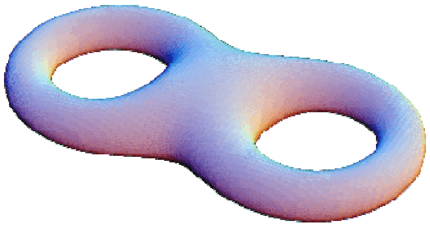
$$\log T(u) = \lim_{u_1 \to u_2 = u} \int \left(\frac{du_1\, du_2}{(u_1 - u_2)^2} - B(u_1, u_2) \right)$$

$$\sum_{m,n} a_{m,n}\, c_{m,n}\, x^m y^n = \frac{1}{2} \left(\frac{\partial_u A}{\partial_v A} \partial_v^2 + \frac{\partial_u T}{T} \partial_v \right) A$$

[S.G., P.Sulkowski]

$$x\ ,\ y \quad \rightsquigarrow \quad \widehat{x}\ ,\ \widehat{y}$$

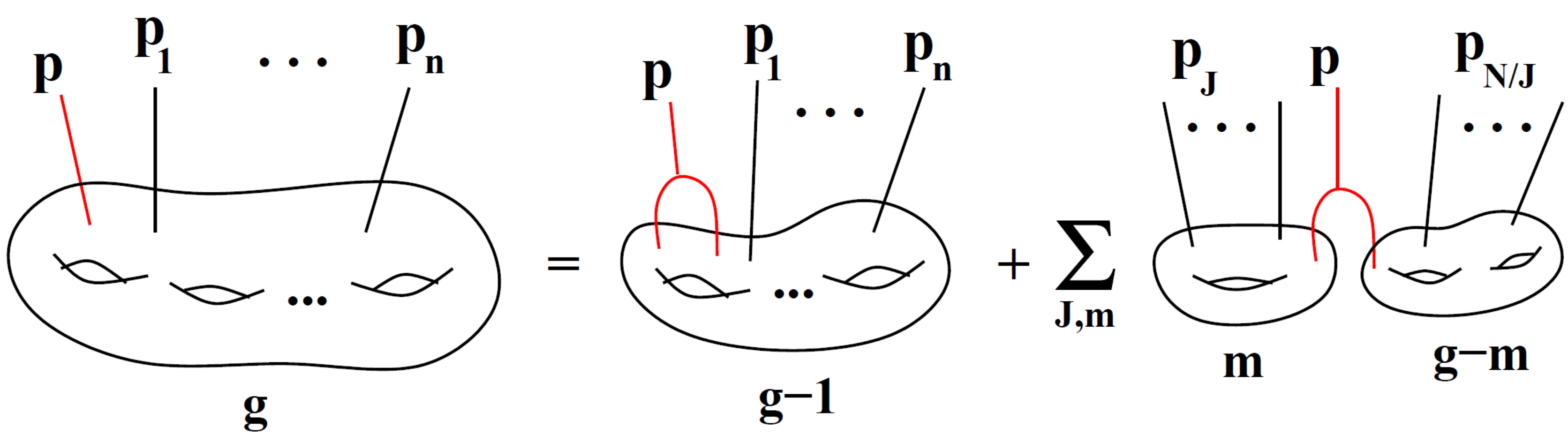
$$A(x,y) = 0 \quad \rightsquigarrow \quad \widehat{A}(\widehat{x}, \widehat{y})\, Z_{\mathrm{CS}}(M) = 0$$



$$\widehat{A}(\widehat{x}, \widehat{y}; q) = \sum_{m,n} a_{m,n}\, q^{c_{m,n}}\, \widehat{x}^m\, \widehat{y}^n$$

Quantization and B-model

[B.Eynard, N.Orantin]
[V.Bouchard, A.Klemm, M.Marino, S.Pasquetti]
[A.S.Alexandrov, A.Mironov, A.Morozov]
[R.Dijkgraaf, H.Fuji, M.Manabe]
:



$x(p)$ and $y(p)$ $\rightsquigarrow$ W_n^g $\rightsquigarrow$ Z $\boxed{\hat{A}(\hat{x},\hat{y})}$

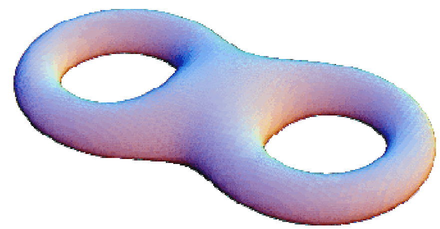
(recall: $\hat{A}(\hat{x},\hat{y})\, Z(\mathbf{M}) = 0$)

Classical A-polynomial

[D.Cooper, M.Culler, H.Gillet, D.Long, P.B.Shalen]

M = 3-manifold
with a toral boundary,
e.g. a knot complement

planar algebraic curve:



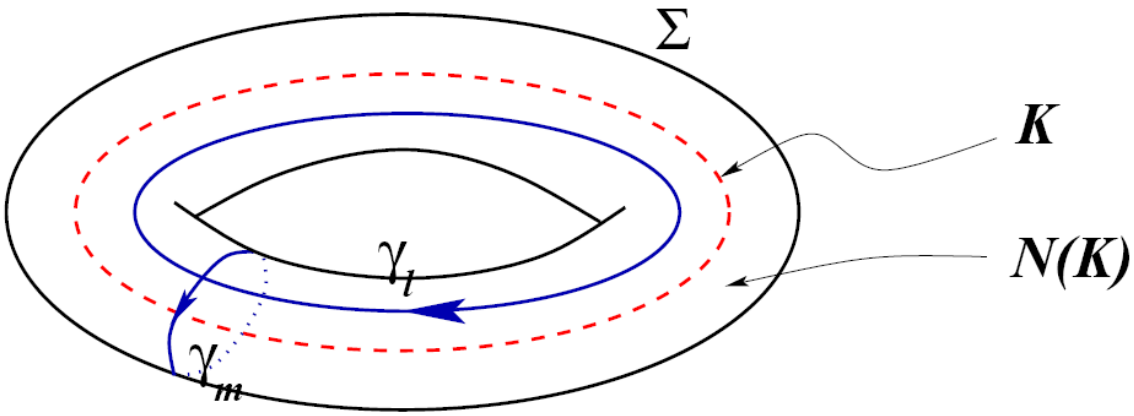
A-polynomial
of a knot K

$$\mathcal{C}: \quad \left\{ (x,y) \in \mathbb{C}^* \times \mathbb{C}^* \,\middle|\, A(x,y) = 0 \right\}$$

representation
variety:

$$\rho\colon \pi_1(M) \to SL(2,\mathbb{C})$$

Consider, for a example, a knot complement:



$$\rho(\gamma_l) = \begin{pmatrix} y & * \\ 0 & y^{-1} \end{pmatrix} \quad , \quad \rho(\gamma_m) = \begin{pmatrix} x & * \\ 0 & x^{-1} \end{pmatrix}$$

$$\pi_1 = < a, b \mid a\,b\,a = b\,a\,b >$$

$$\begin{cases} m = a \\ \ell = b\,a^2\,b\,a^{-4} \end{cases} \Rightarrow \quad A(x,y) = (y-1)(y+x^3)$$

Properties of the A-polynomial

$H_1(M) \cong \mathbb{Z}$ for any knot complement

$\Rightarrow$ $A(x,y) = (y-1)\,(\,\ldots\,)$

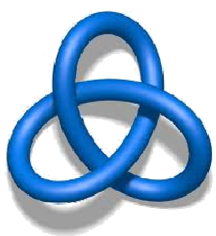
- $(y-1)$: Abelian representations
- $(\,\ldots\,)$: non-Abelian representations

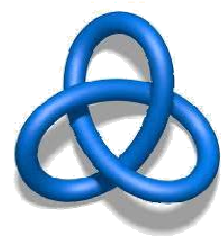
- If **K** is a hyperbolic knot, then $A(x,y) \neq y-1$.
- If **K** is a knot in a homology sphere, then the A-polynomial involves only even powers of x.

Properties of the A-polynomial

- A-polynomial can distinguish mirror knots:

$$A(x,y) = 0 \quad \xleftrightarrow{\text{parity}} \quad A(x^{-1},y) = 0$$



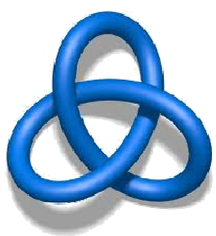


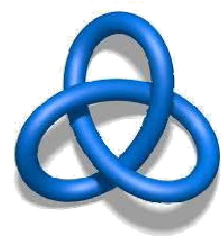
- If K is a hyperbolic knot, then $A(x,y) \neq y-1$.
- If K is a knot in a homology sphere, then the A-polynomial involves only even powers of x.

Properties of the A-polynomial

- A-polynomial can distinguish mirror knots:

$$A(x,y) = 0 \quad \xleftrightarrow{\text{parity}} \quad A(x^{-1},y) = 0$$





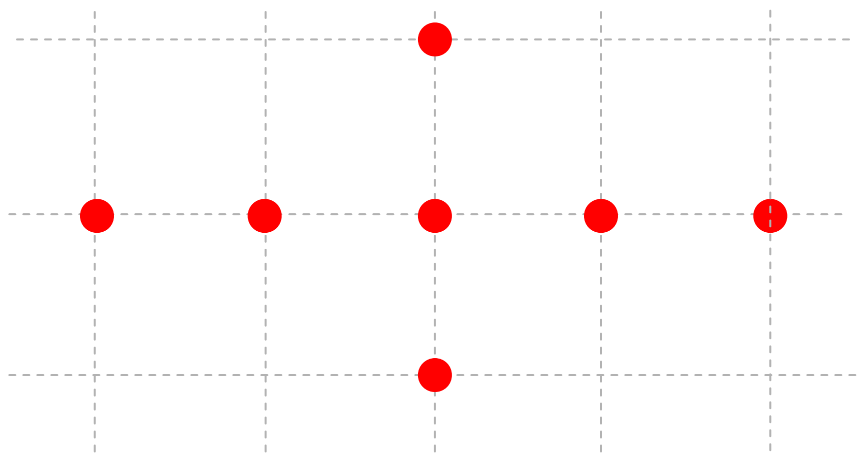
- The A-polynomial is reciprocal:

$$A(x,y) \sim A(x^{-1},y^{-1})$$

- The A-polynomial has integer coefficients

Properties of the A-polynomial

- The A-polynomial is tempered, *i.e.* the faces of the Newton polygon of $A(x,y)$ define cyclotomic polynomials in one variable:



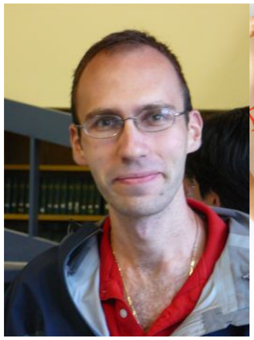
- The slopes of the sides of the Newton polygon of $A(x,y)$ are boundary slopes of incompressible surfaces in **M**.

Connection to Physics

[S.G., 2003]

- Explains known facts and leads to many new ones:
  - the A-polynomial curve should be viewed as a holomorphic Lagrangian submanifold (as opposed to a complex equation)
  - its quantization with symplectic form $\frac{dy}{y} \wedge \frac{dx}{x}$ leads to an interesting wave function
  - has all the attributes to be an analog of the Seiberg-Witten curve for knots and 3-manifolds

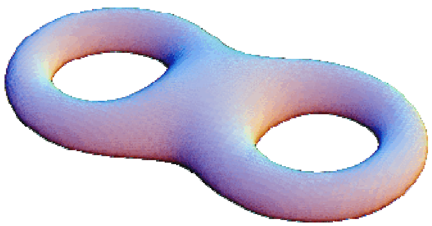
w/ T.Dimofte, L.Hollands



Connection to Physics

[S.G., 2003]

- For any closed cycle:



$$\oint_\Gamma \log y \frac{dx}{x} \in 2\pi^2 \cdot \mathbb{Q}$$

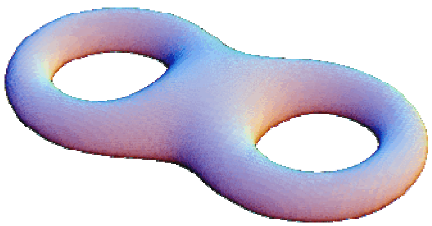
- Has an elegant interpretation in terms of algebraic K-theory and the Bloch group of $\overline{\mathbb{Q}}$

[S.G., P.Sulkowski]

$\mathcal{C}$ is quantizable $\longleftrightarrow$ $\{x, y\} \in K_2(\mathbb{C}(\mathcal{C}))$ is a torsion class

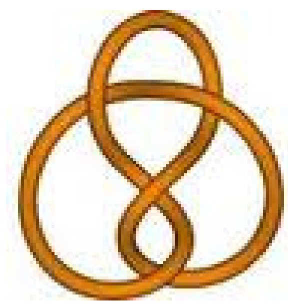
Connection to Physics

- For any closed cycle:



$$\oint_\Gamma \log y \frac{dx}{x} \in 2\pi^2 \cdot \mathbb{Q}$$

<u>Example:</u>



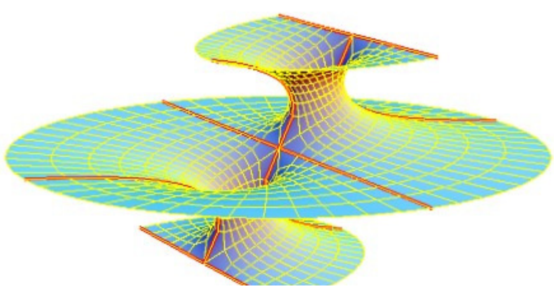
$$A(x,y) = 1 - (x^{-4} - x^{-2} - 2 - x^{2} + x^{4})y + y^{2}$$

$$B(x,y) = 1 - (x^{-6} - x^{-2} - 2 - x^{2} + x^{6})y + y^{2}$$

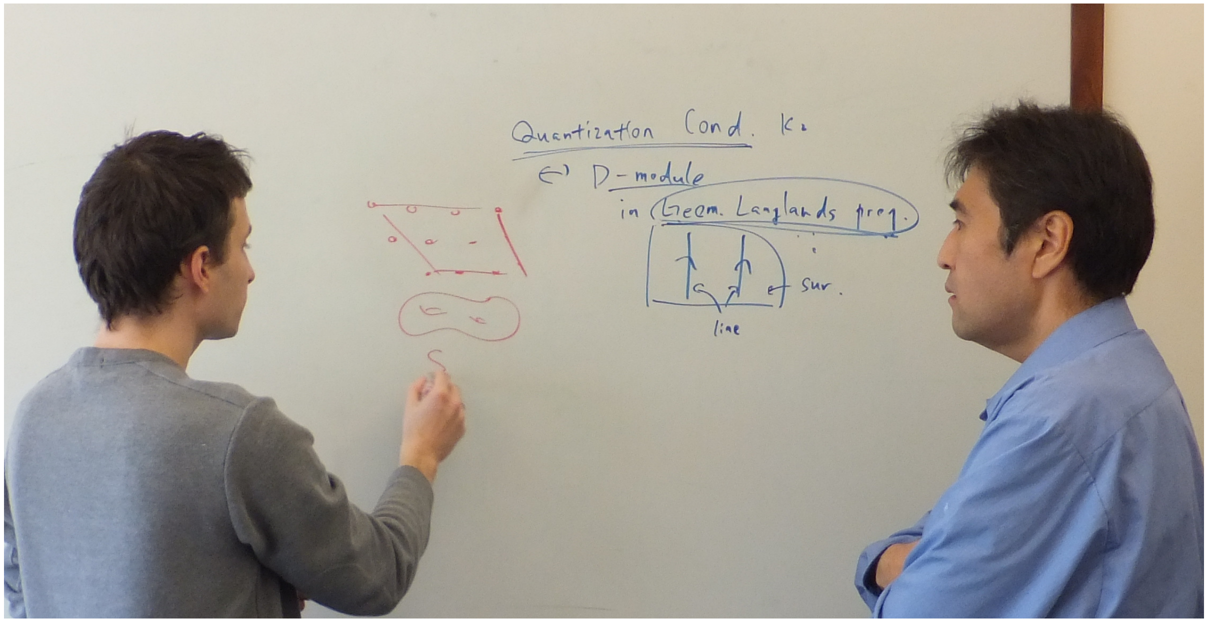
Has all the symmetries, but is NOT A-polynomial of any knot

Lessons

- A-polynomial as a limit shape (in large color limit)
- A-polynomial as a characteristic variety for recursion relations / q-difference equations
- Can be categorified to a homological volume conjecture!



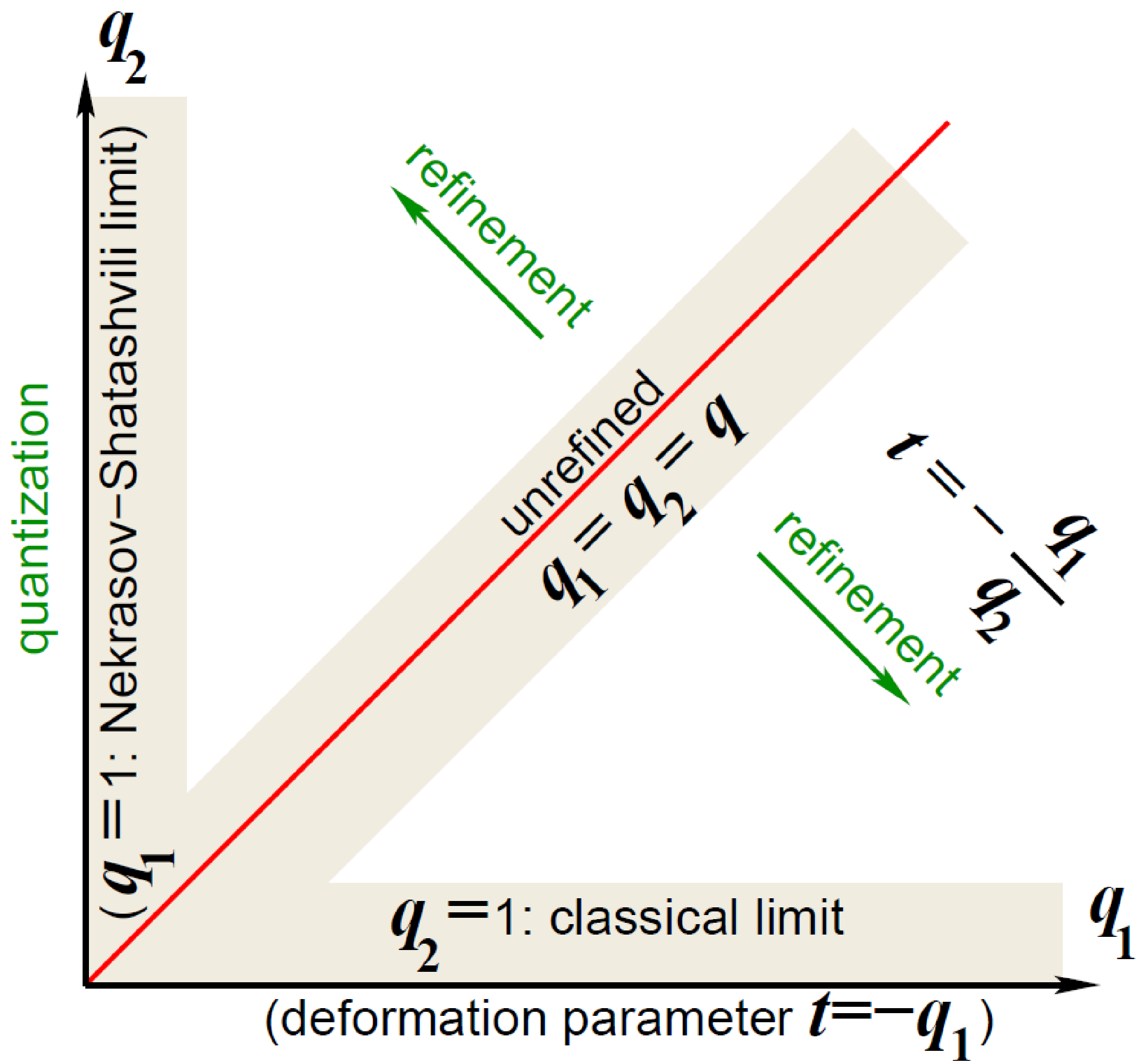
w/ H.Fuji, P.Sulkowski



Refinement / Categorification

	Algebraic curve	Refinement
Knots / SL(2,C) CS	A-polynomial	Homological invariants $P_n(q,t)$
Matrix models	spectral curve	β-deformation
4d gauge theories 3d superconformal indices	Seiberg-Witten curve A-polynomial	refinement $t = -\frac{q_1}{q_2}$
Topological strings (B-model)	mirror Calabi-Yau geometry $A(x,y) = zw$	

Deformation vs Quantization



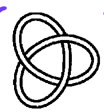
Knot Homologies

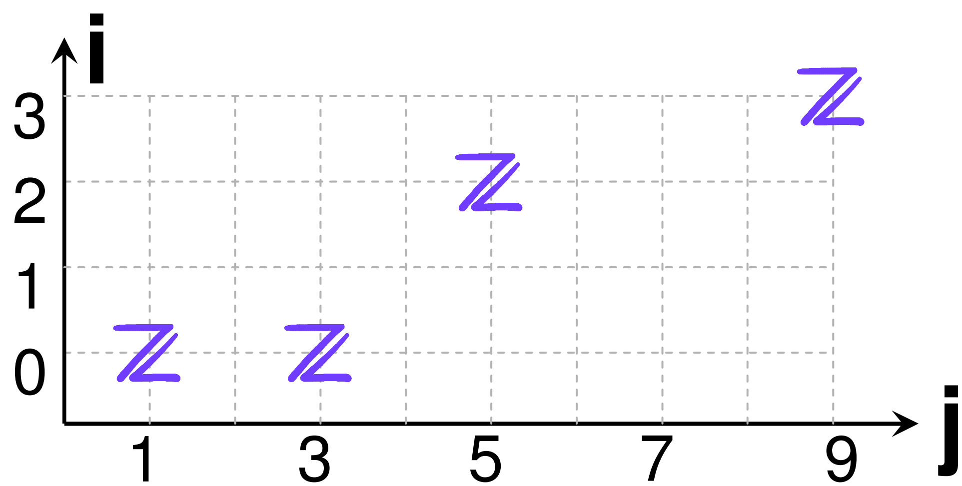
- Khovanov homology: $H^{Kh}_{i,j}(K)$

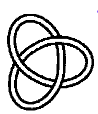
$$J(q) = \sum_{i,j} (-1)^i \, q^j \, \dim H^{Kh}_{i,j}(K)$$

[M.Khovanov]

Example:

$H^{Kh}_{i,j}($  $)$



$J($  $) = q + q^3 + q^5 - q^9$

(Large) Color Behavior of Knot Homologies

- similarly, $\mathcal{H}^{sl(2),V_n}(K)$ is the **n**-colored sl(2) knot homology:

$$P_n(K;q,t) \;=\; \sum_{i,j} q^i t^j \dim \mathcal{H}^{sl(2),V_n}_{i,j}(K)$$

- categorify **n**-colored Jones polynomials:

$$J_n(K;q) \;=\; P_n(K;q,t=-1)$$

(Large) Color Behavior of Knot Homologies

- similarly, $\mathcal{H}^{sl(2),V_n}(K)$ is the **n**-colored sl(2) knot homology:

$$P_n(K;q,t) \; = \; \sum_{i,j} q^i t^j \dim \mathcal{H}_{i,j}^{sl(2),V_n}(K)$$

- satisfy recursion relations
- exhibit beautiful large-**n** asymptotic behavior

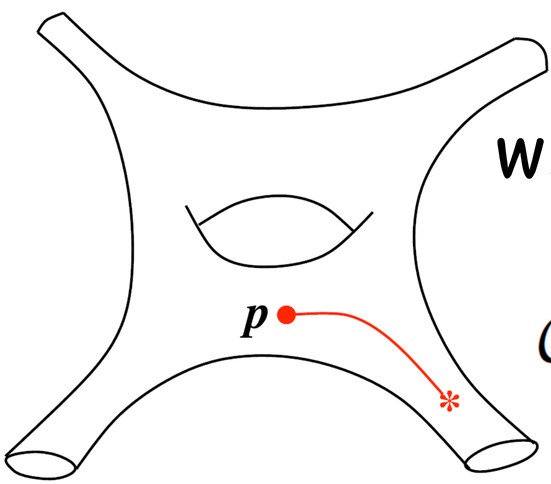
↳ Both controlled by a "refined" algebraic curve

Refinement / Categorification

Generalized Volume Conjecture: [S.G., H.Fuji, P.Sulkowski]

$$q = e^{\hbar} \to 1 \,, \qquad t = \text{fixed} \,, \qquad x \equiv e^{u} = q^{n} = \text{fixed}$$

$$P_n \simeq \exp\left(\frac{1}{\hbar} S_0(u,t) + \sum_{n=0}^{\infty} S_{n+1}(u,t)\, \hbar^n \right)$$

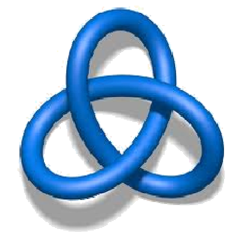


where $S_0(u,t) = \int v du = \int \log y \frac{dx}{x}$

$$\mathcal{C}^{\text{ref}} : \quad \left\{ (x,y) \in \mathbb{C}^* \times \mathbb{C}^* \middle| A^{\text{ref}}(x,y,t) = 0 \right\}$$

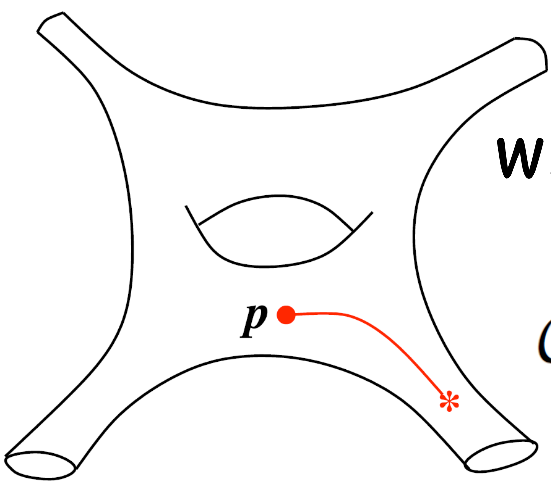
Refined Algebraic Curves

<u>Example:</u> $A(x,y) = (y-1)(y+x^3)$



refinement:

$$A^{\text{ref}}(x,y;t) = y^2 - \frac{1-xt^2+x^3t^5+x^4t^6+2x^2t^2(t+1)}{1+xt^3}\,y + \frac{(x-1)x^3t^4}{1+xt^3}$$



where $S_0(u,t) = \int v\,du = \int \log y \frac{dx}{x}$

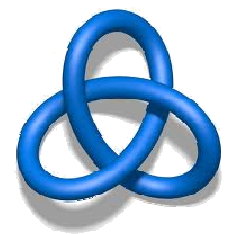
$$\mathcal{C}^{\text{ref}}: \quad \left\{ (x,y) \in \mathbb{C}^* \times \mathbb{C}^* \middle| A^{\text{ref}}(x,y,t) = 0 \right\}$$

Refinement / Categorification

<u>Quantum Volume Conjecture:</u> [S.G., H.Fuji, P.Sulkowski]

$$\widehat{A}^{\mathrm{ref}}(\widehat{x},\widehat{y};q,t)\; P_*(K;q,t) \;\simeq\; 0$$

<u>Example:</u> $\alpha P_{n-1} + \beta P_n + \gamma P_{n+1} \;=\; 0$



$$\alpha \;=\; \frac{x^3(x-q)t^4}{q(q+x^2t^3)(1+xqt^3)}\,,$$

$$\beta \;=\; -\frac{t^2x^2}{1+x^2qt^3} - \frac{q - xqt^2 + x^4t^6 + x^2t^2(1+t+qt)}{(q+x^2t^3)(1+xqt^3)}$$

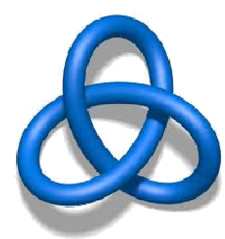
$$\gamma \;=\; \frac{1}{q+x^2q^2t^3}\,.$$

Refinement / Categorification

Quantum Volume Conjecture: [S.G., H.Fuji, P.Sulkowski]

$$\widehat{A}^{\mathrm{ref}}(\widehat{x},\widehat{y};q,t)\, P_*(K;q,t) \;\simeq\; 0$$

Example: $\widehat{A}^{\mathrm{ref}}(\widehat{x},\widehat{y};q,t) \;=\; \alpha \widehat{y}^{-1} + \beta + \gamma \widehat{y}$

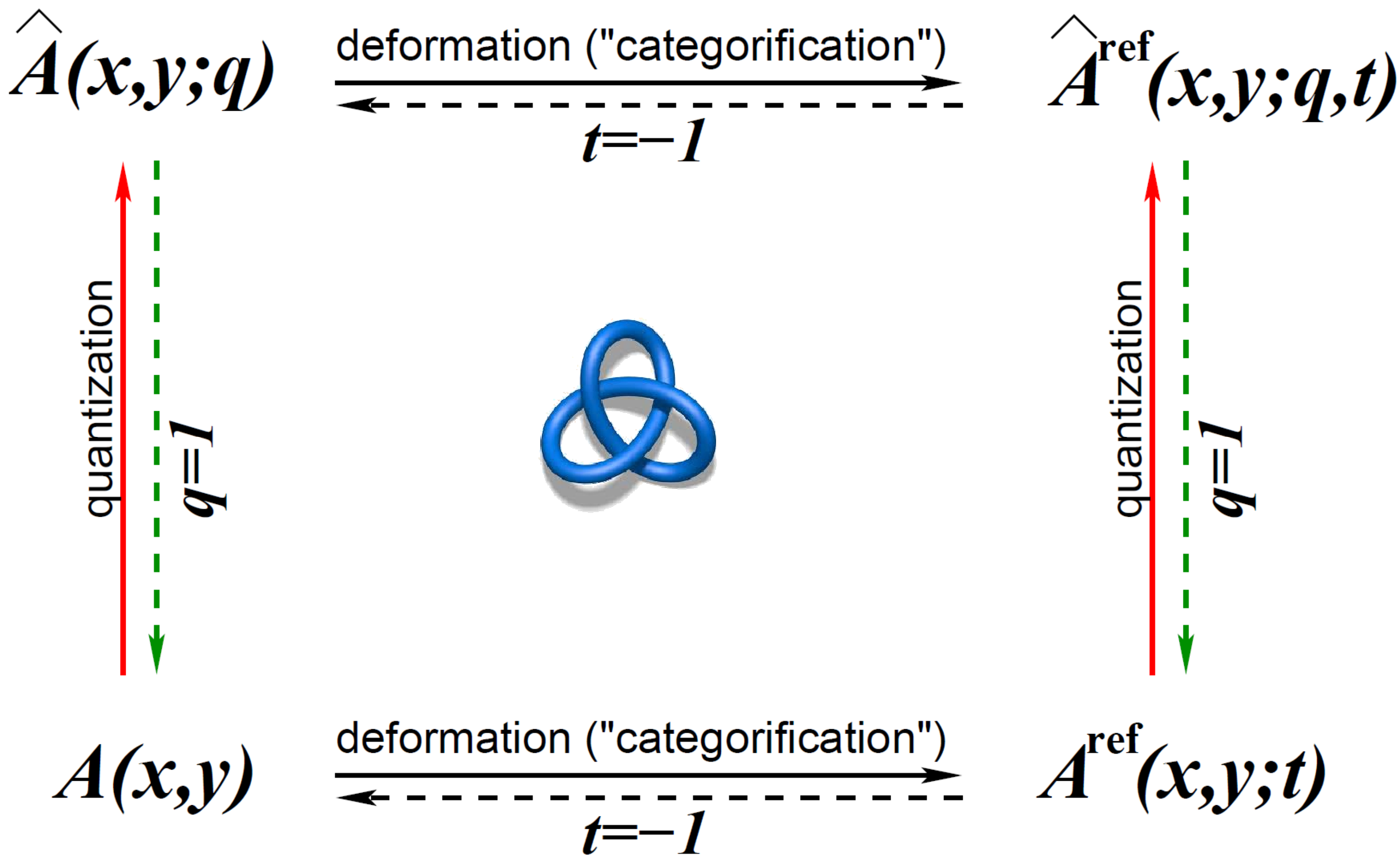


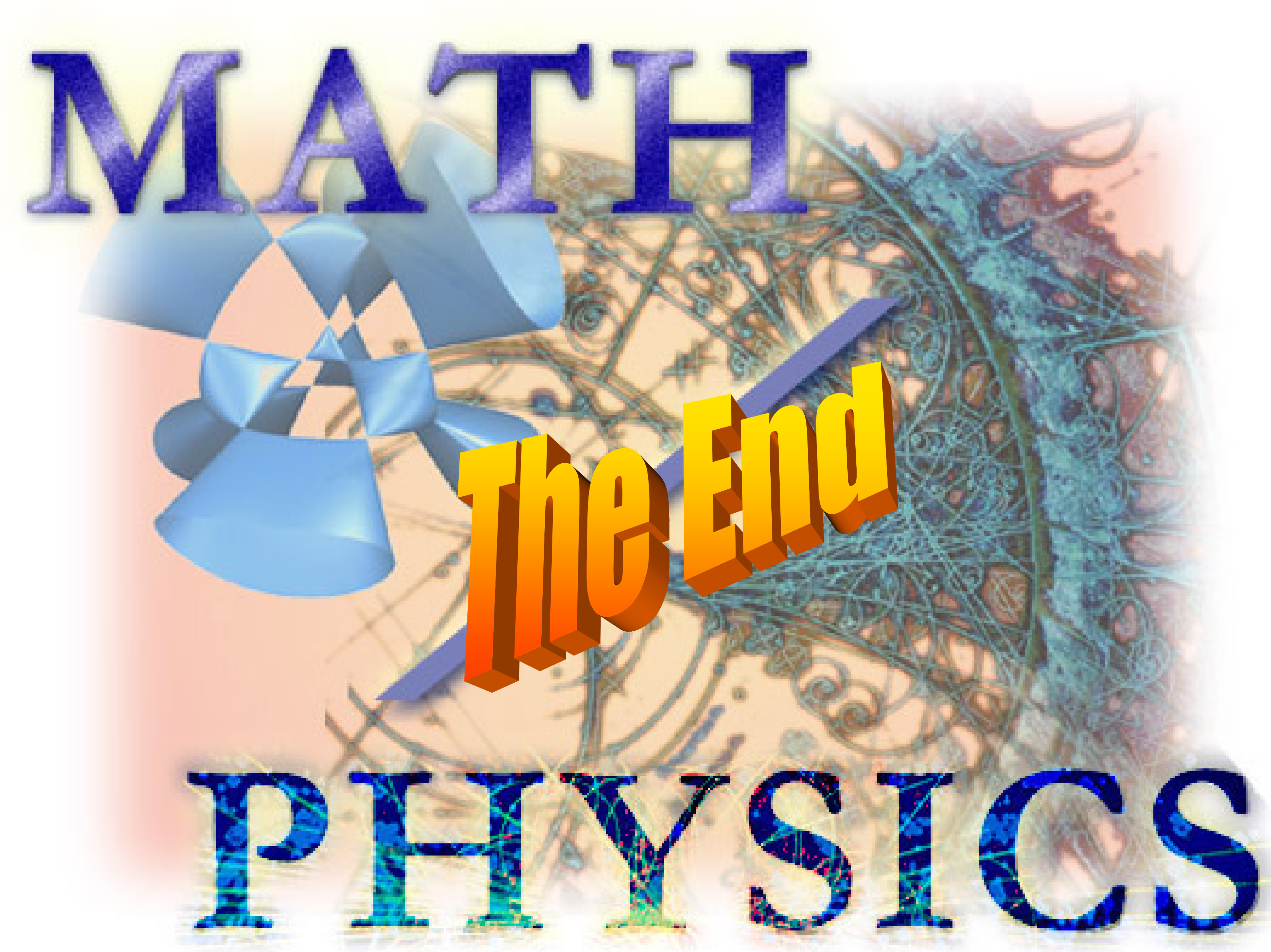
$$\alpha = \frac{x^3(x-q)t^4}{q(q+x^2t^3)(1+xqt^3)}\,,$$

$$\beta = -\frac{t^2x^2}{1+x^2qt^3} - \frac{q - xqt^2 + x^4t^6 + x^2t^2(1+t+qt)}{(q+x^2t^3)(1+xqt^3)}$$

$$\gamma = \frac{1}{q+x^2q^2t^3}\,.$$

Deformation vs Quantization




MATH
The End
PHYSICS